Catalan Numbers

Richard P. Stanley

July 19, 2021

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$C_0 = 1,\ C_1 = 1,\ C_2 = 2,\ C_3 = 5,\ C_4 = 14, \ldots$

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Aside. **A000001**: number of groups of order n

Catalan monograph

R. Stanley, *Catalan Numbers*, Cambridge University Press, 2015.

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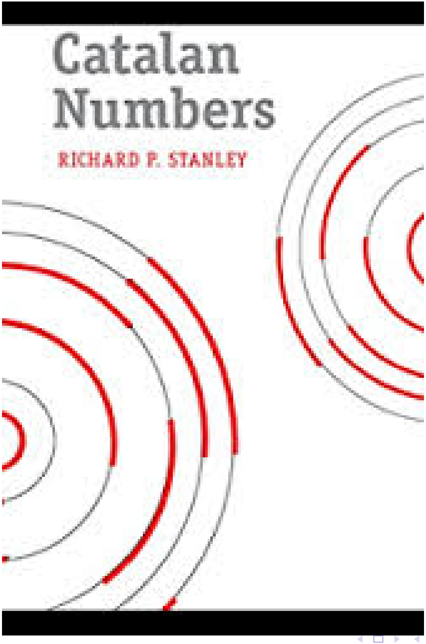
Includes 214 combinatorial interpretations of C_n and 68 additional problems.


Catalan Numbers
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History

Sharabiin Myangat, also known as **Minggatu**, **Ming'antu** (明安图), and **Jing An** (c. 1692–c. 1763): a Mongolian astronomer, mathematician, and topographic scientist who worked at the Qing court in China.

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No combinatorics, no further work in China.

Ming'antu

Manuscript of Ming'antu

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甲乙與丙庚為第一率與
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即如三率乘一率除

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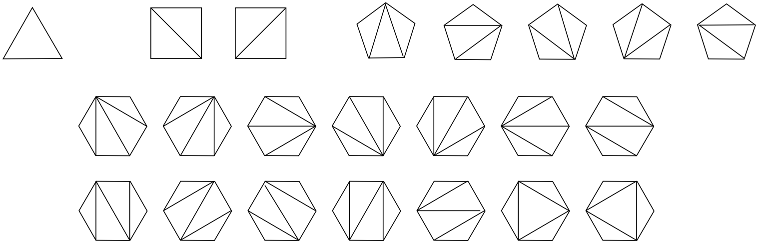
More history, via Igor Pak

- **Euler** (1751): conjectured formula for the number of triangulations of a convex $(n+2)$-gon. In other words, draw $n-1$ noncrossing diagonals of a convex polygon with $n+2$ sides.



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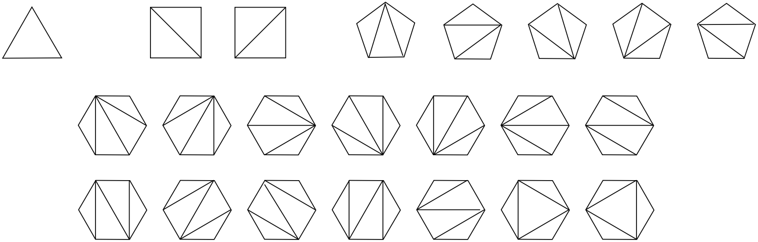

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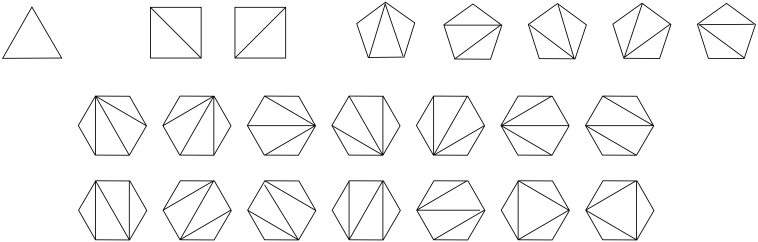

$$1, \quad 2, \quad 5, \quad 14, \quad \ldots$$

We **define** these numbers to be the Catalan numbers C_n.

Completion of proof

- **Goldbach and Segner** (1758–1759): helped Euler complete the proof, in pieces.
- **Lamé** (1838): first self-contained, complete proof.

