TORIC SCHUR FUNCTIONS

Gr_{kn}: **Grassmann variety** of k-subspaces of $\mathbb{C}^n$

$$\dim_{\mathbb{C}} \mathrm{Gr}_{kn} = k(n-k)$$

$H^*(\mathrm{Gr}_{kn})$ $= H^*(\mathrm{Gr}_{kn}; \mathbb{Z})$: cohomology ring (fundamental object for **Schubert calculus**)

basis for $H^*(\mathrm{Gr}_{kn})$: **Schubert classes** $\boldsymbol{\sigma_\lambda}$, where $\lambda = (\lambda_1, \ldots, \lambda_k)$ and

$$\lambda \subseteq \boldsymbol{k \times (n-k)},$$

i.e.,

$$n - k \geq \lambda_1 \geq \cdots \geq \lambda_k \geq 0.$$

Let $\boldsymbol{P_{kn}}$ be the set of all such partitions λ, so

$$\#P_{kn} = \operatorname{rank} H^*(\mathrm{Gr}_{kn}) = \binom{n}{k}.$$

$\boldsymbol{\Omega_\lambda} \subset \mathrm{Gr}_{kn}$: **Schubert variety**, defined by bounds on $\dim X \cap V_i$, for $X \in \mathrm{Gr}_{kn}$, where

$$\{0\} = V_0 \subset V_1 \subset \cdots \subset V_n = \mathbb{C}^n$$

is a fixed flag.

Multiplication in $H^*(\mathrm{Gr}_{kn})$:

$$\sigma_\mu \sigma_\nu = \sum_{\lambda \in P_{kn}} c^\lambda_{\mu\nu} \sigma_\lambda,$$

where $c^\lambda_{\mu\nu}$ is a **Littlewood-Richardson coefficient**.

$$\Rightarrow c^\lambda_{\mu\nu} = \# \left(\tilde{\Omega}_\mu \cap \tilde{\Omega}_\nu \cap \tilde{\Omega}_{\lambda^\vee} \right),$$

where $\boldsymbol{\tilde{\Omega}_\nu}$ is a generic translate of Ω_ν and $\boldsymbol{\lambda^\vee}$ is the **complementary partition**

$$\lambda^\vee = (n-k-\lambda_k, \ldots, n-k-\lambda_1).$$

$\mathrm{QH}^*(\mathrm{Gr}_{kn})$: **quantum deformation** of $H^*(\mathrm{Gr}_{kn})$

Λ_k: ring of symmetric polynomials over $\mathbb{Z}$ in $x_1, \ldots, x_k$.

$$\Lambda_k = \mathbb{Z}[e_1, \ldots, e_k],$$

where e_i is the ith **elementary symmetric function** in $x_1, \ldots, x_k$.

h_i: sum of all monomials of degree i (**complete symmetric function**)

$$H^*(\mathrm{Gr}_{kn}) \cong \Lambda_k/(h_{n-k+1}, \ldots, h_n)$$

$$\mathrm{QH}^*(\mathrm{Gr}_{kn}) \cong \Lambda_k \otimes \mathbb{Z}[q]/(h_{n-k+1}, \ldots, h_{n-1}, h_n + (-1)^k q)$$

classical case: $q = 0$

$$H^*(\mathrm{Gr}_{kn}) \cong \Lambda_k/(h_{n-k+1}, \ldots, h_n)$$

Basis $\boldsymbol{B_{kn}}$ for $\Lambda_k/(h_{n-k+1}, \ldots, h_n)$:

Let λ be a partition.

semistandard Young tableau (SSYT) of shape λ:

$$\begin{array}{c} \leqslant \\ \wedge \;\; \begin{array}{|c|c|c|c|} \hline 1 & 1 & 3 & 4 \\ \hline 2 & 4 & 4 & 6 \\ \hline 4 & 6 & 9 & \\ \cline{1-3} 6 & & & \\ \cline{1-1} \end{array} \end{array}$$

$$\lambda = (4, 4, 3, 1)$$

$$x^T = x_1^2 x_2 x_3 x_4^4 x_6^3 x_9$$

Schur function $\boldsymbol{s_\lambda}$ of shape λ:

$$s_\lambda = \sum_T x^T,$$

summed over all SSYT T of shape λ.

