Smith Normal Form and Combinatorics

Richard P. Stanley

January 19, 2020

Smith normal form

A: $n \times n$ matrix over commutative ring ***R*** (with 1)

Suppose there exist ***P***, ***Q*** $\in \mathrm{GL}(n, R)$ such that

$$PAQ := B = \mathrm{diag}(d_1, d_1 d_2, \ldots, d_1 d_2 \cdots d_n),$$

where $d_i \in R$. We then call B a **Smith normal form (SNF)** of A.

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Note. (1) Can extend to $m \times n$.

$$(2)\ \mathrm{unit} \cdot \det(A) = \det(B) = d_1^n d_2^{n-1} \cdots d_n.$$

Thus SNF is a refinement of det.

Row and column operations

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- Add a multiple of a row to another row.
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Over a field, SNF is **row reduced echelon form** (with all unit entries equal to 1).

Existence of SNF

PIR: principal ideal ring, e.g., $\mathbb{Z}$, $K[x]$, $\mathbb{Z}/m\mathbb{Z}$.

Theorem (**Smith**, for $R = \mathbb{Z}$). *If R is a PIR then A has a unique SNF up to units.*

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Theorem (**Smith**, for $R = \mathbb{Z}$). *If R is a PIR then A has a unique SNF up to units.*

Otherwise A "typically" does not have a SNF but may have one in special cases.

Who is Smith?

Henry John Stephen Smith

- born 2 November 1826 in Dublin, Ireland
- educated at Oxford University (England)
- remained at Oxford throughout his career
- twice president of London Mathematical Society
- 1861: SNF paper in *Phil. Trans. R. Soc. London*
- 1868: Steiner Prize of Royal Academy of Sciences of Berlin

More

- died 9 February 1883
- April 1883: shared *Grand prix des sciences mathématiques* with Minkowski

Algebraic note

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Open: every matrix over a Bézout domain has an SNF.

Algebraic interpretation of SNF

R: a PID

A: an $n \times n$ matrix over R with rows $v_1, \dots, v_n \in R^n$

$\mathrm{diag}(e_1, e_2, \dots, e_n)$: SNF of A

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$R^n/(v_1, \dots, v_n)$: **(Kasteleyn) cokernel** of A

An explicit formula for SNF

R: a PID (so gcd's exist)

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Theorem. *$e_1 e_2 \cdots e_i$ is the gcd of all $i \times i$ minors of A.*

minor: determinant of a square submatrix.

Special case: e_1 is the gcd of all entries of A.

Laplacian matrices

$\boldsymbol{L(G)}$: Laplacian matrix of the (loopless) graph G

rows and columns indexed by vertices of G

$$\boldsymbol{L}(G)_{uv} = \begin{cases} -\#(\text{edges } uv), & u \neq v \\ \deg(u), & u = v. \end{cases}$$

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reduced Laplacian matrix $\boldsymbol{L_0(G)}$: for some vertex v, remove from $\boldsymbol{L}(G)$ the row and column indexed by v

Matrix-tree theorem

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Applications to sandpile models, chip firing, etc.

An example

Reduced Laplacian matrix of K_4:

$$A = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}$$

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What about SNF?

An example (continued)

$$\begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & -1 \\ -4 & 4 & -1 \\ 8 & -4 & 3 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & -1 \\ -4 & 4 & 0 \\ 8 & -4 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 0 & 0 & -1 \\ 0 & 4 & 0 \\ 4 & -4 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & -1 \\ 0 & 4 & 0 \\ 4 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Reduced Laplacian matrix of K_n

$$\begin{aligned} \mathbf{L_0(K_n)} &= nI_{n-1} - J_{n-1} \\ \det L_0(K_n) &= n^{n-2} \end{aligned}$$

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Theorem. $\boldsymbol{L_0}(K_n) \xrightarrow{\text{SNF}} \mathrm{diag}(1, n, n, \ldots, n)$, *a refinement of Cayley's theorem that* $\kappa(K_n) = n^{n-2}$.

