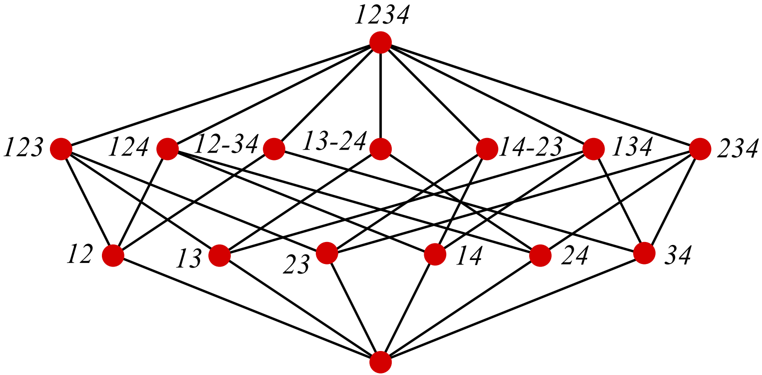
Some of My Favorite Posets

Richard P. Stanley
U. Miami & M.I.T.

September 4, 2023

The partition lattice

Arose from a computation of R. McEliece around 1967 which included a formula whose proof used the Möbius function of Π_n, the lattice of partitions of $[n] = \{1, 2 \ldots, n\}$ ordered by refinement.



Properties

- **Möbius function**: $\mu(\hat{0}, \hat{1}) = (-1)^{n-1}(n-1)!$

Properties

- **Möbius function**: $\mu(\hat{0}, \hat{1}) = (-1)^{n-1}(n-1)!$
- **Characteristic polynomial**:

$$\begin{aligned} \chi_{\Pi_n}(q) &= \sum_{t \in \Pi_n} \mu(\hat{0}, t) q^{n-1-\mathrm{rank}(t)} \\ &= (q-1)(q-2)\cdots(q-n+1) \end{aligned}$$

Properties

- **Möbius function**: $\mu(\hat{0}, \hat{1}) = (-1)^{n-1}(n-1)!$
- **Characteristic polynomial**:

$$\begin{aligned} \chi_{\Pi_n}(q) &= \sum_{t \in \Pi_n} \mu(\hat{0}, t) q^{n-1-\mathrm{rank}(t)} \\ &= (q-1)(q-2)\cdots(q-n+1) \end{aligned}$$

- Π_n is the intersection lattice of the braid arrangement $\mathcal{B}_n$.

Properties

- **Möbius function**: $\mu(\hat{0},\hat{1}) = (-1)^{n-1}(n-1)!$
- **Characteristic polynomial**:

$$\begin{aligned}\chi_{\Pi_n}(q) &= \sum_{t\in\Pi_n} \mu(\hat{0},t)q^{n-1-\mathrm{rank}(t)} \\ &= (q-1)(q-2)\cdots(q-n+1)\end{aligned}$$

- Π_n is the intersection lattice of the braid arrangement $\mathcal{B}_n$.
- $\mathcal{B}_n$ is the graphical arrangement of the complete graph K_n, so $q\cdot\chi_{\Pi_n}(q)$ is the chromatic polynomial of K_n.

Properties

- **Möbius function**: $\mu(\hat{0},\hat{1}) = (-1)^{n-1}(n-1)!$
- **Characteristic polynomial**:

$$\begin{aligned} \chi_{\Pi_n}(q) &= \sum_{t\in\Pi_n} \mu(\hat{0},t) q^{n-1-\mathrm{rank}(t)} \\ &= (q-1)(q-2)\cdots(q-n+1) \end{aligned}$$

- Π_n is the intersection lattice of the braid arrangement $\mathcal{B}_n$.
- $\mathcal{B}_n$ is the graphical arrangement of the complete graph K_n, so $q\cdot\chi_{\Pi_n}(q)$ is the chromatic polynomial of K_n.
- Π_n is **supersolvable** (explains factorization of $\chi_{\Pi_n}(q)$ lattice-theoretically).

