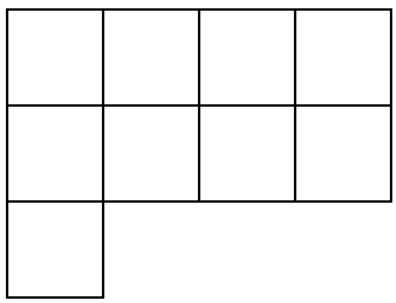
partition λ of $n \geq 0$:

$$\lambda = (\lambda_1, \lambda_2, \ldots)$$

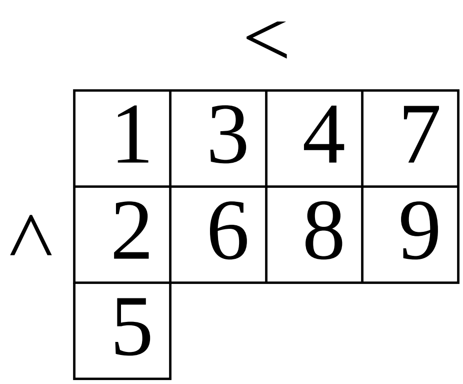
$$\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$$

$$\sum \lambda_i = n$$

Young diagram of $\lambda = (4, 4, 1)$:



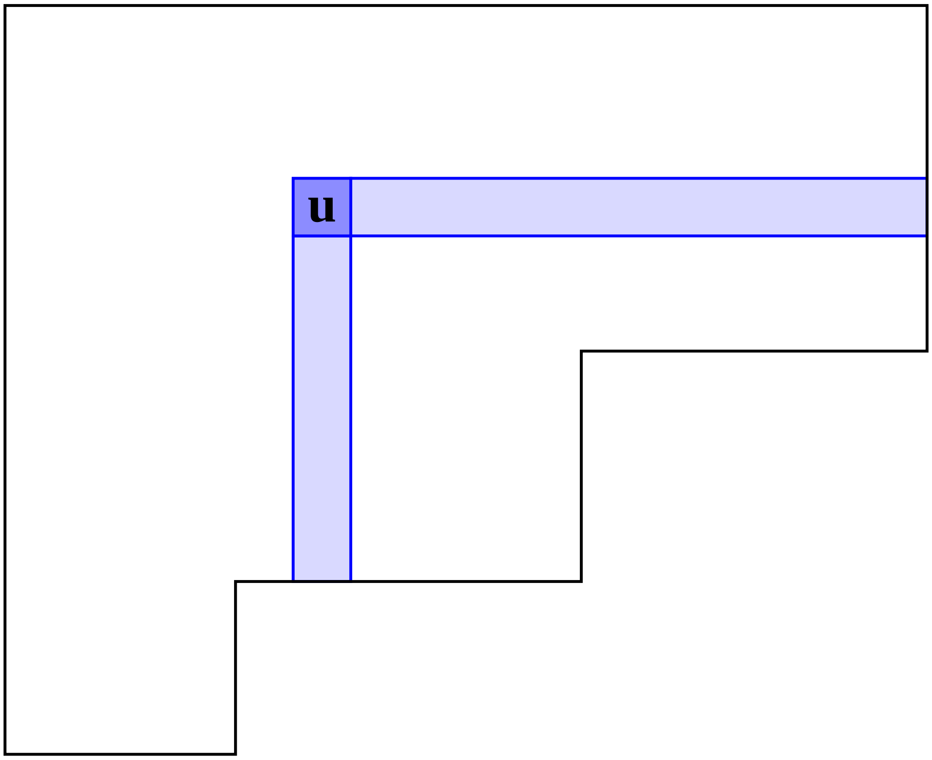
standard Young tableau (**SYT**) of shape $(4, 4, 1)$:



$f^{\boldsymbol{\lambda}} = \#$ of SYT of shape λ

1 2 3 4	1 2 3 5	1 2 3 6
5 6	4 6	4 5
1 2 4 5	1 2 4 6	1 2 5 6
3 6	3 5	3 4
1 3 4 5	1 3 4 6	1 3 5 6
2 6	2 5	2 4

$$f^{4,2} = 9$$



$\boldsymbol{H(u)}$: **hook** at (or of) u

$\boldsymbol{h(u)} = \#H(u)$: **hook length** at u

Frame-Robinson-Thrall hook length formula (1954):

$$f^\lambda = \frac{n!}{\prod_{u\in\lambda} h(u)}$$

$$\text{hook lengths}: \quad \begin{matrix} 5\,4\,2\,1 \\ 2\,1 \end{matrix}$$

$$f^{4,2} = \frac{6!}{5\cdot 4\cdot 2\cdot 2\cdot 1\cdot 1} = 9$$

"nice" bijective proof by Novelli-Pak-Stoyanovskii (1997)