Proof that $L_0(K_n) \xrightarrow{\text{SNF}} \mathrm{diag}(1, n, n, \ldots, n)$

Trick: 2×2 submatrices (up to row and column permutations):

$$\begin{bmatrix} n-1 & -1 \\ -1 & n-1 \end{bmatrix}, \quad \begin{bmatrix} n-1 & -1 \\ -1 & -1 \end{bmatrix}, \quad \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix},$$

with determinants $n(n-2)$, $-n$, and 0. Hence $e_1 e_2 = n$. Since $\prod e_i = n^{n-2}$ and $e_i | e_{i+1}$, we get the SNF $\mathrm{diag}(1, n, n, \ldots, n)$.

Chip firing

Abelian sandpile: a finite collection σ of indistinguishable chips distributed among the vertices V of a (finite) connected graph. Equivalently,

$$\sigma\colon V \to \{0, 1, 2, \dots\}.$$

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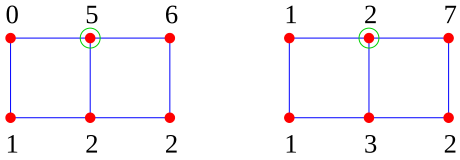
toppling of a vertex v: if $\sigma(v) \geq \deg(v)$, then send a chip to each neighboring vertex.



The sandpile group

Choose a vertex to be a **sink**, and ignore chips falling into the sink.

stable configuration: no vertex can topple

Theorem (easy). *After finitely many topples a stable configuration will be reached, which is independent of the order of topples.*

The monoid of stable configurations

Define a commutative monoid $\boldsymbol{M}$ on the stable configurations by vertex-wise addition followed by stabilization.

ideal of M: subset $J \subseteq M$ satisfying $\sigma J \subseteq J$ for all $\sigma \in M$

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Exercise. The (unique) minimal ideal of a finite commutative monoid is a group.

Sandpile group

sandpile group of G: the minimal ideal $\boldsymbol{K(G)}$ of the monoid M

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Theorem. *Let*

$$L_0(G) \xrightarrow{\text{SNF}} \mathrm{diag}(e_1, \ldots, e_{n-1}).$$

Then

$$K(G) \cong \mathbb{Z}/e_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/e_{n-1}\mathbb{Z}.$$

SNF of random matrices

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Relatively little work on SNF of random matrices over a PID.

Is the question interesting?

$Mat_k(n)$: all $n \times n$ $\mathbb{Z}$-matrices with entries in $[-k, k]$ (uniform distribution, independent entries)

$p_k(n, d)$: probability that if $M \in \mathrm{Mat}_k(n)$ and $\mathrm{SNF}(M) = (e_1, \ldots, e_n)$, then $e_1 = d$.

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Theorem. $\lim_{k\to\infty} p_k(n, d) = \dfrac{1}{d^{n^2}\zeta(n^2)}$

Specifying some e_i

with **Yinghui Wang**

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Two general results.

- Let $\alpha_1, \ldots, \alpha_{n-1} \in \mathbb{P}$, $\alpha_i | \alpha_{i+1}$.

  $\mu_k(n)$: probability that the SNF of a random $A \in \mathrm{Mat}_k(n)$ satisfies $e_i = \alpha_i$ for $1 \leq \alpha_i \leq n-1$.

$$\mu(n) = \lim_{k \to \infty} \mu_k(n).$$

  Then $\mu(n)$ exists, and $0 < \mu(n) < 1$.

Second result

- Let $\alpha_n \in \mathbb{P}$.

  $\boldsymbol{\nu_k(n)}$: probability that the SNF of a random $A \in \mathrm{Mat}_k(n)$ satisfies $e_n = \alpha_n$.

