GENERALIZED RIFFLE SHUFFLES AND QUASISYMMETRIC FUNCTIONS

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Let $\mathbf{x_i}$ = probability of $i \in \mathbb{P} = \{1, 2 \ldots\}$.

Fix $n \in \mathbb{P}$. Define a random $w \in \mathfrak{S}_n$ as follows:

For $1 \le j \le n$, choose independently an integer i_j from the distribution x_i. Then **standardize** the sequence $\boldsymbol{i} = i_1 \cdots i_n$, i.e., replace the 1's with $1, 2, \ldots, a_1$ from left-to-right, then the 2's with $a_1 + 1, a_1 + 2, \ldots, a_1 + a_2$ from left-to-right, etc.

$$\begin{aligned} \boldsymbol{i} &= 311431 \\ w &= 412653 \end{aligned}$$

Call this the **QS-distribution** or **QS($\boldsymbol{x}$)-distribution**.

Previously studied by

- Diaconis-Fill-Pitman
- Fulman
- Its-Tracy-Widom
- Kuperberg,

at least when x_i has finite support.

Example. $w = 213$. The sequence $i_1 i_2 i_3$ has standardization 213 if and only if $i_2 < i_1 \leq i_3$. Hence

$$\mathrm{Prob}(213) = \sum_{a<b\leq c} x_a x_b x_c = L_{(1,2)}(x).$$

Theorem. *Let $w \in \mathfrak{S}_n$. The probability* $\mathrm{Prob}(w)$ *that a permutation in $\mathfrak{S}_n$ chosen from the QS-permutation is equal to w is given by*

$$\mathrm{Prob}(w) = L_{\mathrm{co}(w^{-1})}(x).$$

Example. $w = 74513826$

$$\begin{aligned} w^{-1} &= 47 \cdot 5 \cdot 238 \cdot 16 \\ \mathrm{co}(w^{-1}) &= (2, 1, 3, 2) \\ L_{(2,1,3,2)}(x) &= \sum_{a\leq b<c<d\leq e\leq f<g\leq h} x_a \cdots x_h. \end{aligned}$$

Special cases:

- $x_1 = x_2 = 1/2$: riffle or dovetail shuffle (Bayer-Diaconis), the $\mathbf{U_2}$-distribution
- $x_1 = \cdots = x_q = 1/q$: q-shuffle (Bayer-Diaconis), the $\mathbf{U_q}$-distribution
- $\lim_{q\to\infty} U_q$: the **uniform** distribution U

Note. A q-shuffle followed by an r-shuffle is a qr-shuffle, i.e., $U_q * U_r = U_{qr}$.

Theorem. *Let y have finite support. Then*

$$\mathrm{QS}(x) * \mathrm{QS}(y) = \mathrm{QS}(xy),$$

where xy denotes the variables $x_i y_j$ in lexicographic order.

The QS-distribution defines a Markov chain (or random walk) on $\mathfrak{S}_n$ by

$$\text{Prob}(u, uw) = L_{\text{co}(w^{-1})}(x).$$

Theorem. *The eigenvalues of M_n are the power sum symmetric functions $p_\lambda(x)$ for $\lambda \vdash n$. The eigenvalue $p_\lambda(x)$ occurs with multiplicity $n!/z_\lambda$, the number of elements in $\mathfrak{S}_n$ of cycle type λ.*

(consequence of Bergeron-Garsia or Bidigare-Hanlon-Rockmore)

Sample enumerative results. For $w \in \mathfrak{S}_n$ let

$$\begin{aligned}
\mathbf{inv(w)} &= \#\{(i,j)\ :\ i<j,\ w(i)>w(j)\} \\
\mathbf{maj(w)} &= \sum_{i\,:\,w(i)>w(i+1)} i \\
\mathbf{I_n(j)} &= \mathrm{Prob}(\mathrm{inv}(w)=j) \\
\mathbf{M_n(j)} &= \mathrm{Prob}(\mathrm{maj}(w)=j).
\end{aligned}$$

Theorem. *We have*

$$M_n(j) = I_n(j)$$

$$\sum_{n\geq 0}\sum_{j\geq 0} \frac{M_n(j)t^j z^n}{(1-t)(1-t^2)\cdots(1-t^n)} = \prod_{i,j\geq 1}\left(1-t^{i-1}x_j z\right)^{-1}.$$

MacMahon (1913):

$$\#\{w \in \mathfrak{S}_n \,:\, \mathrm{maj}(w) = j\}$$
$$= \#\{w \in \mathfrak{S}_n \,:\, \mathrm{inv}(w) = j\}.$$

Since $U = \lim_{q\to\infty} U_q$, the result $M_n(j) = I_n(j)$ is a generalization.

In fact, if

$$F_\lambda(t) = \sum_v t^{\mathrm{maj}(v)}$$

$$G_\lambda(t) = \sum_v t^{\mathrm{inv}(v)},$$

where v ranges over all permutations of the multiset $\{1^{\lambda_1}, 2^{\lambda_2}, \ldots\}$, then

$$\sum_j M_n(j)t^j = \sum_{\lambda \vdash n} F_\lambda(t) m_\lambda(x)$$

$$\sum_j I_n(j)t^j = \sum_{\lambda \vdash n} G_\lambda(t) m_\lambda(x).$$

Thus $M_n(j) = I_n(j)$ is equivalent to MacMahon's result for multisets.