Robinson-Schensted-Knuth (RSK) algorithm: $w \stackrel{\text{rsk}}{\rightarrow} (P,Q)$, where $w \in \mathfrak{S}_n$ and P,Q are SYT of same shape $\lambda \vdash n$

Note. Schensted = Ea Ea

(ea.ea.home.mindspring.com)

$$w = 4273615$$

$$\begin{array}{l} \mathbf{4} \end{array} \qquad \begin{array}{l} 1 \end{array}$$

$$\begin{array}{l} \mathbf{2} \\ \mathbf{4} \end{array} \qquad \begin{array}{l} 1 \\ 2 \end{array}$$

$$\begin{array}{ll} 2 & \mathbf{7} \\ 4 & \end{array} \qquad \begin{array}{ll} 1 & 3 \\ 2 & \end{array}$$

$$\begin{array}{ll} 2 & \mathbf{3} \\ 4 & \mathbf{7} \end{array} \qquad \begin{array}{ll} 1 & 3 \\ 2 & 4 \end{array}$$

$$\begin{array}{lll} 2 & 3 & \mathbf{6} \\ 4 & 7 & \end{array} \qquad \begin{array}{lll} 1 & 3 & 5 \\ 2 & 4 & \end{array}$$

$$\begin{array}{lll} \mathbf{1} & 3 & 6 \\ \mathbf{2} & 7 & \\ \mathbf{4} & & \end{array} \qquad \begin{array}{lll} 1 & 3 & 5 \\ 2 & 4 & \\ 6 & & \end{array}$$

$$\begin{array}{lll} 1 & 3 & \mathbf{5} \\ 2 & \mathbf{6} & \\ 4 & \mathbf{7} & \end{array} \qquad \begin{array}{lll} 1 & 3 & 5 \\ 2 & 4 & \\ 6 & 7 & \end{array}$$

An element j bumps the smallest $i > j$.

$\boldsymbol{\chi^{\lambda}}$: irreducible character of $\mathfrak{S}_n$
indexed by $\lambda \vdash n$

$$f^{\lambda} = \chi^{\lambda}(1) = \dim \chi^{\lambda}$$

$$\sum_{\lambda \vdash n} \left(f^{\lambda}\right)^2 = n!$$

First symmetry property.

$$w = 4273615 \xrightarrow{\text{rsk}} \begin{array}{lll} 1 & 3 & 5 \\ 2 & 6 & \\ 4 & 7 & \end{array} \quad \begin{array}{lll} 1 & 3 & 5 \\ 2 & 4 & \\ 6 & 7 & \end{array}$$

$$w^{-1} = 6241753 \xrightarrow{\text{rsk}} \begin{array}{lll} 1 & 3 & 5 \\ 2 & 4 & \\ 6 & 7 & \end{array} \quad \begin{array}{lll} 1 & 3 & 5 \\ 2 & 6 & \\ 4 & 7 & \end{array}$$

Theorem (Schützenberger) *If* $w \xrightarrow{\text{rsk}} (P, Q)$*, then*

$$w^{-1} \xrightarrow{\text{rsk}} (Q, P).$$

Corollary. *Let* $t(n)$ *denote the number of SYT with* n *squares. Then*

$$t(n) = \#\{w \in \mathfrak{S}_n \,:\, w^2 = 1\}$$

$$\sum_{n \geq 0} t(n) \frac{x^n}{n!} = \exp\left(x + \frac{x^2}{2}\right).$$

$$t(n) = \sum_{\lambda \vdash n} f^\lambda = \sum_{\lambda \vdash n} \chi^\lambda(1)$$

Theorem (Frobenius). *Let G be a finite group and $\hat{G}$ its set of (complex) irreducible characters. Then*

$$\sum_{\chi \in \hat{G}} \chi(1) = \#\{w \in G \,:\, w^2 = 1\}$$

if and only if every representation of G is equivalent to a real representation (true for $\mathfrak{S}_n$).

