RECENT PROGRESS IN ALGEBRAIC COMBINATORICS

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The Saturation Conjecture

W. Fulton, *Bull. Amer. Math. Soc.* 37 (2000), 209–249.

Define the group homomorphism (polynomial representation)

$$\varphi : \mathrm{GL}(2,\mathbb{C}) \rightarrow \mathrm{GL}(3,\mathbb{C})$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} a^2 & 2ab & b^2 \\ ac & ad+bc & bd \\ c^2 & 2cd & d^2 \end{bmatrix}$$

If $A \in \mathrm{GL}(2,\mathbb{C})$ has eigenvalues x, y, then $\varphi(A)$ has eigenvalues x^2, xy, y^2.

$$\begin{aligned} \mathrm{char}(\varphi) &= \mathrm{tr}\,\varphi(A) \\ &= x^2 + xy + y^2 \end{aligned}$$

The irreducible polynomial representations φ_λ of $\mathrm{GL}(n, \mathbb{C})$ are indexed by **partitions**

$$\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n,$$

$$\lambda_1 \geq \cdots \geq \lambda_n \geq 0.$$

$$\begin{aligned} \mathbf{char}(\varphi_\lambda) &:= \operatorname{tr} \varphi_\lambda(A) \\ &= s_\lambda(x_1, \ldots, x_n), \end{aligned}$$

the **Schur function** indexed by λ, where A has eigenvalues $x_1, \ldots, x_n$.

Let $n >> 0$ and

$$s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda$$

$$c^\lambda_{\mu\nu} = \mathrm{mult}(\varphi_\lambda, \varphi_\mu \otimes \varphi_\nu),$$

a **Littlewood-Richardson coefficient**.

Littlewood-Richardson rule:

$$c^{654}_{432,321} = 2$$

				1	1
			1	2	
		2	3		

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				1	1
			2	2	
		1	3		

112231

rows weakly increasing
columns strictly increasing

lattice permutation: any initial sequence has at least as many i's as $i+1$'s.

Saturation conjecture:

$$c^{m\lambda}_{m\mu,m\nu} \neq 0 \Rightarrow c^{\lambda}_{\mu\nu} \neq 0$$

E.g.,

$$c^{8442}_{442,62} = 3,\ \ c^{4221}_{221,31} = 2.$$

Proved by A. Knutson and T. Tao (1999) using a new combinatorial description of $c^{\lambda}_{\mu\nu}$ (honeycombs and hives). Later proved by H. Derksen and J. Weyman using representations of quivers.

So what?

Let A, B, C be $n \times n$ Hermitian (or real symmetric) matrices. Eigenvalues of A:

$$\alpha \ : \ \alpha_1 \geq \cdots \geq \alpha_n,$$

and similarly β and γ for B and C.

Problem. Characterize those triples (α, β, γ) for which there exist Hermitian (equivalently, real symmetric) matrices $C = A + B$.

Clearly

$$\sum \gamma_i = \sum \alpha_i + \sum \beta_i.$$

A. Horn conjectured additional necessary and sufficient linear inequalities of the form

$$\sum_{k \in K} \gamma_k \leq \sum_{i \in I} \alpha_i + \sum_{j \in J} \beta_j,$$

$$I, J, K \subset \{1, \ldots, n\}$$

$$|I| = |J| = |K|$$

E.g., $n = 2$:

$$\begin{aligned} \gamma_1 &\le \alpha_1 + \beta_1 \\ \gamma_2 &\le \alpha_2 + \beta_1 \\ \gamma_2 &\le \alpha_1 + \beta_2 \end{aligned}$$

$n = 3$: 12 inequalities

Theorem (Klyachko). *Saturation conjecture* $\Rightarrow$ *Horn's conjecture.*

Theorem (Heckman, Klyachko). *Saturation conjecture implies:* $c^{\gamma}_{\alpha\beta} \neq 0 \Leftrightarrow \exists$ $n \times n$ *Hermitian matrices* $A + B = C$ *with eigenvalues* α, β, γ.

($c^{\gamma}_{\alpha\beta} \neq 0$ determined by linear inequalities)

Let G be an abelian p-group of type λ, i.e., $G \cong$
$$\left(\mathbb{Z}/p^{\lambda_1}\mathbb{Z}\right) \times \left(\mathbb{Z}/p^{\lambda_2}\mathbb{Z}\right) \times \cdots.$$
Let $g^{\lambda}_{\mu\nu}(p)$ be the number of subgroups H of G of type μ such that G/H has type ν.