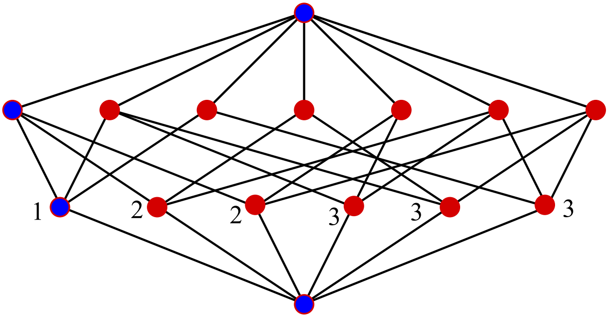
Properties

- **Möbius function**: $\mu(\hat{0},\hat{1}) = (-1)^{n-1}(n-1)!$
- **Characteristic polynomial**:

$$\chi_{\Pi_n}(q) = \sum_{t\in\Pi_n} \mu(\hat{0},t)q^{n-1-\mathrm{rank}(t)} = (q-1)(q-2)\cdots(q-n+1)$$

- Π_n is the intersection lattice of the braid arrangement $\mathcal{B}_n$.
- $\mathcal{B}_n$ is the graphical arrangement of the complete graph K_n, so $q\cdot\chi_{\Pi_n}(q)$ is the chromatic polynomial of K_n.
- Π_n is **supersolvable** (explains factorization of $\chi_{\Pi_n}(q)$ lattice-theoretically).
- $\mathfrak{S}_n$ acts on the top homology of the order complex $\Delta(\Pi_n - \{\hat{0},\hat{1}\})$. Action is isomorphic to sign twist of the action of $\mathfrak{S}_n$ on the multilinear part of the free Lie algebra $\mathrm{Lie}(n)$.

Supersolvability of Π_n

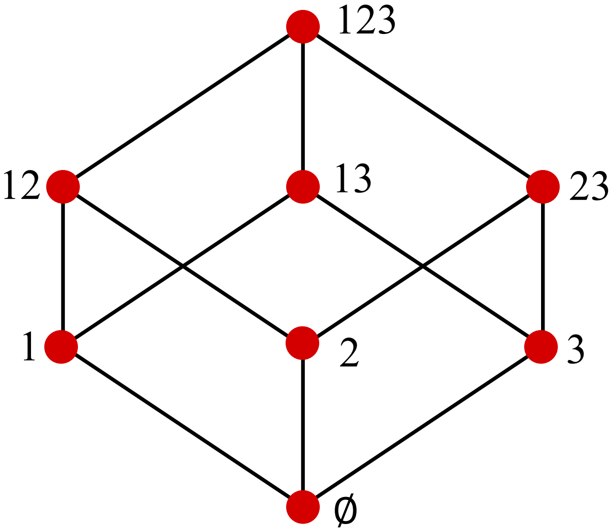


blue vertices: **modular maximal chain** (together with any chain, generates a distributive lattice)

Π_n is (upper) **semimodular**

$\Rightarrow \chi_{\Pi_4}(q) = (q-1)(q-2)(q-3)$

Boolean algebra B_n



Flag f-vectors and h-vectors

P: graded poset of rank n with $\hat{0}$ and $\hat{1}$ and with rank function ρ

$S \subseteq [n-1] := \{1, 2, \ldots, n-1\}$

flag f-vector:

$$\alpha_P(S) = \#\{\hat{0} < t_1 < \cdots < t_k < \hat{1} \, : \, S = \{\rho(t_1), \ldots, \rho(t_k)\}\}$$

flag h-vector: $\beta_P(S) = \sum_{T \subseteq S} (-1)^{\#(S-T)} \alpha_P(T)$

Flag f-vectors and h-vectors

P: graded poset of rank n with $\hat{0}$ and $\hat{1}$ and with rank function ρ

$S \subseteq [n-1] := \{1, 2, \ldots, n-1\}$

flag f-vector:

$$\alpha_P(S) = \#\{\hat{0} < t_1 < \cdots < t_k < \hat{1} : S = \{\rho(t_1), \ldots, \rho(t_k)\}\}$$

flag h-vector: $\beta_P(S) = \sum_{T \subseteq S} (-1)^{\#(S-T)} \alpha_P(T)$

$$\Rightarrow \alpha_P(S) = \sum_{T \subseteq S} \beta_P(T)$$

Flag vectors for B_4

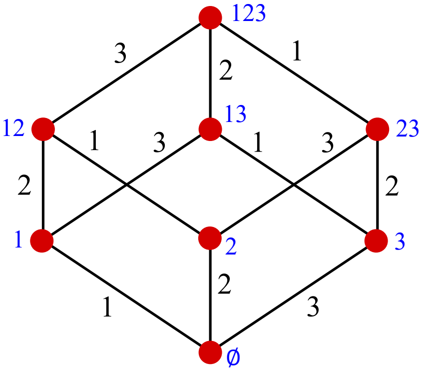
S	$\alpha_{B_4}(S)$	$\beta_{B_4}(S)$
$\varnothing$	1	1
1	4	3
2	6	5
3	4	3
1,2	12	3
1,3	12	5
2,3	12	3
1,2,3	24	1

Edge labelling of B_n

Label the Hasse diagram edge $(S, S \cup i)$ with i.