$$w = 3\mathbf{1}8\mathbf{4}9\mathbf{6}\mathbf{7}25$$

$$\boldsymbol{is}(w) := \text{length of longest increasing subsequence of } w \in \mathfrak{S}_n$$

$$\text{is}(318496725) = 4$$

$$318496725 \xrightarrow{\text{rsk}} \begin{array}{llll} 1 & 2 & 5 & 7 \\ 3 & 4 & 6 & \\ 8 & 9 & & \end{array} \qquad \begin{array}{llll} 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & \\ 8 & 9 & & \end{array}$$

Theorem. *Let* $w \xrightarrow{\text{rsk}} (P, Q)$,

$$\text{shape}(P) = (\lambda_1, \lambda_2, \ldots).$$

Then

$$\text{is}(w) = \lambda_1.$$

Proof (sketch). Let the first row be

$$b_1, b_2, \ldots, b_k.$$

Straightforward to show by induction on n that b_i is the rightmost element j of w for which the longest increasing subsequence of w ending at j has length i. $\square$

Second symmetry property.

$$w = a_1 a_2 \cdots a_n, \quad \overline{w} := a_n \cdots a_2 a_1$$

Theorem (Schensted). *If* $w \xrightarrow{\text{rsk}} (P, Q)$, *then*

$$\overline{w} \xrightarrow{\text{rsk}} (P^t, \text{evac}(Q)^t).$$

Corollary. *Let* $w \xrightarrow{\text{rsk}} (P, Q)$, $\text{shape}(P) = (\lambda_1, \lambda_2, \ldots)$. *Then*

$$\text{ds}(w) = \lambda_1' = \ell(\lambda).$$

Corollary (Erdős-Szekeres). *Let* $w \in \mathfrak{S}_{pq+1}$. *Then either*

$$\text{is}(w) > p \;\text{ or }\; \text{ds}(w) > q.$$

Corollary. *Let $p \le q$ (say). Then*

$$\#\{w \in \mathfrak{S}_{pq} \ : \ \mathrm{is}(w) = p, \ \mathrm{ds}(w) = q\}$$

$$= \left(f^{q \times p}\right)^2$$

$$= \left(\frac{(pq)!}{1^1 2^2 \cdots p^p (p+1)^p \cdots q^p (q+1)^{p-1} \cdots (p+q-1)^1}\right)^2.$$

7	6	5	4	3
6	5	4	3	2
5	4	3	2	1

$$\begin{bmatrix} 0&0&0&1&0 \\ 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&0&0&1 \\ 0&0&1&0&0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&0&1 \\ 1&0&0&0&0 \\ 0&0&0&1&0 \end{bmatrix}$$

$$w = 41253 \qquad w^{-1} = 23514$$

$$\downarrow$$

$$\begin{bmatrix} 0&0&1&0&0 \\ 0&0&0&0&1 \\ 0&1&0&0&0 \\ 1&0&0&0&0 \\ 0&0&0&1&0 \end{bmatrix}$$

$$\overline{w} = 35214$$

These two reflections generate the dihedral group D_4 (symmetries of a square).

A **reverse semistandard tableau** T of shape $(5, 4, 3)$:

$$\begin{array}{lllll} 6 & 6 & 4 & 2 & 2 \\ 4 & 4 & 3 & 1 & \\ 2 & 1 & 1 & & \end{array}$$

$$x^T = x_1^3 x_2^3 x_3 x_4^3 x_6^2$$

(weakly decreasing in rows, strictly decreasing in columns)

$$A = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 2 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$$w_A = \begin{pmatrix} 3 & 3 & 2 & 2 & 2 & 1 & 1 \\ 3 & 1 & 3 & 2 & 2 & 4 & 4 \end{pmatrix}$$

$$\begin{array}{l} \mathbf{3} \end{array} \qquad \begin{array}{l} 3 \end{array}$$

$$\begin{array}{l} 3\,\mathbf{1} \end{array} \qquad \begin{array}{l} 3\,3 \end{array}$$

$$\begin{array}{l} 3\,\mathbf{3} \\ \mathbf{1} \end{array} \qquad \begin{array}{l} 3\,3 \\ 2 \end{array}$$

$$\begin{array}{l} 3\,3\,\mathbf{2} \\ 1 \end{array} \qquad \begin{array}{l} 3\,3\,2 \\ 2 \end{array}$$

$$\begin{array}{l} 3\,3\,2\,\mathbf{2} \\ 1 \end{array} \qquad \begin{array}{l} 3\,3\,2\,2 \\ 2 \end{array}$$

$$\begin{array}{l} \mathbf{4}\,3\,2\,2 \\ \mathbf{3} \\ \mathbf{1} \end{array} \qquad \begin{array}{l} 3\,3\,2\,2 \\ 2 \\ 1 \end{array}$$

$$\begin{array}{l} 4\,\mathbf{4}\,2\,2 \\ 3\,\mathbf{3} \\ 1 \end{array} \qquad \begin{array}{l} 3\,3\,2\,2 \\ 2\,1 \\ 1 \end{array}$$