Theorem (P. Hall, M. Maley)

$$g^{\lambda}_{\mu\nu}(t) \in \mathbb{Z}[t],$$

and for any prime p,

$$g^{\lambda}_{\mu\nu}(p) \neq 0 \Leftrightarrow c^{\lambda}_{\mu\nu} \neq 0.$$

II. The $n!$ and $(n+1)^{n-1}$ Conjectures

math.ucsd.edu/∼mhaiman

$\mathfrak{S}_n$ **acts on** $R = \mathbb{C}[x_1, \ldots, x_n]$ **by permuting variables.**

Invariants:

$$R^{\mathfrak{S}_n} = \mathbb{C}[e_1, \ldots, e_n]$$

$$e_k = \sum_{i_1 < \cdots < i_k} x_{i_1} \cdots x_{i_k}$$

$$\begin{aligned}\Delta &= R/(e_1, \ldots, e_n) \\ &= \Delta_0 \oplus \Delta_1 \oplus \cdots\end{aligned}$$

$$\dim \Delta = n!$$

$$\sum_i \dim(\Delta_i)\, q^i =$$

$$(1+q)(1+q+q^2)\cdots$$
$$(1+q+\cdots+q^{n-1})$$

Δ affords the regular representation of $\mathfrak{S}_n$.

Irreducible representations M_λ of $\mathfrak{S}_n$ indexed by partitions λ of n $(\sum \lambda_i = n)$.

$$\dim M_\lambda = f^\lambda,$$

the number of standard Young tableaux of shape λ.

E.g., $n = 5$, $\lambda = (3, 2) \vdash 5$:

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2}\end{array}$$

$$\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2}\end{array} \qquad f^{(3,2)} = 5$$

$$\mathbf{MAJ}(T) = \sum_{\substack{i+1 \text{ below } i \\ \text{in } T}} i$$

1	2	5	9
3	7	8	11
4	10	12	
6			

$$\begin{aligned} \mathbf{MAJ}(T) &= 2+3+5+9+11 \\ &= 30 \end{aligned}$$

Recall

$$\Delta = R/(e_1, \ldots, e_n).$$

Theorem (Lusztig, RS).

$$\mathrm{mult}\,(M_\lambda, \Delta_i) =$$

$$\#\{\text{SYT } T \text{ of shape } \lambda,$$

$$\mathrm{MAJ}(T) = i\}$$

Example. $n = 5$

$$\Delta_3 \cong M_{41} \oplus M_{32} \oplus M_{311}$$

$$\begin{array}{lll} 1\,2\,3\,5 & 1\,2\,3 & 1\,4\,5 \\ 4 & 4\,5 & 2 \\ & & 3 \end{array}$$

$$\begin{aligned} \dim \Delta_3 &= \dim M_{41} \\ &\quad + \dim M_{32} + \dim M_{311} \\ &= 4 + 5 + 6 \\ &= 15 \end{aligned}$$

Another description of Δ.

If $P(x_1, \ldots, x_n)$ is a polynomial, define

$$\partial P = \mathrm{span}_{\mathbb{C}}\{P \text{ and all partial derivatives}\}.$$

Let

$$V = \prod_{1 \leq i < j \leq n} (x_i - x_j).$$

Then

$$\Delta \cong \partial V,$$

as graded $\mathfrak{S}_n$-modules.

A. Garsia, M. Haiman: Generalize to **two** sets of variables

$$\boldsymbol{x} = (x_1, \ldots, x_n)$$
$$\boldsymbol{y} = (y_1, \ldots, y_n).$$

1. $\mathfrak{S}_n$ acts **diagonally** on $\boldsymbol{S} = \mathbb{C}[x, y]$, i.e.,

$$w \cdot x_i = x_{w(i)}, \quad w \cdot y_i = y_{w(i)}.$$

$$S^{\mathfrak{S}_n} = \{f \in S : w \cdot f = f \quad \forall w \in \mathfrak{S}_n\}$$

I = ideal of S generated by all $f \in S^{\mathfrak{S}_n}$ with 0 constant term

$$\Delta^{(2)} = S/I$$

$(n+1)^{n-1}$-conjecture (Garsia-Haiman):

$$\dim \Delta^{(2)} = (n+1)^{n-1}.$$

$\Delta^{(2)}$ is a bigraded (by x and y degree) $\mathfrak{S}_n$-module. E.g., $n=4$, $\Delta^{(2)}_{2,1} \cong$

$$2M_{211} \oplus M_{22} \oplus M_{31}.$$

A (complicated) conjecture for $\operatorname{mult}\left(M_\lambda, \Delta^{(2)}_{r,s}\right)$ was given by Haiman.

In particular, let

$$\Gamma = \{f \in \Delta^{(2)} :$$
$$w \cdot f = (\operatorname{sgn} w) f \;\; \forall w \in \mathfrak{S}_n\}.$$

Then

$$\dim \Gamma = \frac{1}{n+1}\binom{2n}{n},$$

a Catalan number.