Catalan

- **Eugène Charles Catalan** (1838): wrote C_n in the form $\frac{(2n)!}{n!\,(n+1)!}$ and showed it counted (nonassociative) **bracketings** (or **parenthesizations**) of a string of $n+1$ letters.

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Born in 1814 in Bruges (now in Belgium, then under Dutch rule). Studied in France and worked in France and Liège, Belgium. Died in Liège in 1894.

Why "Catalan numbers"?

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- **Martin Gardner** (1976): used the term in his Mathematical Games column in *Scientific American*. Real popularity began.

The primary recurrence

$$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}, \quad C_0 = 1$$

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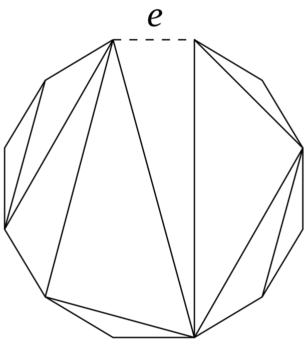

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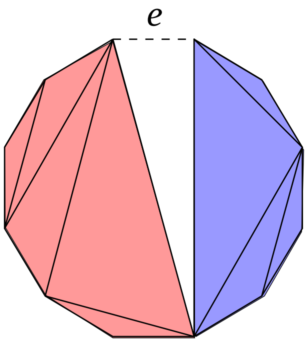

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Let $\boldsymbol{y} = \sum_{n\geq 0} C_n x^n$ (**generating function**).

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Multiply both sides by x^n and sum on $n \geq 0$:

$$\begin{aligned} \sum_{n\geq 0} C_{n+1} x^n &= \frac{y-1}{x} \\ \sum_{n\geq 0} \left(\sum_{k=0}^{n} C_k C_{n-k} \right) x^n &= y^2 \end{aligned}$$

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$$\Rightarrow xy^2 - y + 1 = 0$$

Solving the recurrence (cont.)

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The $-$ sign is correct:

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where

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}.$$

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where

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}.$$

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{n!\,(n+1)!}$$

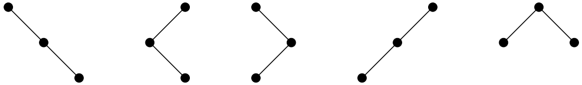
Other combinatorial interpretations

$$\begin{aligned} \mathcal{P}_n &:= \{\text{triangulations of convex } (n+2)\text{-gon}\} \\ \Rightarrow \#\mathcal{P}_n &= C_n \text{ (where } \#S = \text{number of elements of } S) \end{aligned}$$

We want other combinatorial interpretations of C_n, i.e., other sets $\mathcal{S}_n$ for which $C_n = \#\mathcal{S}_n$.

“Transparent” interpretations

4. Binary trees with n vertices (each vertex has a left subtree and a right subtree, which may be empty)



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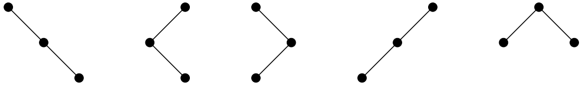

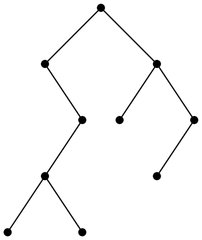

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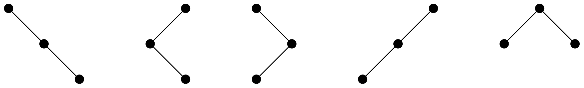

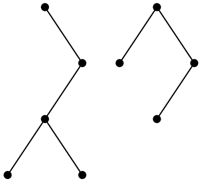

Binary parenthesizations

3. Binary **parenthesizations** or **bracketings** of a string of $n+1$ letters

$$(xx\cdot x)x \quad x(xx\cdot x) \quad (x\cdot xx)x \quad x(x\cdot xx) \quad xx\cdot xx$$

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$$((\boldsymbol{x}(\boldsymbol{xx}))\boldsymbol{x})(\boldsymbol{x}((\boldsymbol{xx})(\boldsymbol{xx})))$$

The ballot problem

Bertrand's ballot problem: first published by **W. A. Whitworth** in 1878 but named after **Joseph Louis François Bertrand** who rediscovered it in 1887 (one of the first results in probability theory).