$$B_{kn} = \{s_\lambda \,:\, \lambda \subseteq k \times (n-k)\},$$

$$\begin{aligned} H^*(\mathrm{Gr}_{kn}) &\xrightarrow{\cong} \Lambda_k/(h_{n-k+1}, \ldots, h_n) \\ \sigma_\lambda &\mapsto s_\lambda \end{aligned}$$

$$s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda$$

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}$$

$$\begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$$

$$\begin{aligned}
s_{21}(a,b,c) &= a^2b + ab^2 + a^2c + ac^2 \\
&\quad + b^2c + bc^2 + 2abc \\
s_{21} &= \sum_{i \neq j} x_i^2 x_j + 2 \sum_{i<j<k} x_i x_j x_k \\
s_{21}^2 &= s_{42} + s_{33} + s_{411} + 2s_{321} + s_{222} \\
&\quad + s_{3111} + s_{2211} \\
&\to s_{42} + s_{33} \text{ in } H^*(\mathrm{Gr}_{26}).
\end{aligned}$$

basis for $\mathrm{QH}^*(\mathrm{Gr}_{kn})$ remains

$$\{\sigma_\lambda : \lambda \subseteq k \times (n-k)\}$$

quantum multiplication:

$$\sigma_\mu * \sigma_\nu = \sum_{d \geq 0} \sum_{\substack{\lambda \vdash |\mu|+|\nu|-dn \\ \lambda \in P_{kn}}} q^d \boldsymbol{C_{\mu\nu}^{\lambda,d}} \sigma_\lambda,$$

where $C_{\mu\nu}^{\lambda,d} \in \mathbb{Z}$.

$C_{\mu\nu}^{\lambda,d}$: number of rational curves of degree d in Gr_{kn} meeting $\tilde{\Omega}_\mu \cap \tilde{\Omega}_\nu \cap \tilde{\Omega}_{\lambda^\vee}$, a **3-point Gromov-Witten invariant**

Naively, a **rational curve of degree r in Gr_{kn}** is a set

$$C = \left\{ (f_1(s,t), f_2(s,t), \ldots, f_{\binom{n}{k}}(s,t)) \in P^{\binom{n}{k}-1}(\mathbb{C}) \;:\; s,t \in \mathbb{C} \right\},$$

where $f_1(x,y), \ldots, f_{\binom{n}{k}}(x,y)$ are homogeneous polynomials of degree d such that $C \subset \mathrm{Gr}_{kn}$.

Rational curve of degree $d = 0$ is a point.

Let λ/μ be a **skew partition**, i.e., $\mu \subseteq \lambda$.

semistandard Young tableau (SSYT) of shape λ/μ:

$$\begin{array}{cccc} \multicolumn{4}{r}{\leqslant} \\ & & 2 & 3 \\ \wedge & 1 & 4 & 4 \\ & 3 & 6 & \\ 2 & & & \end{array}$$

$$\lambda/\mu = (4,4,3,1)/(2,1,1)$$

$$x^T = x_1 x_2^2 x_3^2 x_4^2 x_6$$

skew Schur function $\boldsymbol{s_{\lambda/\mu}}$ of shape λ/μ:

$$s_{\lambda/\mu} = \sum_T x^T,$$

summed over all SSYT T of shape λ/μ.