(For 66 combinatorial interpretations of the Catalan numbers, see *Enumerative Combinatorics*, vol. 2, Exercise 6.19.)

$(n+1)^{n-1}$-conjecture and its refinements proved by M. Haiman (July, 2000).

2. Let $\mu = (3, 2) \vdash 5$.

0,0	1,0	2,0
0,1	1,1	

$$D_\mu = \begin{vmatrix} 1 & x_1 & x_1^2 & y_1 & x_1y_1 \\ 1 & x_2 & x_2^2 & y_2 & x_2y_2 \\ 1 & x_3 & x_3^2 & y_3 & x_3y_3 \\ 1 & x_4 & x_4^2 & y_4 & x_4y_4 \\ 1 & x_5 & x_5^2 & y_5 & x_5y_5 \end{vmatrix}$$

The $n!$ conjecture
(Garsia-Haiman)

$$\dim \partial D_\mu = n!$$

More generally, D_μ affords the regular representation of $\mathfrak{S}_n$.

In fact D_μ is a bigraded (by x and y degree) $\mathfrak{S}_n$-module. A (complicated) conjecture for

$$\text{mult}\left(M_\lambda, (D_\mu)_{r,s}\right)$$

was given by Haiman and shown to be equivalent to the $n!$-conjecture.

$n!$-conjecture (and its refinements) proved by M. Haiman (July, 2000).

Coefficient of

$$x^{\mu} = x_1^{\mu_1} x_2^{\mu_2} \cdots$$

in the Schur function s_λ is the Kostka number $K_{\lambda\mu}$ (nice combinatorial interpretation in terms of "semistandard Young tableaux").

$K_{\lambda\mu}(q,t)$: two-parameter generalization of $K_{\lambda\mu}$ conjectured to be a polynomial with nonnegative integer coefficients (I. G. Macdonald, 1988).

$$K_{\lambda\mu}(0,1) = K_{\lambda\mu}$$

Haiman:

$$t^{n(\mu)}K_{\lambda\mu}(q,1/t) =$$

$$\sum_{r,s} t^r q^s \operatorname{mult}\left(M_\lambda, (D_\mu)_{r,s}\right)$$

Proof based on the geometry of $\mathrm{Hilb}^n(\mathbb{C}^2)$, the Hilbert scheme of n points in the plane (Procesi).

$$\mathrm{Hilb}^n(\mathbb{C}^2) = \{I \subseteq \mathbb{C}[x,y] : \dim \mathbb{C}[x,y]/I = n\}$$

Let $X = \{z_1, \ldots, z_n\}$ be n distinct points in $\mathbb{C}^2$. Then

$$I_X = \{f \in \mathbb{C}[x,y] : f(z_1) = \cdots = f(z_n) = 0\} \in \mathrm{Hilb}^n(\mathbb{C}^2).$$

$\mathrm{Hilb}^n(\mathbb{C}^2)$ has the structure of a smooth irreducible algebraic variety, of dimension $2n$.

What about more than two sets of variables? Obvious conjectures are false.

One difficulty: $\mathrm{Hilb}^n(\mathbb{C}^k)$ isn't smooth for $k > 2$.

III. Longest Increasing Subsequences

P. Deift, *Notices Amer. Math. Soc.* 47 (2000), 631–640.

Let $w = a_1 a_2 \cdots a_n \in \mathfrak{S}_n$.

increasing subsequence of w:

$$a_{i_1} a_{i_2} \cdots a_{i_k}$$

$$i_1 < i_2 < \cdots < i_k$$

$$a_{i_1} < a_{i_2} < \cdots < a_{i_k}$$

Let $\mathrm{is}_n(w)$ = length of longest increasing subsequence of $w \in \mathfrak{S}_n$.

$w = 627419583 \Rightarrow \mathrm{is}_9(w) = 4$

$$E(n) = \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \mathrm{is}_n(w)$$

(expected value of $\mathrm{is}_n(w)$).

Elementary:

$$\tfrac{1}{2}\sqrt{n} \leq E(n) \leq e\sqrt{n}.$$

Hammersley (1972):

$$c = \lim_{n\to\infty} \frac{E(n)}{\sqrt{n}} \text{ exists.}$$

Recall: f^{λ} = #SYT of shape λ.

Logan-Shepp, Vershik-Kerov (1977): $c = 2$.

Proof based on:

$$E(n) = \frac{1}{n!} \sum_{\lambda \vdash n} \lambda_1 \left(f^{\lambda}\right)^2,$$

where $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$.

What about stronger results on the behavior of $\mathrm{is}_n(w)$?