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Special case: there are two candidates A and B in an election. Each receives n votes. What is the probability that A will never trail B during the count of votes?

Example. $AABABBBAAB$ is bad, since after seven votes, A receives 3 while B receives 4.

Definition of ballot sequence

Encode a vote for A by 1, and a vote for B by -1 (abbreviated $-$). Clearly a sequence $a_1 a_2 \cdots a_{2n}$ of n each of 1 and -1 is allowed if and only if $\sum_{i=1}^{k} a_i \geq 0$ for all $1 \leq k \leq 2n$. Such a sequence is called a **ballot sequence**.

Ballot sequences

77. Ballot sequences, i.e., sequences of n 1's and n -1's such that every partial sum is nonnegative (with -1 denoted simply as $-$ below)

$$111---\quad 11-1--\quad 11--1-\quad 1-11--\quad 1-1-1-$$

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Note. Answer to original problem (probability that a sequence of n each of 1's and -1's is a ballot sequence) is therefore

$$\frac{C_n}{\binom{2n}{n}} = \frac{\frac{1}{n+1}\binom{2n}{n}}{\binom{2n}{n}} = \frac{1}{n+1}.$$

The ballot recurrence

$$1\,1 - 1\,1 - 1 - - - 1 - 1\,1 - 1 - -$$

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$$1\,1 - 1\,1 - 1 - - - \; 1 - 1\,1 - 1 - -$$

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The ballot recurrence

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A combinatorial proof

$B(n)$: number of ballot sequences of length $2n$

Goal: a direct combinatorial proof that $B(n) = \frac{1}{n+1}\binom{2n}{n}$

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***B*(*n*)**: number of ballot sequences of length $2n$

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Note. Let $C(n)$ denote the number of sequences $b_1 b_2 \dots b_{2n+1}$ with $n+1$ occurrences of 1 and n occurrences of -1, such that $b_1 + b_2 + \cdots + b_i > 0$, $1 \le i \le 2n+1$ (**strict ballot sequence**). In particular, $b_1 = 1$. Then $C(n) = B(n)$.

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Proof. $b_1 b_2 \cdots b_{2n+1}$ is counted by $C(n)$ if and only if $b_2 b_3 \cdots b_{2n+1}$ is a ballot sequence. $\square$

Crucial lemma

Lemma. *Every sequence* $b_1 b_2 \cdots b_{2n+1}$ *where 1 occurs* $n+1$ *times and* -1 *occurs* n *times, with* $b_1 = 1$*, has a unique cyclic shift* $b_i b_{i+1} \cdots b_{2n+1} b_1 \cdots b_{i-1}$ *that is a strict ballot sequence.*

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Proof #1. Induction on n. Clear for $n = 0$. Assume for $n - 1$. Let $\beta = b_1 b_2 \cdots b_{2n+1}$ be a sequence with $b_1 = 1$, 1 occurring $n+1$ times and -1 occuring n times. Let $b_j = 1$, $b_{j+1} = -1$ (subscripts mod $2n+1$). Remove b_j, b_{j+1} from β, obtaining β'.

By induction, β' has a unique cyclic shift, say beginning with b_k, that is a strict ballot sequence.

Easy to check: the cyclic shift of β beginning with b_k is a strict ballot sequence, and no other cyclic shift has this property. □

Geometric proof.

Proof #2. **Example.** $(1, -1, -1, 1, -1, 1, 1)$

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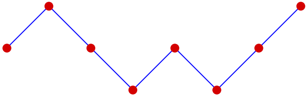

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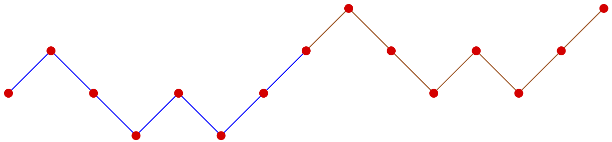