  Then
  $$\lim_{k\to\infty} \nu_k(n) = 0.$$

Sample result

$\mu_k(n)$: probability that the SNF of a random $A \in \mathrm{Mat}_k(n)$ satisfies $e_1 = 2$, $e_2 = 6$.

$$\mu(n) = \lim_{k \to \infty} \mu_k(n).$$

Conclusion

$$e_1 = 2, \quad e_2 = 6 = 2 \cdot 3$$

$$\begin{aligned}
\mu(n) &= 2^{-n^2}\left(1 - \sum_{i=(n-1)^2}^{n(n-1)} 2^{-i} + \sum_{i=n(n-1)+1}^{n^2-1} 2^{-i}\right) \\
&\cdot \frac{3}{2} \cdot 3^{-(n-1)^2}(1 - 3^{(n-1)^2})(1 - 3^{-n})^2 \\
&\cdot \prod_{p>3}\left(1 - \sum_{i=(n-1)^2}^{n(n-1)} p^{-i} + \sum_{i=n(n-1)+1}^{n^2-1} p^{-i}\right).
\end{aligned}$$

Cyclic cokernel

$\boldsymbol{\kappa(n)}$: probability that an $n \times n$ $\mathbb{Z}$-matrix has SNF $\operatorname{diag}(e_1, e_2, \ldots, e_n)$ with $e_1 = e_2 = \cdots = e_{n-1} = 1$

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Theorem (**T. Ekedahl**, 1991)

$$\kappa(n) = \frac{\prod_p \left(1 + \frac{1}{p^2} + \frac{1}{p^3} + \cdots + \frac{1}{p^n}\right)}{\zeta(2)\zeta(3)\cdots}$$

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Corollary.

$$\begin{aligned} \lim_{n\to\infty} \kappa(n) &= \frac{1}{\zeta(6)\prod_{j\geq 4}\zeta(j)} \\ &\approx 0.846936\cdots. \end{aligned}$$

Small number of generators

g: number of generators of cokernel (number of entries of SNF $\neq 1$) as $n \to \infty$

previous slide: $\text{Prob}(g = 1) = 0.846936\cdots$

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Theorem. $\mathrm{Prob}(g \leq \ell) =$

$$1 - (3.46275\cdots)2^{-(\ell+1)^2}(1 + O(2^{-\ell}))$$

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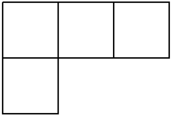
$$3.46275\cdots = \frac{1}{\prod_{j\geq 1}\left(1-\frac{1}{2^j}\right)}$$

Example of SNF computation

$\boldsymbol{\lambda}$: a partition $(\lambda_1, \lambda_2, \dots)$, identified with its Young diagram



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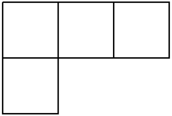

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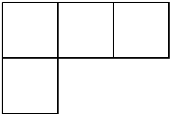

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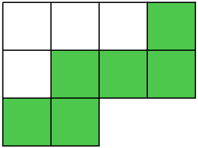
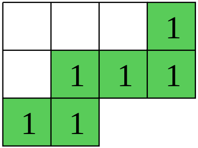


(3,1)* = (4,4,2)

Initialization

Insert 1 into each square of λ^*/λ.



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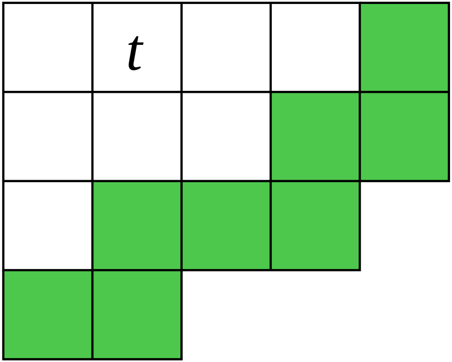

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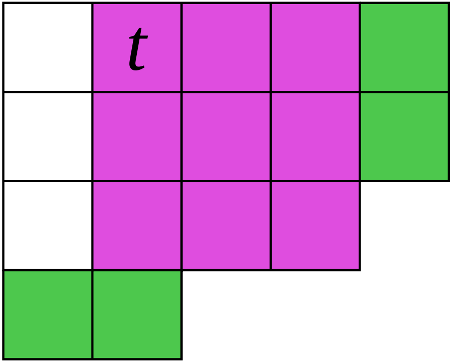