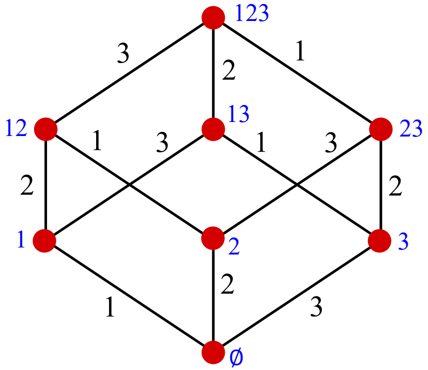
Edge labelling of B_n

Label the Hasse diagram edge $(S, S \cup i)$ with i.



Edge labelling of B_n

Label the Hasse diagram edge $(S, S \cup i)$ with i.



Key property: every interval $[s, t]$ has a unique weakly increasing saturated chain from s to t.

Consequence

Theorem. *Let* $S \subseteq [n-1]$*. Then* $\beta_{B_n}(S)$ *is equal to the number of maximal chains of* B_n *whose labels* $w = (a_1, a_2, \ldots, a_n)$ *(from bottom to top) have* descent set S*, i.e.,*

$$S = \mathrm{Des}(\alpha) := \{i \,:\, a_i > a_{i+1}\}.$$

Consequence

Theorem. *Let $S \subseteq [n-1]$. Then $\beta_{B_n}(S)$ is equal to the number of maximal chains of B_n whose labels $w = (a_1, a_2, \ldots, a_n)$ (from bottom to top) have* descent set *S, i.e.,*

$$S = \mathrm{Des}(\alpha) := \{i : a_i > a_{i+1}\}.$$

Since the labels of the $n!$ maximal chains of B_n are just the elements of $\mathfrak{S}_n$, we get:

Corollary. $\beta_{B_n}(S) = \#\{w \in \mathfrak{S}_n : \mathrm{Des}(w) = S\}$

Consequence

Theorem. *Let* $S \subseteq [n-1]$. *Then* $\beta_{B_n}(S)$ *is equal to the number of maximal chains of* B_n *whose labels* $w = (a_1, a_2, \ldots, a_n)$ *(from bottom to top) have* descent set S, *i.e.,*

$$S = \mathrm{Des}(\alpha) := \{i \,:\, a_i > a_{i+1}\}.$$

Since the labels of the $n!$ maximal chains of B_n are just the elements of $\mathfrak{S}_n$, we get:

Corollary. $\beta_{B_n}(S) = \#\{w \in \mathfrak{S}_n \,:\, \mathrm{Des}(w) = S\}$

The first glimpse of the theory of flag vectors, edge labellings, lexicographic shellability, topological combinatorics, ...

The Sperner property

$\boldsymbol{P} = P_0 \dot\cup P_1 \dot\cup \cdots \dot\cup P_n$: finite graded poset of rank n

antichain $A \subseteq P$: no two elements of A are comparable

Each P_i is an antichain.

The Sperner property

$\boldsymbol{P} = P_0 \dot\cup P_1 \dot\cup \cdots \dot\cup P_n$: finite graded poset of rank n

antichain $A \subseteq P$: no two elements of A are comparable

Each P_i is an antichain.

Sperner property: $\max_{\text{antichain } A} |A| = \max_i |P_i|$

The Sperner property

P $= P_0 \dot\cup P_1 \dot\cup \cdots \dot\cup P_n$: finite graded poset of rank n

antichain $A \subseteq P$: no two elements of A are comparable

Each P_i is an antichain.

Sperner property: $\max_{\text{antichain } A} |A| = \max_i |P_i|$

Sperner's theorem (1927): B_n has the Sperner property.

The Sperner property

$\boldsymbol{P} = P_0 \dot\cup P_1 \dot\cup \cdots \dot\cup P_n$: finite graded poset of rank n

antichain $A \subseteq P$: no two elements of A are comparable

Each P_i is an antichain.

Sperner property: $\max_{\text{antichain } A} |A| = \max_i |P_i|$

Sperner's theorem (1927): B_n has the Sperner property.