$$s_\lambda = s_{\lambda/\emptyset}$$

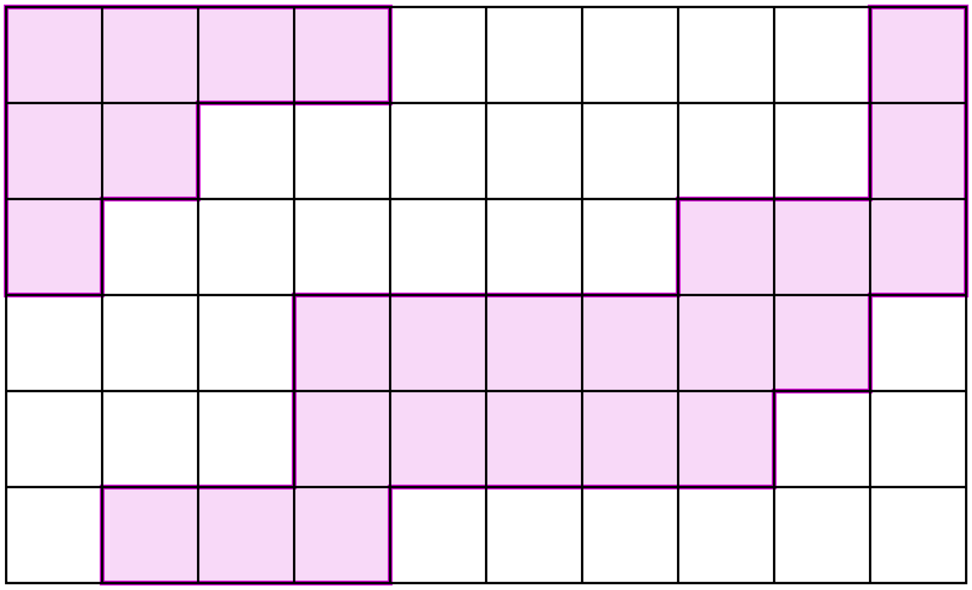
$$s_{\lambda/\mu} = \sum_\nu c^\lambda_{\mu\nu} s_\nu, \qquad (1)$$

where $c^\lambda_{\mu\nu}$ is a Littlewood-Richardson coefficient, i.e.,

$$s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda.$$

Want to generalize (1) to $C^{\lambda,d}_{\mu\nu}$.

toric shape τ in a 6×10 rectangle:



semistandard toric tableau (SSTT):

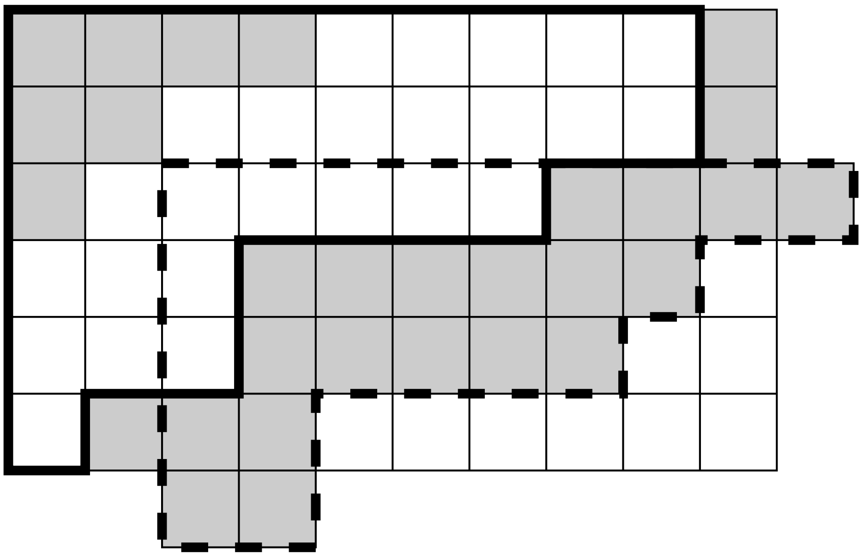
$\leqslant$

$\wedge$

2	2	4	6						
3	5								
4							1	2	4
			1	2	2	2	2	5	
			3	3	4	4	4		
	1	2	4						

the toric shape

$$\tau = \boldsymbol{\lambda/d/\mu}$$
$$= (9,7,6,2,2,0)/2/(9,9,7,3,3,1):$$



toric Schur function:

$$\boldsymbol{s_{\lambda/d/\mu}} = \sum_T x^T,$$

summed over all SSTT of shape $\lambda/d/\mu$

Theorem. *Let $\lambda/d/\mu$ be a toric shape contained in a $k \times (n-k)$ torus. Then*

$$s_{\lambda/d/\mu}(x_1, \ldots, x_k) = \sum_{\nu \in P_{kn}} C_{\mu\nu}^{\lambda,d} s_\nu(x_1, \ldots, x_k).$$

Compare the case $d = 0$: If

$$\lambda/\mu \subseteq k \times (n-k),$$

then

$$s_{\lambda/\mu}(x_1, , \ldots, x_k) = \sum_{\nu \in P_{kn}} c_{\mu\nu}^{\lambda} s_\nu(x_1, \ldots, x_k).$$