Determinantal algorithm

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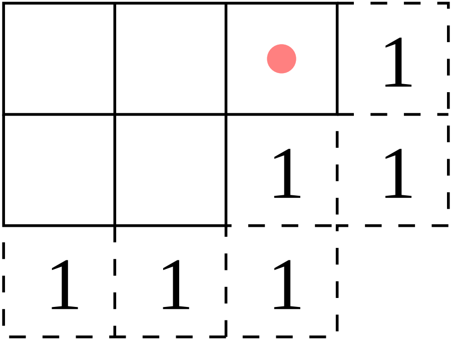

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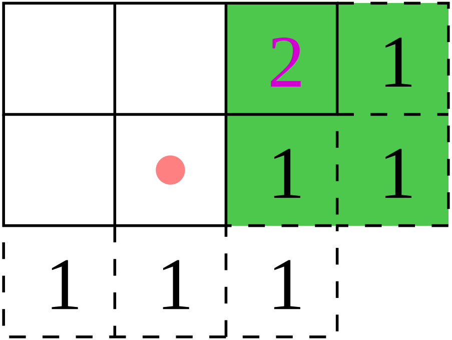

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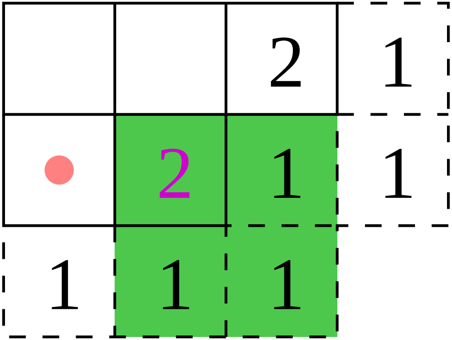

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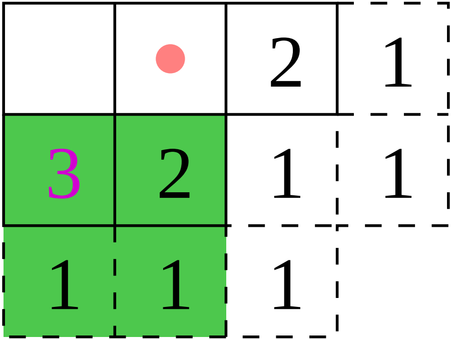

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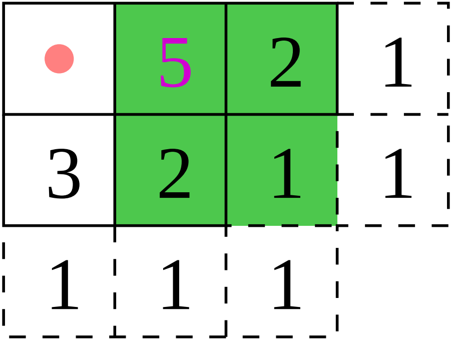

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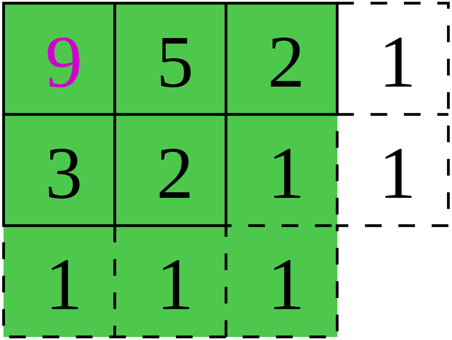

Uniqueness

Easy to see: the numbers n_t are well-defined and unique.

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Why? Expand $\det M_t$ by the first row. The coefficient of n_t is 1 by induction.