Many proofs, including a **linear algebraic method**.

Linear algebraic method

$\boldsymbol{P} = P_0 \dot\cup P_1 \dot\cup \cdots \dot\cup P_n$: finite graded poset of rank n

$\mathbb{Q}\boldsymbol{P_i}$: $\mathbb{Q}$-vector space with basis P_i

order-raising operator $\boldsymbol{U_i} : \mathbb{Q}P_i \to \mathbb{Q}P_{i+1}$: if $t \in P_i$, then $U(t) \in \mathrm{span}_{\mathbb{Q}}\{u \in P_{i+1} : u > t\}$

Linear algebraic method

$\boldsymbol{P} = P_0 \dot\cup P_1 \dot\cup \cdots \dot\cup P_n$: finite graded poset of rank n

$\mathbb{Q}\boldsymbol{P_i}$: $\mathbb{Q}$-vector space with basis P_i

order-raising operator $\boldsymbol{U_i} : \mathbb{Q}P_i \to \mathbb{Q}P_{i+1}$: if $t \in P_i$, then $U(t) \in \mathrm{span}_{\mathbb{Q}}\{u \in P_{i+1} : u > t\}$

Theorem. *Suppose that for some j, U_i is injective for $i < j$ and surjective for $i > j$. Then P has the Sperner property.*

Definition of U for B_n

For $\#S = j$, define $U(S) = \sum_{\substack{S \subset T \\ \#T = j+1}} T$.

Definition of U for B_n

For $\#S = j$, define $U(S) = \sum_{\substack{S \subset T \\ \#T = j+1}} T$.

This works!

The weak (Bruhat) order

Let $w = a_1 \cdots a_n \in \mathfrak{S}_n$.

inversion set $\boldsymbol{I(w)} := \{(a_j, a_i) : i < j,\ a_i > a_j\}$

length $\ell(w) := \# I(w)$

$\boldsymbol{s_i}$: the **adjacent transposition** $(i, i+1)$ for $1 \le i \le n-1$

The weak (Bruhat) order

Let $w = a_1 \cdots a_n \in \mathfrak{S}_n$.

inversion set $I(w) := \{(a_j, a_i) : i < j,\ a_i > a_j\}$

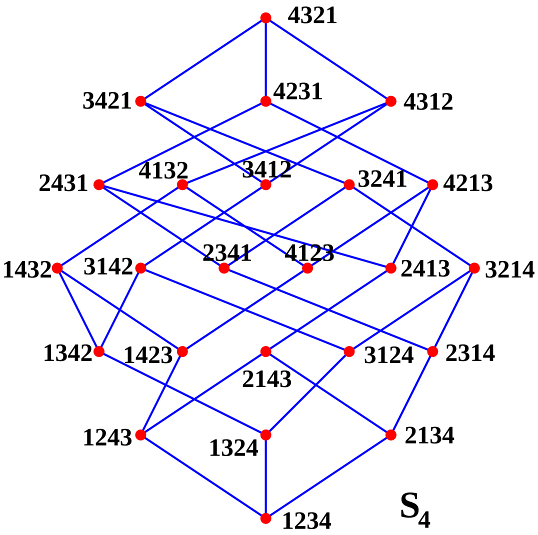
length $\ell(w) := \#I(w)$

s_i: the **adjacent transposition** $(i, i+1)$ for $1 \le i \le n-1$

weak order $W(\mathfrak{S}_n)$ on $\mathfrak{S}_n$: v covers u if for some i, we have $v = s_i u$ and $\ell(v) = \ell(u) + 1$.

Equivalently, $u < v$ if $I(u) \subset I(v)$.

$W(\mathfrak{S}_4)$



Reduced decompositions

Let $w \in \mathfrak{S}_n$, $\ell(w) = p$.

reduced decomposition of w: $(c_1, \dots, c_p)$ such that $w = s_{c_1} \cdots s_{c_p}$

$\boldsymbol{r(w)}$: number of reduced decompositions of w

Reduced decompositions

Let $w \in \mathfrak{S}_n$, $\ell(w) = p$.

reduced decomposition of w: $(c_1, \ldots, c_p)$ such that $w = s_{c_1} \cdots s_{c_p}$

$\boldsymbol{r(w)}$: number of reduced decompositions of w

For $w \in \mathfrak{S}_n$, let $\boldsymbol{e(w)}$ denote the number of saturated chains from $\hat{0} = \mathrm{id}$ to w in $W(\mathfrak{S}_n)$.