An element j bumps the largest element $i < j$.

Lemma (simple). Let $A \xrightarrow{\text{rsk}} (P, Q)$. Equal elements of Q are inserted left-to-right (allowing the construction of the inverse map $(P, Q) \rightarrow A$).

Schur function:

$$\boldsymbol{s_\lambda} = \sum_{\substack{\text{RSSYT } T \\ \text{shape}(T)=\lambda}} x^T$$

3 3 2	3 3 1	3 2 2	3 1 1
2 2 1	2 1 1	3 2 1	3 1 2

$$s_{2,1}(x_1, x_2, x_3) = x_2x_3^2 + x_1x_3^2 + x_2^2x_3 + x_1^2x_3 + x_1x_2^2 + x_1^2x_2 + 2x_1x_2x_3$$

Cauchy identity:

$$\prod_{i,j}(1 - x_iy_j)^{-1} = \sum_\lambda s_\lambda(x)s_\lambda(y)$$

(analytic formulation of RSK for (reverse) SSYT)

Plane partition of $n \geq 0$:

$$\pi = (\pi_{ij})_{i,j\geq 1}$$

$$\pi_{ij} \geq \pi_{i+1,j}, \quad \pi_{ij} \geq \pi_{i,j+1}, \quad \sum \pi_{ij} = n$$

$\boldsymbol{a(n)}$ = # plane partitions of n

$$\begin{matrix} 5 & 5 & 3 & 2 & 2 & 1 \\ 5 & 3 & 3 & 1 & 1 & \\ 5 & 2 & 2 & 1 & 1 & \\ 2 & 2 & 1 & & & \end{matrix}$$

$$3 \qquad 2\,1 \qquad 1\,1\,1 \qquad \begin{matrix} 2 \\ 1 \\ \end{matrix} \qquad \begin{matrix} 1 \\ 1 \\ 1 \end{matrix} \qquad \begin{matrix} 1\,1 \\ 1 \\ \end{matrix}$$

$$a(3) = 6$$

MacMahon:

$$\sum_{n\geq 0} a(n)x^n = \prod_{i\geq 1}(1-x^i)^{-i}.$$

$$A \quad = \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix}$$

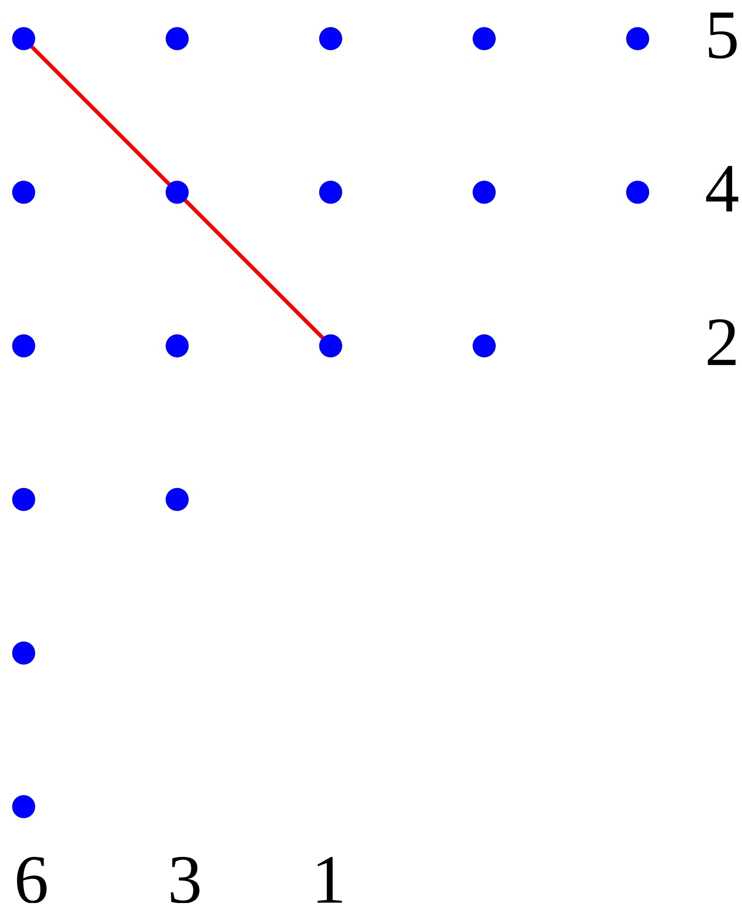
$$\xrightarrow{\text{rsk}} \quad \begin{matrix} 3\,3\,3\,3\,2 \\ 2\,2\,2 \\ 1 \end{matrix} \qquad \begin{matrix} 4\,4\,3\,2\,1 \\ 3\,2\,2 \\ 1 \end{matrix}$$