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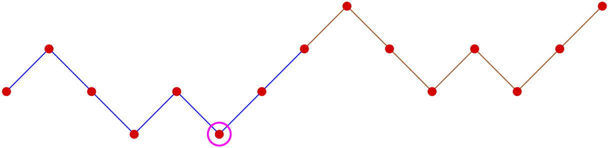

rightmost minimum

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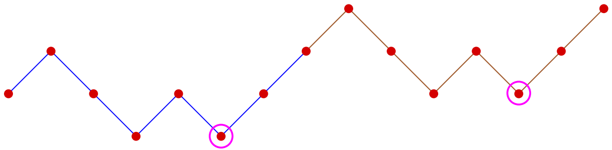

Proof that $C(n) = \frac{1}{n+1}\binom{2n}{n}$

- There are $\binom{2n}{n}$ sequences with 1 occurring $n+1$ times and -1 occurring n times, beginning with a 1.

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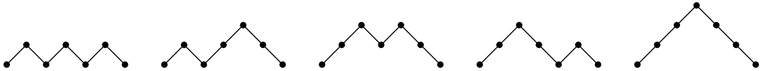
- Exactly one of these cyclic shifts is a strict ballot sequence (previous lemma).
- $\Rightarrow C(n) = \dfrac{1}{n+1}\dbinom{2n}{n} = \dfrac{1}{2n+1}\dbinom{2n+1}{n}$ $\square$

Dyck paths

25. Dyck paths of length $2n$, i.e., lattice paths from $(0,0)$ to $(2n,0)$ with steps $(1,1)$ and $(1,-1)$, never falling below the x-axis



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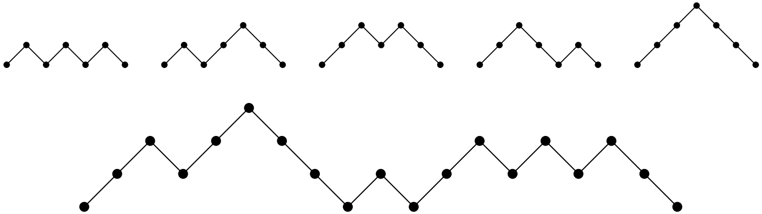

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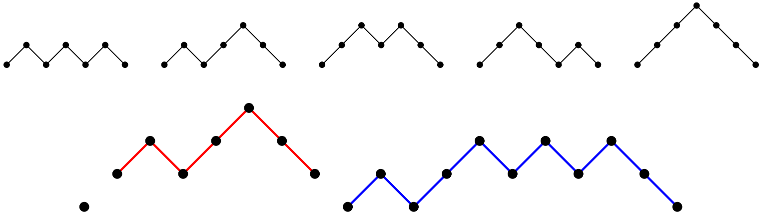

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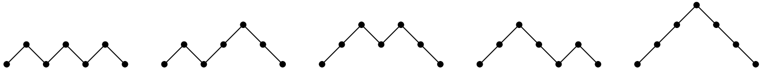
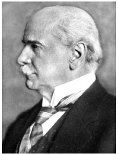
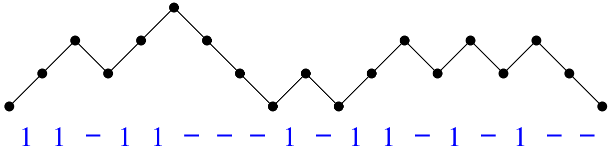


Walther von Dyck (1856–1934)



Bijection with ballot sequences



For each upstep, record 1.

For each downstep, record −1.

312-avoiding permutations

116. Permutations $a_1 a_2 \cdots a_n$ of $1, 2, \ldots, n$ for which there does not exist $i < j < k$ and $a_j < a_k < a_i$ (called 312-**avoiding**) permutations)

123 132 213 231 321

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34251768

312-avoiding permutations

116. Permutations $a_1a_2\cdots a_n$ of $1, 2, \ldots, n$ for which there does not exist $i < j < k$ and $a_j < a_k < a_i$ (called 312-**avoiding** permutations)

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3425 768

312-avoiding permutations

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part of the subject of **pattern avoidance**

321-avoiding permutations

Another example of pattern avoidance:

115. Permutations $a_1 a_2 \cdots a_n$ of $1, 2, \ldots, n$ with longest decreasing subsequence of length at most two (i.e., there does not exist $i < j < k$, $a_i > a_j > a_k$), called **321-avoiding** permutations