TORIC h-VECTORS AND INTERSECTION COHOMOLOGY

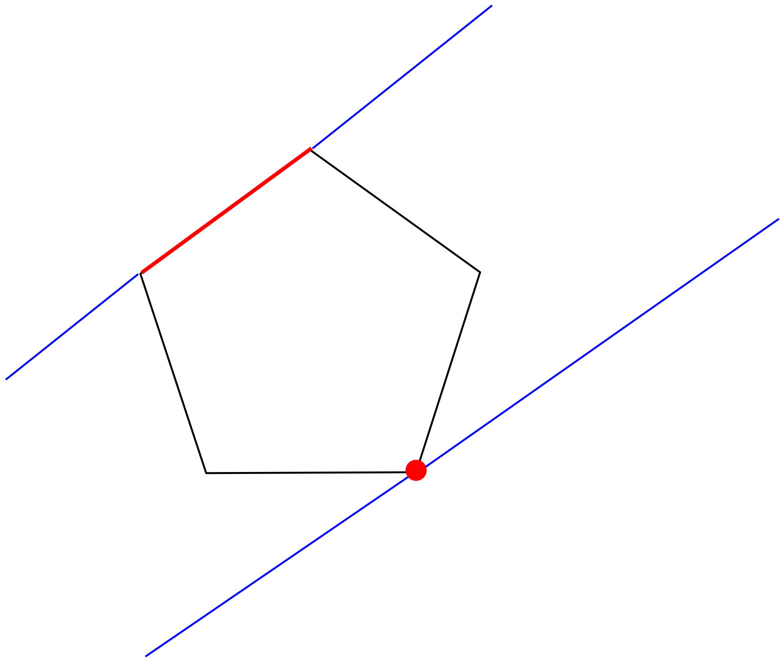
convex polytope: convex hull $\mathcal{P}$ of a finite set in $\mathbb{R}^n$

$\boldsymbol{d} = \dim \mathcal{P}$

face: intersection of $\mathcal{P}$ with a supporting hyperplane

$\boldsymbol{f_i}$: number of i-dimensional faces $(f_{-1} = 1)$

$\boldsymbol{f}$**-vector**: $f(\mathcal{P}) = (f_0, f_1, \ldots, f_{d-1})$



$$
\begin{aligned}
f(\text{pentagon}) &= (5, 5) \\
f(\text{3-cube}) &= (8, 12, 6)
\end{aligned}
$$

simplicial polytope: every proper face is a simplex (e.g., tetrahedron, octahedron, icosahedron)

***h*-vector:** $\boldsymbol{h(\mathcal{P})} = (h_0, \ldots, h_d)$ defined by:

$$\sum_{i=0}^{d} f_{i-1}(x-1)^{d-i} = \sum_{i=0}^{d} \boldsymbol{h_i} x^{d-i}.$$

g-theorem: $(h_0, \ldots, h_d) \in \mathbb{Z}^{d+1}$ is $h(\mathcal{P})$ for some simplicial $\mathcal{P}$ if and only if:

(G$_1$) $h_0 = 1$

(G$_2$) $h_i = h_{d-i}$ (**Dehn-Sommerville equations**)