Let

$$L_n(x) = \frac{1}{n}\sum_{d|n}\mu(d)p_d^{n/d}(x)$$
$$= \mathrm{ch}\ \mathrm{ind}_{C_n}^{\mathfrak{S}_n} e^{2\pi i/n}.$$

Theorem. *Let w be a random permutation in $\mathfrak{S}_n$, chosen from the QS-distribution. The probability* $\mathrm{Prob}(\rho(w) = \lambda)$ *that w has cycle type $\lambda = \langle 1^{m_1}2^{m_2}\cdots\rangle \vdash n$ (i.e., m_i cycles of length i) is given by*

$$\mathrm{Prob}(\rho(w) = \lambda) = \prod_{i\geq 1} h_{m_i}[L_i],$$

where brackets denote plethysm.

Connections with the RSK algorithm

Let $w \in \mathfrak{S}_n$, and let $w \xrightarrow{\text{RSK}} (P, Q)$ denote the RSK algorithm, so P and Q are SYT of the same shape $\lambda \vdash n$. Write

$$\mathbf{sh}(\boldsymbol{w}) = \lambda.$$

Theorem. *Choose $w \in \mathfrak{S}_n$ from the QS-distribution, and let $w \xrightarrow{\text{RSK}} (P, Q)$. Let T be a SYT of shape $\lambda \vdash n$. Then*

$$\text{Prob}(P = T) = s_\lambda(x),$$

where $s_\lambda(x)$ denotes a Schur function.

Corollary. *Choose* $w \in \mathfrak{S}_n$ *from the QS-distribution, and let* $\lambda \vdash n$*. Then*

$$\mathrm{Prob}(\mathrm{sh}(w) = \lambda) = f^\lambda s_\lambda(x),$$

where f^λ *denotes the number of SYT of shape* λ *(given explicitly by the Frame-Robinson-Thrall hook-length formula).*

Longest increasing subsequences

Let $\mathrm{is}(w)$ be the length of the longest increasing subsequence of $w = w_1 \cdots w_n$.

Theorem (Schensted). *If*

$$\mathrm{sh}(w) = (\lambda_1, \lambda_2, \ldots),$$

then $\lambda_1 = \mathrm{is}(w)$. *Hence*

$$E_U(\mathrm{is}(w)) = \frac{1}{n!} \sum_{\lambda \vdash n} \lambda_1 \left(f^\lambda\right)^2.$$

Theorem (Vershik-Kerov):

$$E_U(\mathrm{is}(w)) \sim 2\sqrt{n}.$$

For the QS-distribution,

$$E(\mathrm{is}(w)) = \sum_{\lambda\vdash n} \lambda_1 f^\lambda s_\lambda(x).$$

$$\begin{aligned} E_{U_q}(\mathrm{is}(w)) &= \frac{1}{n!}\sum_{\lambda\vdash n} \lambda_1 \left(f^\lambda\right)^2 \prod_{u\in\lambda}\left(1+q^{-1}c(u)\right) \\ &= E_U(\mathrm{is}(w)) \end{aligned}$$

$$+\frac{1}{n!}\sum_{\lambda\vdash n} \lambda_1 \left(f^\lambda\right)^2 \left(\sum_{u\in\lambda} c(u)\right)\frac{1}{q}+\cdots.$$

Let

$$F_1(n) = \frac{1}{n!}\sum_{\lambda\vdash n} \lambda_1 \left(f^\lambda\right)^2 \left(\sum_{u\in\lambda} c(u)\right).$$

Numerical evidence suggests that $F_1(n)/n$ is slowly increasing, possibly to a finite limit. We computed $F_1(47)/47 \approx 0.6991$.

Logan-Shepp, Vershik-Kerov: "asymptotic shape" of a "typical" $w \in \mathfrak{S}_n$ (uniform distribution) as $n \to \infty$.

Baik-Deift-Johansson: Asymptotic distribution of $\mathrm{sh}(w)$ for $w \in \mathfrak{S}_n$ (uniform distribution) as $n \to \infty$.

Theorem. *For each $n \in \mathbb{P}$ let $w^{(n)} \in \mathfrak{S}_n$ be chosen from the QS-distribution. Let $\mathrm{sh}(w^{(n)}) = (\lambda_1^{(n)}, \lambda_2^{(n)}, \ldots)$, and let $y_1 \geq y_2 \geq \cdots$ be the decreasing rearrangement of $x_1, x_2, \ldots$. Then almost surely (i.e., with probability 1) for all i there holds*

$$\lim_{n\to\infty} \frac{\lambda_i^{(n)}}{n} = y_i.$$

Corollary. *Fix* $x = (x_1, x_2, \ldots)$, *with* $x_i \geq 0$ *and* $\sum x_i = 1$ *as usual. Let* $\mu^{(n)}$ *be a partition* $\nu \vdash n$ *that maximizes* $f^\nu s_\nu(x)$. *Then*

$$\lim_{n\to\infty} \frac{\mu_i^{(n)}}{n} = y_i.$$

Theorem (Its-Tracy-Widom) *Let*

$$x_1 > x_2 > \cdots .$$

Then

$$E(\mathrm{is}(w)) = x_1 n + \sum_{j>1} \frac{p_j}{p_1 - p_j} + O\left(\frac{1}{\sqrt{n}}\right).$$

Open: Find an asymptotic formula for the expected value of λ_1 (where $\mathrm{sh}(w) = \lambda$ under the $\mathrm{QS}(x)$-distribution) that specializes to both the Vershik-Kerov result (uniform distribution) and the case x fixed, $n \to \infty$.