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321-avoiding permutations

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123 213 132 312 231

more subtle: no obvious decomposition into two pieces

Bijection with ballot sequences

$$w = 412573968$$

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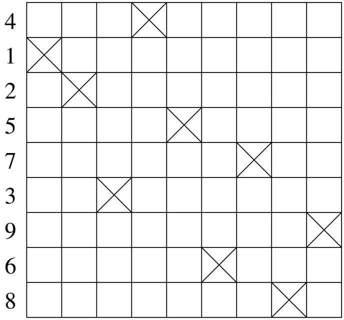

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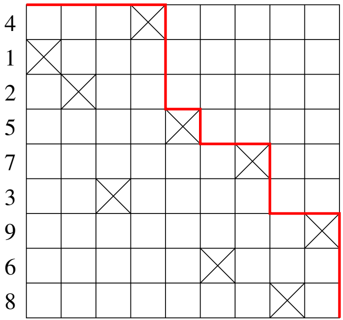

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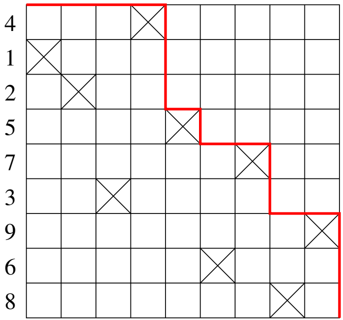

$$1\,1\,1\,1 - - - 1 - 1\,1 - - 1\,1 - - -$$

An unexpected interpretation

92. n-tuples $(a_1, a_2, \ldots, a_n)$ of integers $a_i \geq 2$ such that in the sequence $1a_1a_2 \cdots a_n1$, each a_i divides the sum of its two neighbors

14321 13521 13231 12531 12341

Bijection with ballot sequences

remove largest; insert bar before the element to its left; continue until only 1's remain; then replace bar with 1 and an original number with -1, except last two

$$1 \quad 2 \quad 5 \quad 3 \quad 4 \quad 1$$

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|	1	|	|	2	5	|	3	4	1
1	−	1	1	−	−	1	−		

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1	–	1	1	–	–	1	–		

tricky to prove

A8. Algebraic interpretations

(a) Number of two-sided ideals of the algebra of all $(n-1) \times (n-1)$ upper triangular matrices over a field

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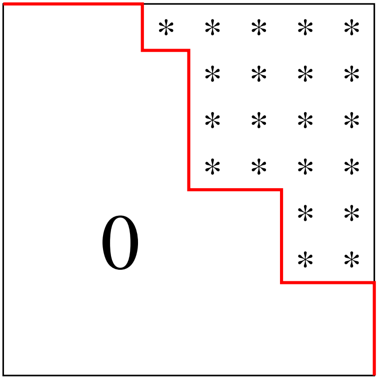

Diagonal harmonics

(i) Let the symmetric group $\mathfrak{S}_n$ act on the polynomial ring $A = \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ by $w \cdot f(x_1, \ldots, x_n, y_1, \ldots, y_n) = f(x_{w(1)}, \ldots, x_{w(n)}, y_{w(1)}, \ldots, y_{w(n)})$ for all $w \in \mathfrak{S}_n$. Let I be the ideal generated by all invariants of positive degree, i.e.,

$$I = \langle f \in A : w \cdot f = f \text{ for all } w \in \mathfrak{S}_n, \text{ and } f(0) = 0 \rangle.$$

Diagonal harmonics (cont.)

Then C_n is the dimension of the subspace of A/I affording the sign representation, i.e.,

$$C_n = \dim\{f \in A/I : w \cdot f = (\operatorname{sgn} w) f \text{ for all } w \in \mathfrak{S}_n\}.$$

Diagonal harmonics (cont.)

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$$C_n = \dim\{f \in A/I : w \cdot f = (\operatorname{sgn} w)f \text{ for all } w \in \mathfrak{S}_n\}.$$

Very deep proof by **Mark Haiman**, 1994.