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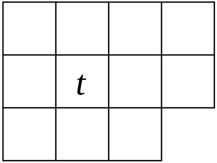

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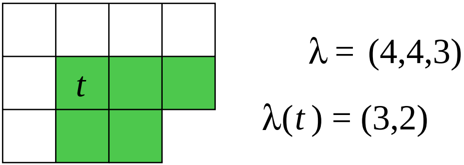

u_λ

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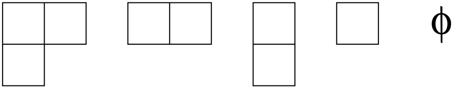
Example. $u_{(2,1)} = 5$:



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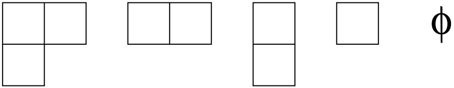

There is a determinantal formula for u_λ, due essentially to **MacMahon** and later **Kreweras** (not needed here).

Carlitz-Scoville-Roselle theorem

- **Berlekamp** (1963) first asked for $n_t \pmod 2$ in connection with a coding theory problem.
- **Carlitz-Roselle-Scoville** (1971): combinatorial interpretation of n_t (over $\mathbb{Z}$).

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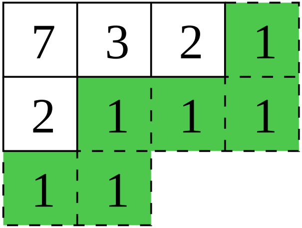
Proofs. 1. Induction (row and column operations).

2. Nonintersecting lattice paths.

An example



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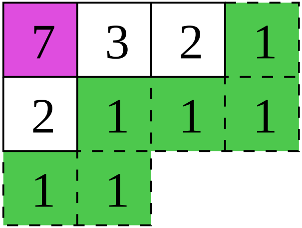

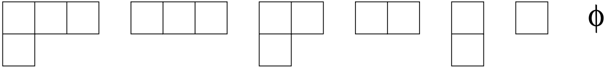

A q-analogue

Weight each $\mu \subseteq \lambda$ by $q^{|\lambda/\mu|}$.

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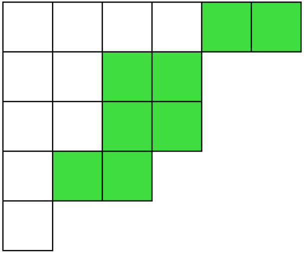
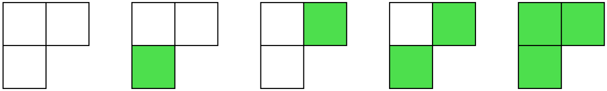


$$\lambda = 64431, \quad \mu = 42211, \quad q^{\lambda/\mu} = q^8$$

$u_\lambda(q)$

$$u_\lambda(q) = \sum_{\mu \subseteq \lambda} q^{|\lambda/\mu|}$$

$u_{(2,1)}(q) = 1 + 2q + q^2 + q^3$:



$M_\lambda(q)$

$M_\lambda(q)$: the largest square submatrix of λ with upper-left corner $(1,1)$ and entry in square t equal to $u_{\lambda(t)}(q)$.

t N	$1+2q+q^2+q^3$	$1+q$	1
$1+q+q^2$	$1+q$	1	1
1	1	1	

$$\lambda = (3,2)$$

$$N = 1 + 2q + 2q^2 + 2q^3 + q^4 + q^5$$

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det $M_t(q)$

$$M_t(q) = M_{(3,2)}(q) = \begin{bmatrix} N & 1+2q+q^2+q^3 & 1+q \\ 1+q+q^2 & 1+q & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

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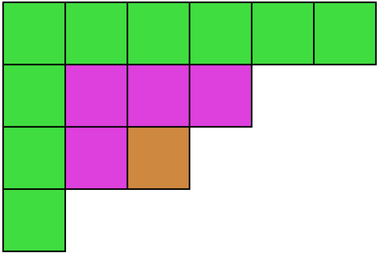
Known: $\det M_\lambda(q) = q^*$ (exponent $*$ to be explained). E.g.,

$$\det M_{3,2}(q) = q^6.$$

What is the SNF?