Reduced decompositions

Let $w \in \mathfrak{S}_n$, $\ell(w) = p$.

reduced decomposition of w: $(c_1, \ldots, c_p)$ such that $w = s_{c_1} \cdots s_{c_p}$

$\boldsymbol{r(w)}$: number of reduced decompositions of w

For $w \in \mathfrak{S}_n$, let $\boldsymbol{e(w)}$ denote the number of saturated chains from $\hat{0} = \mathrm{id}$ to w in $W(\mathfrak{S}_n)$.

Observation. $e(w) = r(w)$

Reduced decompositions

Let $w \in \mathfrak{S}_n$, $\ell(w) = p$.

reduced decomposition of w: $(c_1, \ldots, c_p)$ such that $w = s_{c_1} \cdots s_{c_p}$

$\boldsymbol{r(w)}$: number of reduced decompositions of w

For $w \in \mathfrak{S}_n$, let $\boldsymbol{e(w)}$ denote the number of saturated chains from $\hat{0} = \mathrm{id}$ to w in $W(\mathfrak{S}_n)$.

Observation. $e(w) = r(w)$

Connects $W(\mathfrak{S}_n)$ with the theory of reduced decompositions, stable Schubert polynomials (aka "Stanley symmetric functions"), quiver varieties, etc.

Reduced decompositions

Let $w \in \mathfrak{S}_n$, $\ell(w) = p$.

reduced decomposition of w: $(c_1, \ldots, c_p)$ such that $w = s_{c_1} \cdots s_{c_p}$

$\boldsymbol{r(w)}$: number of reduced decompositions of w

For $w \in \mathfrak{S}_n$, let $\boldsymbol{e(w)}$ denote the number of saturated chains from $\hat{0} = \mathrm{id}$ to w in $W(\mathfrak{S}_n)$.

Observation. $e(w) = r(w)$

Connects $W(\mathfrak{S}_n)$ with the theory of reduced decompositions, stable Schubert polynomials (aka "Stanley symmetric functions"), quiver varieties, etc.

Theorem. $r(n, n-1, \ldots, 1)) = \frac{\binom{n}{2}!}{1^{n-1}3^{n-2}5^{n-3}\cdots(2n-3)^1}$

The Sperner property for $W(\mathfrak{S}_n)$

Theorem. *$W(\mathfrak{S}_n)$ has the Sperner property.*

The Sperner property for $W(\mathfrak{S}_n)$

Theorem. $W(\mathfrak{S}_n)$ *has the Sperner property.*

Two proofs using the linear algebraic method:

- **C. Gaetz** and **Y. Gao** in 2018, by constructing an $\mathfrak{sl}_2$ representation on $\mathbb{Q}W(\mathfrak{S}_n)$

The Sperner property for $W(\mathfrak{S}_n)$

Theorem. $W(\mathfrak{S}_n)$ *has the Sperner property.*

Two proofs using the linear algebraic method:

- **C. Gaetz** and **Y. Gao** in 2018, by constructing an $\mathfrak{sl}_2$ representation on $\mathbb{Q}W(\mathfrak{S}_n)$
- **Z. Hamaker**, **O. Pechenik**, **D. Speyer**, and **A. Weigandt** in 2018, by looking at a differential operator on Schubert polynomials

Pascal poset

$\mathbb{N} \times \mathbb{N}$, where $\mathbb{N} = \{0 < 1 < 2 < \cdots\}$

Pascal poset

$\mathbb{N} \times \mathbb{N}$, where $\mathbb{N} = \{0 < 1 < 2 < \cdots\}$

An infinite **graded** poset, with rank function $\rho(i,j) = i + j$.

Let $e(i,j)$ be the number of saturated chains from $\hat{0} = (0,0)$ to (i,j). Clearly