Generalizations & refinements

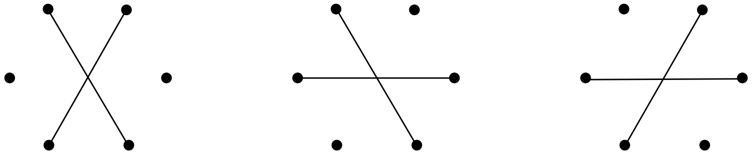
A12. ***k*-triangulation** of n-gon: maximal collections of diagonals such that no $k+1$ of them pairwise intersect in their interiors

$k = 1$: an ordinary triangulation

superfluous edge: an edge between vertices at most k steps apart (along the boundary of the n-gon). They appear in all k-triangulations and are irrelevant.

An example

Example. 2-triangulations of a hexagon (superfluous edges omitted):



Some theorems

Theorem (**Nakamigawa**, **Dress-Koolen-Moulton**). *All k-triangulations of an n-gon have $k(n-2k-1)$ nonsuperfluous edges.*

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Theorem (**Jonsson**, **Serrano-Stump**). *The number $T_k(n)$ of k-triangulations of an n-gon is given by*

$$\begin{aligned} T_k(n) &= \det\left[C_{n-i-j}\right]_{i,j=1}^{k} \\ &= \prod_{1\le i<j\le n-2k} \frac{2k+i+j-1}{i+j-1}. \end{aligned}$$

Representation theory?

Note. The number $T_k(n)$ is the dimension of an irreducible representation of the symplectic group $\mathrm{Sp}(2n-4)$.

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Is there a direct connection?

Number theory

A61. Let $b(n)$ denote the number of 1's in the binary expansion of n. Using Kummer's theorem on binomial coefficients modulo a prime power, show that the exponent of the largest power of 2 dividing C_n is equal to $b(n+1)-1$.

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Kummer's theorem. *Let* $\boldsymbol{p}$ *be prime,* $0 \leq k \leq n$*. Then the exponent of the largest power of* p *dividing* $\binom{n}{k}$ *is equal to the number of carries in adding* k *and* $n-k$*.*

Sums of three squares

Let $f(n)$ denote the number of integers $1 \le k \le n$ such that k is the sum of three squares (of nonnegative integers). Well-known:

$$\lim_{n\to\infty} \frac{f(n)}{n} = \frac{5}{6}.$$

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$$\lim_{n\to\infty} \frac{g(n)}{n} = \frac{7}{8}.$$

Why?

Theorem. *A positive integer n is* not *the sum of three squares if and only if $n = 4^k(8m + 7)$.*

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$$1 - \frac{1}{2} \cdot \frac{1}{4} = \frac{7}{8}$$

Analysis

A65.(b)

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$$2 + \frac{4\sqrt{3}\pi}{27} = 2.806133\cdots$$

Why?

A65.(a)

$$\sum_{n\geq 0} \frac{x^n}{C_n} = \frac{2(x+8)}{(4-x)^2} + \frac{24\sqrt{x}\sin^{-1}\left(\frac{1}{2}\sqrt{x}\right)}{(4-x)^{5/2}}.$$

Why?

A65.(a)

$$\sum_{n\geq 0}\frac{x^n}{C_n}=\frac{2(x+8)}{(4-x)^2}+\frac{24\sqrt{x}\sin^{-1}\left(\frac{1}{2}\sqrt{x}\right)}{(4-x)^{5/2}}.$$

Sketch of solution. Calculus exercise: let

$$y=2\left(\sin^{-1}\frac{1}{2}\sqrt{x}\right)^2.$$

Then $y=\sum_{n\geq 1}\frac{x^n}{n^2\binom{2n}{n}}$.

Completion of proof

Recall

$$y = 2\left(\sin^{-1}\frac{1}{2}\sqrt{x}\right)^2 = \sum_{n\geq 1}\frac{x^n}{n^2\binom{2n}{n}}.$$

Note that:

Completion of proof

Recall

$$y = 2\left(\sin^{-1}\frac{1}{2}\sqrt{x}\right)^2 = \sum_{n\geq 1}\frac{x^n}{n^2\binom{2n}{n}}.$$

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$$\frac{d}{dx}y = \sum_{n\geq 1}\frac{x^{n-1}}{n\binom{2n}{n}}$$

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$$= -1 + \sum_{n\geq 0}\frac{x^n}{C_n},$$

etc.

The last slide

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The last slide