Diagonal hooks

$$d_i(\lambda) = \lambda_i + \lambda'_i - 2i + 1$$



$$d_1 = 9, \quad d_2 = 4, \; d_3 = 1$$

Main result (with C. Bessenrodt)

Theorem. *$M_t(q)$ has an SNF over $\mathbb{Z}[q]$. Write $d_i = d_i(\lambda_t)$. If $M_t(q)$ is a $(k+1) \times (k+1)$ matrix then $M_t(q)$ has SNF*

$$\mathrm{diag}(1, q^{d_k}, q^{d_{k-1}+d_k}, \ldots, q^{d_1+d_2+\cdots+d_k}).$$

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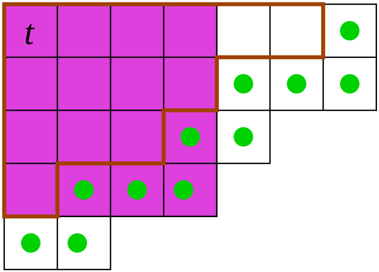
Corollary. $\det M_t(q) = q^{\sum id_i}$.

Note. There is a multivariate generalization.

An example



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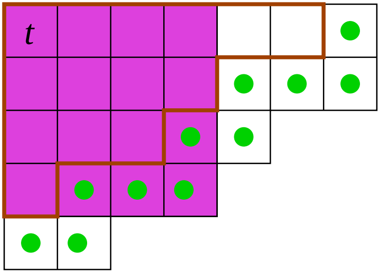
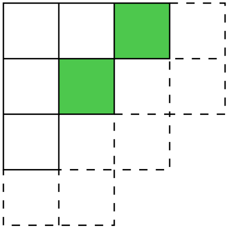


$$\lambda = 6431, \quad d_1 = 9, \quad d_2 = 4, \quad d_3 = 1$$

$$\text{SNF of } M_t(q): \ (1, q, q^5, q^{14})$$

A special case

Let λ be the **staircase** $\delta_n = (n-1, n-2, \ldots, 1)$.



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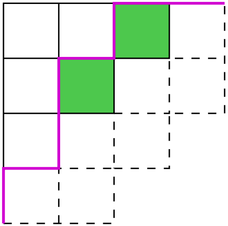
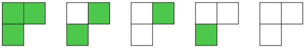


$u_{\delta_{n-1}}(q)$ counts Dyck paths of length $2n$ by (scaled) area, and is thus the well-known q-analogue $\boldsymbol{C_n(q)}$ of the Catalan number C_n.

A q-Catalan example



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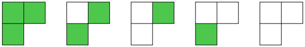

$$C_3(q) = q^3 + q^2 + 2q + 1$$

$$\begin{vmatrix} C_4(q) & C_3(q) & 1+q \\ C_3(q) & 1+q & 1 \\ 1+q & 1 & 1 \end{vmatrix} \overset{\text{SNF}}{\sim} \mathrm{diag}(1, q, q^6)$$

since $d_1(3,2,1) = 1$, $d_2(3,2,1) = 5$.

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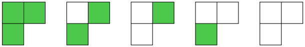

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- q-Catalan determinant previously known
- SNF is new

Ramanujan

$\sum_{n\geq 0} C_n(q)x^n =$

$$\cfrac{1}{1-\cfrac{x}{1-\cfrac{qx}{1-\cfrac{q^2x}{1-\cdots}}}}.$$

Open problem #1: a q-Varchenko matrix

$\ell(w)$: length (number of inversions) of $w = a_1 \cdots a_n \in \mathfrak{S}_n$, i.e.,

$$\ell(w) = \#\{(i,j) \,:\, i < j,\ w_i > w_j\}.$$

$\boldsymbol{V(n)}$: the $n! \times n!$ matrix with rows and columns indexed by $w \in \mathfrak{S}_n$, and

$$V(n)_{uv} = q^{\ell(uv^{-1})}.$$

$n = 3$

$$\det \begin{bmatrix} 1 & q & q & q^2 & q^2 & q^3 \\ q & 1 & q^2 & q & q^3 & q^2 \\ q & q^2 & 1 & q^3 & q & q^2 \\ q^2 & q & q^3 & 1 & q^2 & q \\ q^2 & q^3 & q & q^2 & 1 & q \\ q^3 & q^2 & q^2 & q & q & 1 \end{bmatrix} = (1-q^2)^6(1-q^6)$$