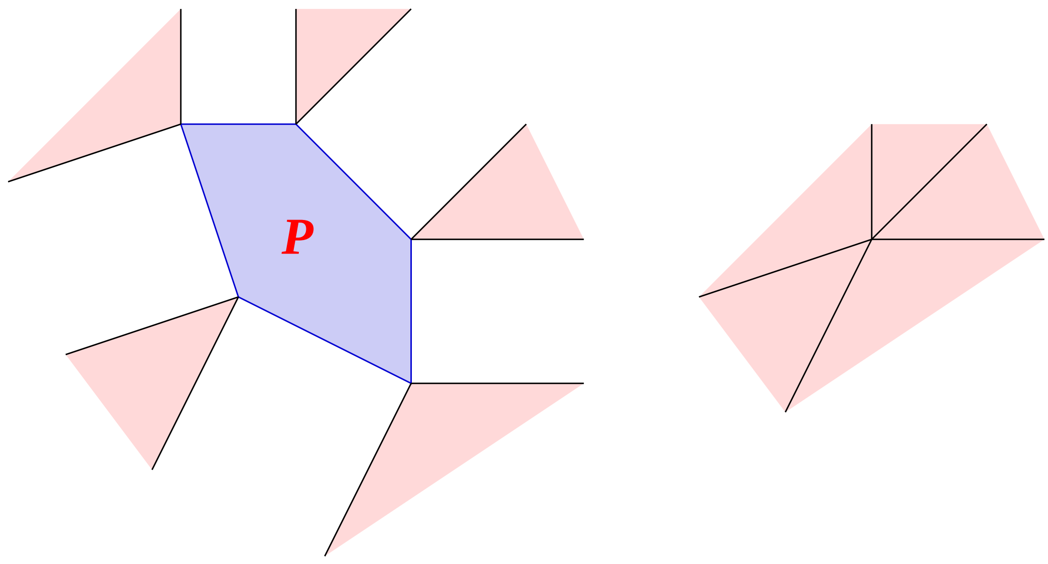
(G$_3$) $h_0 \leq h_1 \leq \cdots \leq h_{\lfloor d/2 \rfloor}$ (**GLBC**)

(G$_4$) Non-polynomial inequalities (**g-inequalities**) on rate of growth of $\boldsymbol{g_i} := h_i - h_{i-1}$

Proof of necessity. **($\mathbf{G_1}$)** trivial

($\mathbf{G_2}$) "classical" (not difficult)

($\mathbf{G_3}$) Perturb $\mathcal{P}$ to have rational vertices. Let $\boldsymbol{X_{\mathcal{P}}}$ be the **toric variety** corresponding to the **normal fan** $\boldsymbol{\Sigma_{\mathcal{P}}}$ of $\mathcal{P}$.



Cohomology ring:

$$H^*(X_{\mathcal{P}};\mathbb{R}) = H^0(X_{\mathcal{P}};\mathbb{R}) \oplus H^2(X_{\mathcal{P}};\mathbb{R}) \oplus \cdots \oplus H^{2d}(X_{\mathcal{P}};\mathbb{R}),$$

where

$$\dim H^{2i}(X_{\mathcal{P}};\mathbb{R}) = h_i(\mathcal{P}).$$

Hard Lefschetz theorem for $X_{\mathcal{P}}$: if $\omega \in H^2$ is the class of a hyperplane section, then

$$\omega^{d-i} : H^{2i} \to H^{2(d-i)}$$

is a bijection, $0 \le i < d/2$. Hence

$$\omega : H^i \to H^{i+1}$$

is injective for $0 \le i < d/2$, so

$$h_i \le h_{i+1}.$$

($\mathbf{G_4}$) Use that $H^*(X_{\mathcal{P}};\mathbb{R})$ is generated by H^2 as an $\mathbb{R}$-algebra.

If $\mathcal{P}$ is nonsimplicial and rational, can still define $X_{\mathcal{P}}$, but $H^*(X_{\mathcal{P}};\mathbb{R})$ is "bad." Instead use **intersection cohomology** (Goresky-MacPherson):

$$\mathbf{IH}(X_{\mathcal{P}};\mathbb{R}) = \mathrm{IH}^0(X_{\mathcal{P}};\mathbb{R}) \oplus \mathrm{IH}^2(X_{\mathcal{P}};\mathbb{R}) \oplus \cdots \oplus \mathrm{IH}^{2d}(X_{\mathcal{P}};\mathbb{R}).$$

Let $\boldsymbol{h_i} = h_i(\mathcal{P}) = \dim \mathrm{IH}^{2i}(X_{\mathcal{P}};\mathbb{R})$ (independent of embedding of $\mathcal{P}$).

toric *h*-vector:

$$\boldsymbol{h(\mathcal{P})} = (h_0, h_1, \ldots, h_d)$$

Computation of $\boldsymbol{h(\mathcal{P})}$. Define $\boldsymbol{f(\mathcal{P},x)}$ and $\boldsymbol{g(\mathcal{P},x)}$ by