$$\xrightarrow{\text{merge}} \quad \begin{matrix} 3\,3\,3\,3\,2 \\ 3\,3\,3\,1 \\ 3\,3\,2 \\ 2\,1 \end{matrix}$$

$$\xrightarrow{\text{rowconj}} \quad \begin{matrix} 5\,5\,4 \\ 4\,3\,3 \\ 3\,3\,2 \\ 2\,1 \end{matrix} = \pi_A$$

Merge column-by-column.

merge of $(5,4,2)$ and $(6,3,1)$ is $(5,5,4,2,1,1)$:



$$|\pi_A| = \sum_{i,j} (i+j-1)a_{ij}$$

$$\#(\text{rows of } \pi_A) = \#(\text{rows of } A)$$

$$\#(\text{columns of } \pi_A) = \#(\text{columns of } A)$$

$$\begin{aligned}
\Rightarrow \sum_{\pi} x^{|\pi|} &= \sum_{A} x^{\sum (i+j-1)a_{ij}} \\
&= \prod_{i,j\geq 1} \left(\sum_{a_{ij}\geq 0} x^{(i+j-1)a_{ij}} \right) \\
&= \prod_{i,j\geq 1} (1 - x^{i+j-1})^{-1} \\
&= \prod_{i\geq 1} (1 - x^{i})^{-i}
\end{aligned}$$

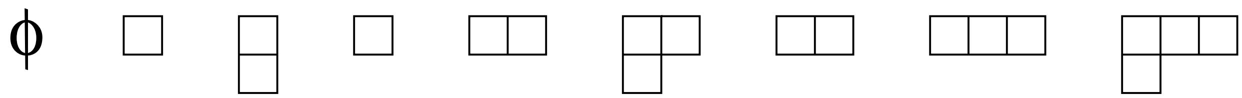
RSK $\Rightarrow$

$$\sum_{\substack{\pi \\ \leq r \text{ rows} \\ \leq s \text{ cols}}} x^{|\pi|} = \prod_{i=1}^{r} \prod_{j=1}^{s} (1 - x^{i+j-1})^{-1}$$

More difficult (MacMahon):

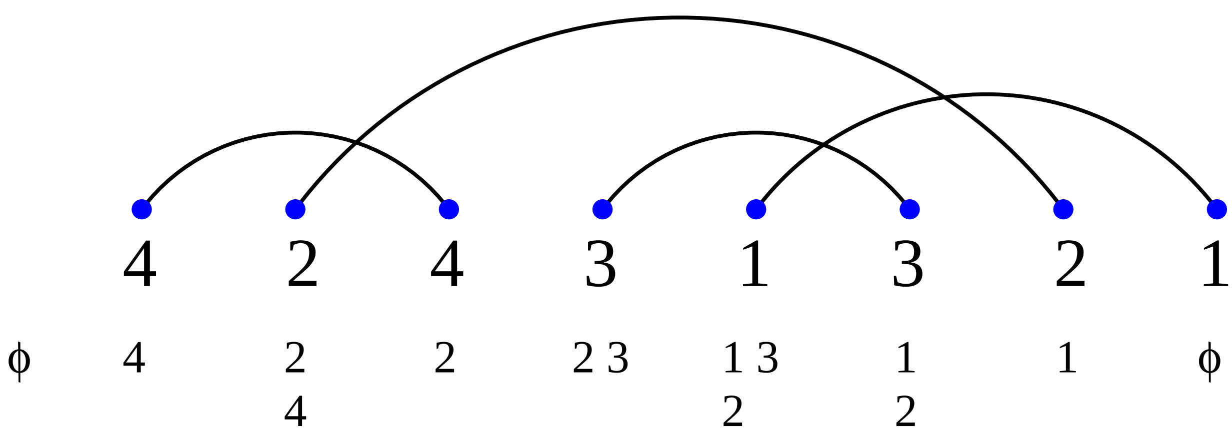
$$\sum_{\substack{\pi \\ \leq r \text{ rows} \\ \leq s \text{ cols} \\ \max \leq t}} x^{|\pi|} = \prod_{i=1}^{r} \prod_{j=1}^{s} \prod_{k=1}^{t} \frac{1 - x^{i+j+k-1}}{1 - x^{i+j+k-2}}$$