Main results (Baik-Deift-Johansson). Let $\mathrm{Ai}(x)$ denote the Airy function:

$$\frac{d^2}{dx^2}\mathrm{Ai}(x) = x\,\mathrm{Ai}(x)$$

$$\mathrm{Ai}(x) \sim \frac{e^{-\frac{2}{3}x^{3/2}}}{2\sqrt{\pi}x^{1/4}}\ (x \to \infty)$$

Define $u(x)$ by

$$\frac{d^2}{dx^2}u(x) = 2u(x)^3 + xu(x) \quad (*)$$
$$u(x) \sim -\mathrm{Ai}(x) \ (x \to \infty).$$

$(*)$ is the **Painlevé II** equation (roughly, the branch points and essential singularities are independent of the initial conditions).

Paul Painlevé

1863: born in Paris.

1890: Grand Prix des Sciences Mathématiques

1908: first passenger of Wilbur Wright; set flight duration record of one hour, 10 minutes.

1917, 1925: Prime Minister of France.

1933: died in Paris.

Tracy-Widom distribution:

$$F(t)$$

$$= \exp\left(-\int_t^\infty (x-t)u(x)^2\,dx\right)$$

Let χ be a random variable with distribution F, and let χ_n be the random variable on $\mathfrak{S}_n$:

$$\chi_n(w) = \frac{\mathrm{is}_n(w) - 2\sqrt{n}}{n^{1/6}}.$$

Theorem. *As* $n \to \infty$,

$\chi_n \to \chi$ **in distribution,**

i.e.,

$$\lim_{n\to\infty} \text{Prob}(\chi_n \le t) = F(t).$$

Theorem. *For any $m = 0, 1, 2, \ldots,$*

$$\lim_{n\to\infty} E(\chi_n^m) = E(\chi^m).$$

Corollary. $\displaystyle\lim_{n\to\infty} \frac{\mathrm{Var}(\mathrm{is}_n)}{n^{1/3}}$

$$= \int t^2\, dF(t) - \left(\int t\, dF(t)\right)^2$$

$$= 0.8132\cdots$$

$$\lim_{n\to\infty} \frac{E(\mathrm{is}_n) - 2\sqrt{n}}{n^{1/6}}$$

$$= \int t\, dF(t)$$

$$= -1.7711\cdots$$

A word on the proof.

Theorem (Gessel 1990, et al.) *Let* $u_k(n) =$

$\#\{w \in \mathfrak{S}_n : \mathrm{is}_n(w) \le k\}$

$$U_k = \sum_{n \ge 0} u_k(n) \frac{x^{2n}}{n!^2}$$

$$b_i = \sum_{n \ge 0} \frac{x^{2n+i}}{n!\,(n+i)!}.$$

Then

$$U_k = \det\left(b_{|i-j|}\right)_{i,j=1}^{k}.$$

Example.

$$\begin{aligned} U_2 &= b_0^2 - b_1^2 \\ &= \sum_{n\geq 0} \frac{1}{n+1}\binom{2n}{n}\frac{x^{2n}}{n!^2} \end{aligned}$$

$$\Rightarrow u_2(n) = \frac{1}{n+1}\binom{2n}{n},$$

a **Catalan number** (Hammersley 1972, D. G. Rogers 1978).

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Gessel's theorem reduces the problem to "just" analysis, viz., the Riemann-Hilbert problem in the theory of integrable systems, and the method of steepest descent to analyze the asymptotic behavior of integrable systems.

Where did the Tracy-Widom distribution come from?

$$\boldsymbol{F(t)} \\ = \exp\left(-\int_t^\infty (x-t)u(x)^2\,dx\right)$$

Gaussian Unitary Ensemble (GUE): Consider an $n \times n$ hermitian matrix $M = (M_{ij})$ with probability distribution

$$Z_n^{-1} e^{-\mathrm{tr}(M^2)} dM,$$

$$dM = \prod_i dM_{ii}$$

$$\cdot \prod_{i<j} d(\Re M_{ij}) d(\Im M_{ij}),$$

where Z_n is a normalization constant.

Let α_1 denote the largest eigenvalue of M. Then

$$\lim_{n\to\infty} \mathrm{Prob}\left(\left(\alpha_1 - \sqrt{2n}\right)\sqrt{2}n^{1/6} \le t\right) = F(t).$$

More generally, let

$$\lambda_1 + \cdots + \lambda_k$$

be the largest number of elements in the union of k increasing subsequences of $w \in \mathfrak{S}_n$, and let α_k be the kth largest eigenvalue of the GUE matrix M. Then as $n \to \infty$, λ_k and α_k are equidistributed, up to scaling (Okounkov, Borodin-Okounkov-Olshanski, Johansson).

Is the connection between is(w) and GUE a coincidence?

The proof of Okounkov provides a connection, *via* the theory of random topologies on surfaces.