- $f(\emptyset,x)=g(\emptyset,x)=1$
- If $\mathcal{P}\neq\emptyset$ then
$$f(\mathcal{P},x)=\sum_{\mathcal{Q}} g(\mathcal{Q},x)(x-1)^{\dim\mathcal{P}-\dim\mathcal{Q}-1},$$
where $\mathcal{Q}$ ranges over all faces of $\mathcal{P}$ (including $\emptyset$) except $\mathcal{P}$.
- If $\dim\mathcal{P}=d\geq 0$, $\boldsymbol{m}=\lfloor d/2\rfloor$, and $f(\mathcal{P},x)=h_0+h_1x+\cdots$, then
$$g(\mathcal{P},x)=h_0+(h_1-h_0)x+(h_2-h_1)x^2+\cdots+(h_m-h_{m-1})x^m.$$

Example. Let $\boldsymbol{\sigma_j}$ = j-simplex, $\boldsymbol{\mathcal{C}_j}$ = j-cube. Say we know

$$g(\sigma_0, x) = g(\sigma_1, x) = 1$$

$$g(\mathcal{C}_2, x) = 1 + x.$$

Then

$$\begin{aligned} f(\mathcal{C}_3, x) &= 6(x+1) + 12(x-1) \\ &\quad +8(x-1)^2 + (x-1)^3 \\ &= x^3 + 5x^2 + 5x + 1 \end{aligned}$$

$$g(\mathcal{C}_3, x) = 1 + 4x.$$

Note. $f(\mathcal{C}_n, 1) = 2\binom{2n-2}{n-1}$

$$g(\mathcal{C}_n, 1) = \frac{1}{n+1}\binom{2n}{n}$$

(Catalan number)

For **any** $\mathcal{P}$, define the **toric h-vector**

$$\boldsymbol{h}(\boldsymbol{\mathcal{P}}) = (h_0, \cdots, h_d),$$

where $f(\mathcal{P}, x) = h_0 + \cdots + h_d x^d$ (easy: $\deg f = d$).

Trivial: $h_0 = 1$ $\mathbf{(G_1)}$

Not difficult: $h_i = h_{d-i}$ $\mathbf{(G_2)}$

If $\mathcal{P}$ is rational, then

$$\dim \mathrm{IH}^{2i}(X_{\mathcal{P}}; \mathbb{R}) = h_i \Rightarrow h_i \geq 0.$$

Moreover, $\mathrm{IH}(X_{\mathcal{P}}; \mathbb{R})$ is a module over $H^*(X_{\mathcal{P}}; \mathbb{R})$, and hard Lefschetz holds. Thus

$$h_0 \leq h_1 \leq \cdots \leq h_{\lfloor d/2 \rfloor}. \quad \mathbf{(G_3)}$$

$\mathrm{IH}(X_{\mathcal{P}}; \mathbb{R})$ is not a ring, so $\mathbf{(G_4)}$ remains open even for $\mathcal{P}$ rational.

Extend to nonrational $\mathcal{P}$:

"Nice" generalization of $X_{\mathcal{P}}$ not known.

Nice generalization of $\mathrm{IH}(X_{\mathcal{P}};\mathbb{R})$ defined by Barthel-Brasslet-Fiesler-Kaup and Bressler-Lunts. Connection with h_i and hard Lefschetz proved by Karu, with improvements by Bressler-Lunts and Barthel et al.

$\boldsymbol{\Sigma}$: complete fan in $\mathbb{R}^d$

$\boldsymbol{\mathcal{A}_{\Sigma}}$: **structure sheaf** of Σ. For each cone $\sigma \in \Sigma$ define the stalk

$$\boldsymbol{\mathcal{A}_{\Sigma,\sigma}} = \mathrm{Sym}(\mathrm{span}\,\sigma)^*,$$

the space of polynomial functions on σ. Restriction map

$$\mathcal{A}_{\Sigma,\sigma} \to A_{\Sigma}(\partial\sigma)$$

defined by restriction of functions.

$\mathcal{A}_\Sigma$ is a sheaf of algebras. Multiplication with elements of $A = \mathrm{Sym}(\mathbb{R}^d)^*$ (all polynomial functions on $\mathbb{R}^d$) gives $\mathcal{A}_\Sigma$ the structure of a sheaf of A-modules.