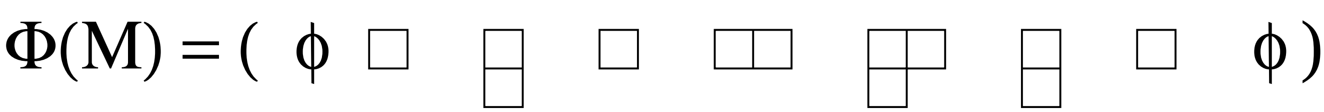
Oscillating tableaux: start with $\emptyset$, add or remove a square at each step.



shape $(3,1)$, length 8

$\tilde{f}_n^{\lambda} = \#\{\text{osc. tab. of shape } \lambda, \text{ length } n\}$





Hence

$$\begin{aligned}\tilde{f}^{\emptyset}_{2n} &= \sum_{\lambda}\left(\tilde{f}^{\lambda}_{n}\right)^2\\ &= (2n-1)!!\\ &:= (2n-1)(2n-3)\cdots 1.\end{aligned}$$

$\tilde{f}^{\lambda}_{n}$ is the dimension of an irreducible representation of the **Brauer algebra** $\mathfrak{B}_n$, a semisimple algebra of dimension $(2n-1)!!$.

$$s_i = (i, i+1) \in \mathfrak{S}_n$$

reduced decomposition $(a_1, \ldots, a_p)$ of $w \in \mathfrak{S}_n$:

$$w = s_{a_1} \cdots s_{a_p},$$

where p is minimal, i.e.,

$$p = \ell(w) = \#\{(i,j) \,:\, i < j,\ w(i) > w(j)\}.$$

$\boldsymbol{R(w)}$: set of reduced decomps. of w

$$\boldsymbol{r(w)} := \#R(w)$$

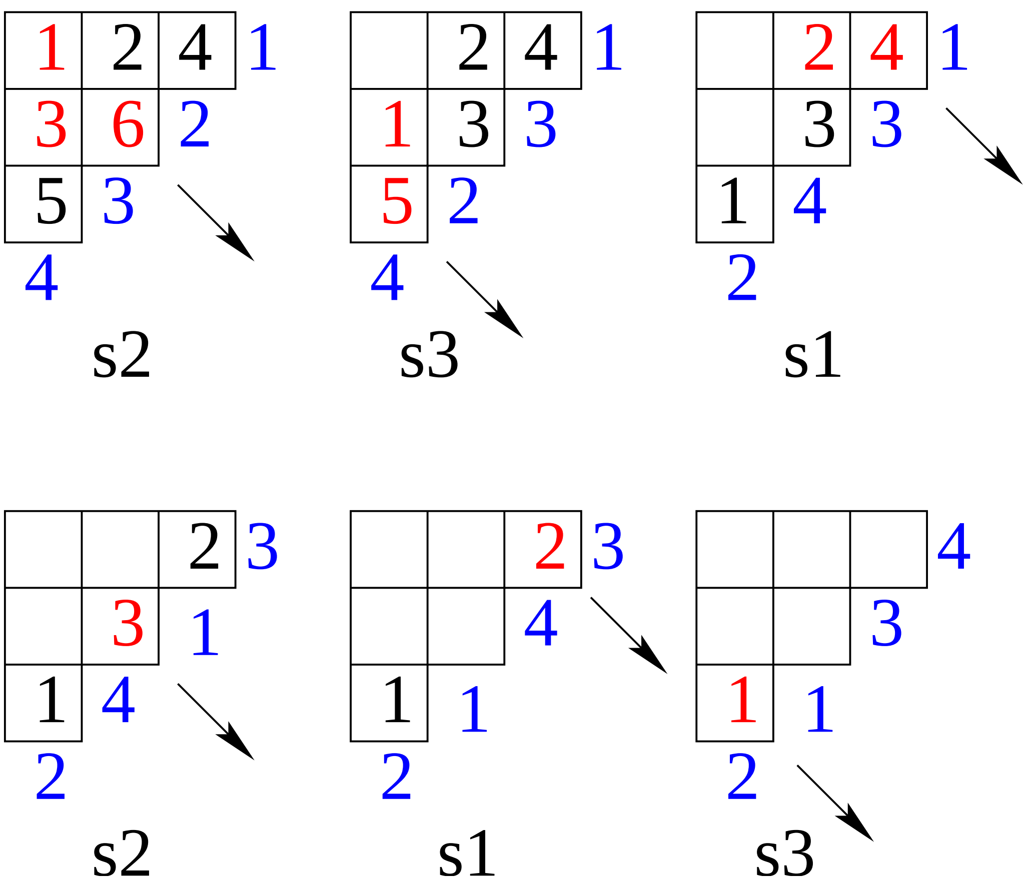
E.g., $w = 321$, $R(w) = \{(1,2,1), (2,1,2)\}$, $r(w) = 2$.