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$$V(3) \overset{\text{snf}}{\to} \mathrm{diag}(1, 1-q^2, 1-q^2, 1-q^2, (1-q^2)^2, (1-q^2)(1-q^6))$$

$n = 3$

$$\det \begin{bmatrix} 1 & q & q & q^2 & q^2 & q^3 \\ q & 1 & q^2 & q & q^3 & q^2 \\ q & q^2 & 1 & q^3 & q & q^2 \\ q^2 & q & q^3 & 1 & q^2 & q \\ q^2 & q^3 & q & q^2 & 1 & q \\ q^3 & q^2 & q^2 & q & q & 1 \end{bmatrix} = (1-q^2)^6(1-q^6)$$

$$V(3) \stackrel{\text{snf}}{\rightarrow} \mathrm{diag}(1, 1-q^2, 1-q^2, 1-q^2, (1-q^2)^2, (1-q^2)(1-q^6))$$

special case of **q-Varchenko matrix**

Zagier's theorem

Theorem (**D. Zagier**, 1992; generalized by **A. Varchenko**, 1993)

$$\det V(n) = \prod_{j=2}^{n} \left(1 - q^{j(j-1)}\right)^{\binom{n}{j}(j-2)!\,(n-j+1)!}$$

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SNF is open. Partial result:

Theorem (**Denham-Hanlon**, 1997) *Let*

$$V(n) \stackrel{\mathrm{snf}}{\to} \mathrm{diag}(e_1, e_2, \ldots, e_{n!}).$$

The number of e_i's exactly divisible by $(q-1)^j$ (or by $(q^2-1)^j$) is the number $c(n, n-j)$ of $w \in \mathfrak{S}_n$ with $n-j$ cycles (signless Stirling number of the first kind).

Open problem #2: $\mathfrak{S}_n$ conjugacy class actions

$\mathbb{Q}\mathfrak{S}_n$: group algebra of $\mathfrak{S}_n$ over $\mathbb{Q}$

K_λ: sum of all $w \in \mathfrak{S}_n$ of cycle type λ (basis for center Z_n of $\mathbb{Q}\mathfrak{S}_n$)

K_λ acts on Z_n by left multiplication. What is the SNF with respect to the basis $\{K_\mu\}$?

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Looks difficult.

The case $\lambda = (n)$

Note $K_{(n)}$ is the sum of all $(n-1)!$ n-cycles.

Easy. The SNF of $K_{(n)}$ has n nonzero diagonal elements.

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Easy. The SNF of $K_{(n)}$ has n nonzero diagonal elements.

Empirical observation: the kth diagonal element of the SNF $(0 \leq k \leq n-1)$ is $k!$ times a rational number with small numerator and denominator.

Two examples

We divide the kth entry by $k!$, $0 \leq k \leq n-1$.

$\boldsymbol{n = 9:}$ $1,\ 2,\ 1,\ \frac{2}{3},\ 1,\ 2,\ \frac{1}{3},\ 2,\ 1$

$\boldsymbol{n = 12:}$ $1,\ 1,\ 1,\ \frac{1}{3},\ \frac{1}{2},\ 1,\ 2,\ 1,\ \frac{1}{2},\ \frac{1}{3},\ 1,\ 1$

Two conjectures

Conjecture. If n is an odd prime then the nonzero SNF terms are $k!$ for k even and $2 \cdot k!$ for k odd $(0 \leq k \leq n-1)$.

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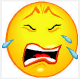
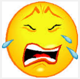
Conjecture. If n is twice an odd prime, then the nonzero SNF terms are $k!$ for all $0 \leq k \leq n-1$, except that $(n/2)!$ is omitted, and $\left(\frac{n}{2}-1\right)!$ appears twice.