$\boldsymbol{\mathcal{L}_\Sigma}$: **equivariant intersection cohomology sheaf**, a sheaf of $\mathcal{A}_\Sigma$-modules (technical definition)

$\mathbf{IH(\Sigma)} = \bar{\mathcal{L}}_\Sigma$ ($\mathcal{A}$-module of global sections of $\mathcal{L}_\Sigma$ modulo the ideal I of A generated by homogeneous linear functions): **intersection homology** of Σ

AXIOMS FOR $\mathcal{L}_\Sigma$

(E$_1$) (normalization) $\mathcal{L}_{\Sigma,0} = \mathbb{R}$

(E$_2$) (local freeness) $\mathcal{L}_{\Sigma,\sigma}$ is a free $\mathcal{A}_{\Sigma,\sigma}$-module for any $\sigma \in \Sigma$.

(E$_3$) (minimal flabbiness) Let I be the ideal of A generated by homogeneous linear functions, and for any A-module M write $\overline{M} = M/IM$. Then modulo the ideal I the restriction map induces an isomorphism

$$\overline{\mathcal{L}}_{\Sigma,\sigma} \to \overline{\mathcal{L}_\Sigma(\partial\sigma)}.$$

Bressler-Lunts:

- $\mathrm{IH}(\Sigma) = \mathrm{IH}^0 \oplus \mathrm{IH}^2 \oplus \cdots \oplus \mathrm{IH}^{2d}$
- Poincaré duality so
$$\mathrm{IH}^{2i}(\Sigma) \equiv \mathrm{IH}^{2(d-i)}(\Sigma)$$
- **Conjecture.** If $\Sigma = \Sigma_{\mathcal{P}}$ (normal fan of the polytope $\mathcal{P}$), then $\mathrm{IH}(\mathcal{P})$ satisfies hard Lefschetz: for strictly convex $l \in \mathcal{A}^2_{\Sigma_{\mathcal{P}}}$ and $i < d/2$,
$$l^{d-i} : \mathrm{IH}^{2i}(\mathcal{P}) \xrightarrow{\cong} \mathrm{IH}^{2(d-i)}.$$
- Above conjecture $\Rightarrow \dim \mathrm{IH}^{2i}(\mathcal{P}) = h_i(\mathcal{P})$, proving $\mathbf{(G_3)}$:
$$h_0 \leq h_1 \leq \cdots \leq h_{\lfloor d/2 \rfloor}$$

Karu: proved conjecture of Bressler-Lunts.

Stronger result: **Hodge-Riemann-Minkowski bilinear relations.** Poincaré duality $\Rightarrow$

$$\mathrm{IH}^{d-i}(\mathcal{P}) \times \mathrm{IH}^{d+i}(\mathcal{P}) \to \mathbb{R},$$

denoted $\langle \boldsymbol{x}, \boldsymbol{y} \rangle$. If $l \in A^2_\Sigma$ is strictly convex, define a quadratic form $\boldsymbol{Q_l}$ on $\mathrm{IH}^{d-i}(\mathcal{P})$ by

$$Q_l(x) = \langle l^i x, x \rangle.$$

Primitive intersection cohomology:

$$\mathbf{IP}^{\boldsymbol{d-i}}(\mathcal{P}) = \ker(l^{i+1}, \mathrm{IH}^{d-i}(\mathcal{P}))$$

$$l^{i+1} : \mathrm{IH}^{d-i}(\mathcal{P}) \to \mathrm{IH}^{d+i+2}(\mathcal{P}).$$

H-R-M: $(-1)^{(d-i)/2} Q_l$ is positive definite on $\mathrm{IP}^{d-i}(\mathcal{P})$ for all $i \geq 0$.

(proved by McMullen for simplicial $\mathcal{P}$)

Extremely rough sketch of proof: find a suitable triangulation of the fan $\Sigma_{\mathcal{P}}$ and "lift" H-R-M from Δ to Σ.

Bressler-Lunts: **canonical** pairing $\langle \cdot, \cdot \rangle$, independent of choice of Δ.

Barthel-Brasselet-Fiesler-Kaup: "direct" approach to proof of Bressler-Lunts, replacing derived categories with elementary sheaf theory and commutative algebra.