$$w_0 = n, n-1, \ldots, 1 \in \mathfrak{S}_n$$

Theorem.

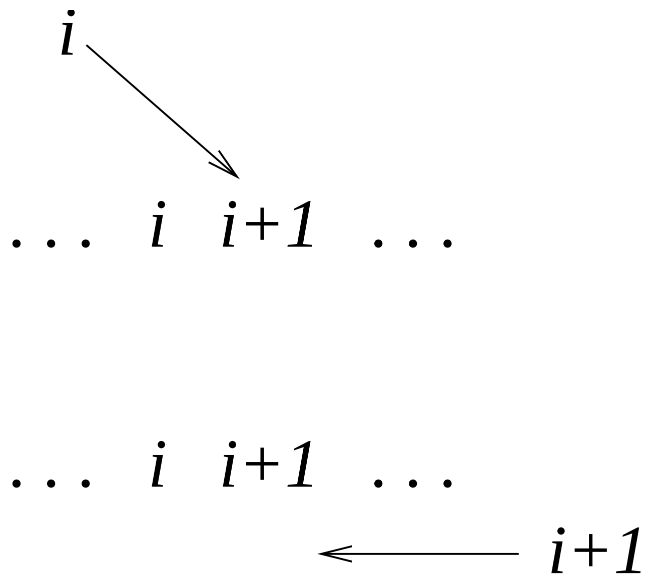
$$r(w_0) = f^{(n-1,n-2,\ldots,1)}$$
$$= \frac{\binom{n}{2}!}{1^{n-1}\,3^{n-2}\,5^{n-3}\cdots(2n-3)}$$

Proof (Edelman-Greene).



$$(2,3,1,2,1,3) \in R(4321)$$

Inverse:



Example. $312132 \in R(4321)$

P	Q
3	1

P	Q
1	1
3	2

P	Q
1 **2**	1 3
3	2

P	Q
1 **2**	1 3
2	2
3	4

P	Q
1 2 **3**	1 3 5
2	2
3	4

P	Q
1 2 **3**	1 3 5
2 **3**	2 6
3	4

P is always the same, and Q runs through all SYT of shape $(n-1, n-2, \ldots, 1)$.

Variant:

$$f(w) = \sum_{(a_1,\ldots,a_p)\in R(w)} a_1 a_2 \cdots a_p$$

E.g., $w = 321$, $R(321) = \{121, 212\}$,

$$f(321) = 1\cdot 2\cdot 1 + 2\cdot 1\cdot 2 = 6 = 3!$$

Theorem (Macdonald, Fomin-S).

$$f(w_0) = \binom{n}{2}!$$

More generally, $f(w) = \mathfrak{S}_w(1,1,\ldots,1)\ell(w)!$, where $\mathfrak{S}_w$ is a **Schubert polynomial**.

Corollary. $f(w) = \ell(w)!$ if and only if there never holds

$$i < j < k \Rightarrow w(i) < w(k) < w(j).$$

Number of such $w \in \mathfrak{S}_n$ is the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$.