The last slide

The last slide 

The last slide 

Encore: Jacobi-Trudi specialization

Jacobi-Trudi identity:

$$s_\lambda = \det[h_{\lambda_i - i + j}],$$

where $\boldsymbol{s_\lambda}$ is a **Schur function** and $\boldsymbol{h_i}$ is a **complete symmetric function**.

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Jacobi-Trudi identity:

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where s_λ is a **Schur function** and h_i is a **complete symmetric function**.

We consider the specialization $x_1 = x_2 = \cdots = x_n = 1$, other $x_i = 0$. Then

$$h_i \to \binom{n+i-1}{i}.$$

Specialized Schur function

$$s_\lambda \to \prod_{u \in \lambda} \frac{n + c(u)}{h(u)}.$$

$c(u)$: **content** of the square u

0	1	2	3	4
−1	0	1	2	
−2	−1	0	1	
−3	−2			

Diagonal hooks $D_1, \ldots, D_m$

0	1	2	3	4
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$$\lambda = (5,4,4,2)$$

Diagonal hooks $D_1, \ldots, D_m$

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D_1

Diagonal hooks $D_1, \ldots, D_m$

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−3	−2			

D_2

Diagonal hooks $D_1, \ldots, D_m$

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−1	0	1	2	
−2	−1	0	1	
−3	−2			

D_3

SNF result

$$\boldsymbol{R} = \mathbb{Q}[n]$$

Let

$$\mathrm{SNF}\left[\binom{n+\lambda_i-i+j-1}{\lambda_i-i+j}\right] = \mathrm{diag}(e_1,\ldots,e_m).$$

Then

$$e_i = \prod_{u\in D_{m-i+1}} \frac{n+c(u)}{h(u)}.$$

Idea of proof

We will use the fact that if

$$\mathrm{SNF}(A) = \mathrm{diag}(e_1, e_2, \ldots, e_n),$$

then $e_1 e_2 \cdots e_i$ is the gcd of the $i \times i$ minors of A.

Idea of proof (cont.)

$$f_i = \prod_{u \in D_{m-i+1}} \frac{n + c(u)}{h(u)}$$

Then $f_1 f_2 \cdots f_i$ is the value of the "lower-leftmost" nonzero $i \times i$ minor.

Idea of proof (cont.)

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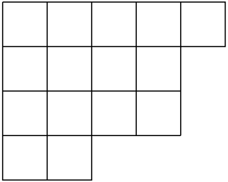
Then $f_1 f_2 \cdots f_i$ is the value of the "lower-leftmost" nonzero $i \times i$ minor.

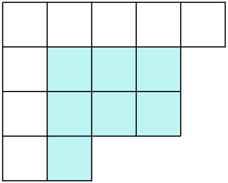
Every $i \times i$ minor is a specialized skew Schur function $s_{\mu/\nu}$. Let s_α correspond to the lower left $i \times i$ minor.

An example



$$s_{5442} = \begin{bmatrix} h_5 & h_6 & h_7 & h_9 \\ h_3 & h_4 & h_5 & h_6 \\ h_2 & h_3 & h_4 & h_5 \\ 0 & 1 & h_1 & h_2 \end{bmatrix}$$

An example



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$$s_{331} = \begin{vmatrix} h_3 & h_4 & h_5 \\ h_2 & h_3 & h_4 \\ 0 & 1 & h_1 \end{vmatrix}$$

Conclusion of proof

Let

$$s_{\mu/\nu} = \sum_{\rho} c^{\mu}_{\nu\rho} s_{\rho}.$$

By Littlewood-Richardson rule,

$$c^{\mu}_{\nu\rho} \neq 0 \;\Rightarrow\; \alpha \subseteq \rho.$$

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Hence

$$f_1 \cdots f_i = \mathrm{gcd}(i \times i \text{ minors}) = e_1 \cdots e_i.$$