$$e(i,j) = \binom{i+j}{i}.$$

Pascal poset

$\mathbb{N} \times \mathbb{N}$, where $\mathbb{N} = \{0 < 1 < 2 < \cdots\}$

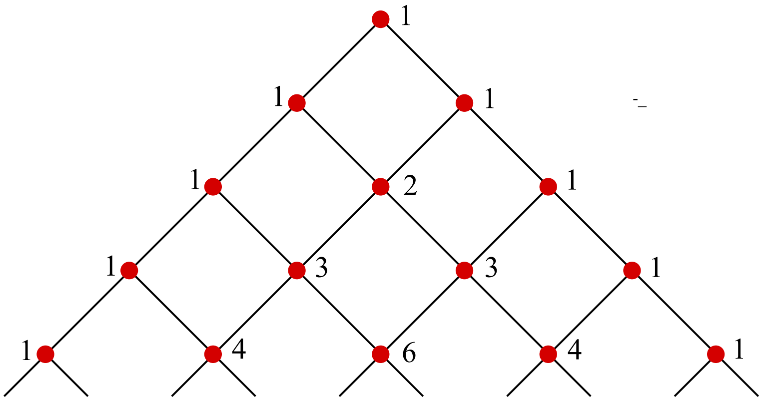
An infinite **graded** poset, with rank function $\rho(i, j) = i + j$.

Let $e(i, j)$ be the number of saturated chains from $\hat{0} = (0, 0)$ to (i, j). Clearly

$$e(i, j) = \binom{i + j}{i}.$$

The poset $\mathbb{N} \times \mathbb{N}$, with the element (i, j) labelled $e(i, j)$, is **Pascal's triangle**.

Pascal's triangle



Properties of $\binom{n}{k}$

$f_r(n) := \sum_{k=0}^{n} \binom{n}{k}^r$

Properties of $\binom{n}{k}$

$f_r(n) := \sum_{k=0}^{n} \binom{n}{k}^r$

- $f_0(n) = n + 1$ (not "spread out")

Properties of $\binom{n}{k}$

$f_r(n) := \sum_{k=0}^{n} \binom{n}{k}^r$

- $f_0(n) = n + 1$ (not "spread out")
- $f_1(n) = 2^n$, $\sum_{n \geq 0} f_1(n) x^n = \frac{1}{1-2x}$ (rational)

Properties of $\binom{n}{k}$

$f_r(n) := \sum_{k=0}^{n} \binom{n}{k}^r$

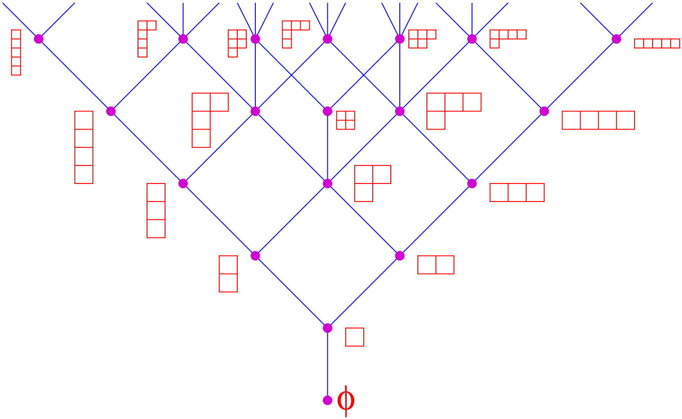
- $f_0(n) = n + 1$ (not "spread out")
- $f_1(n) = 2^n, \ \sum_{n \geq 0} f_1(n) x^n = \frac{1}{1-2x}$ (rational)
- $f_2(n) = \binom{2n}{n}, \ \sum_{n \geq 0} f_2(n) x^n = \frac{1}{\sqrt{1-4x}}$ (algebraic, not rational)

Properties of $\binom{n}{k}$

$f_r(n) := \sum_{k=0}^{n} \binom{n}{k}^r$

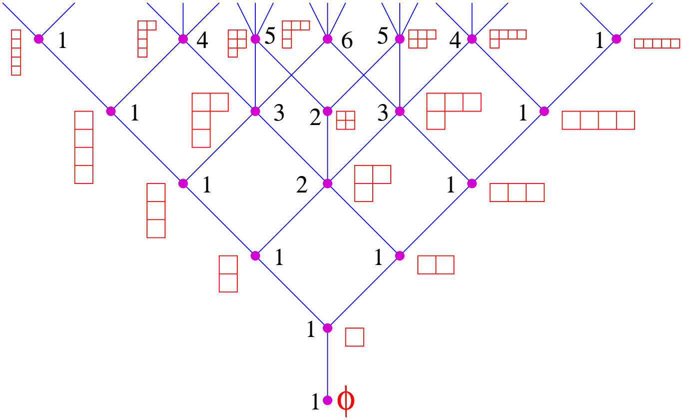
- $f_0(n) = n + 1$ (not "spread out")
- $f_1(n) = 2^n$, $\displaystyle\sum_{n\geq 0} f_1(n)x^n = \frac{1}{1-2x}$ (rational)
- $f_2(n) = \binom{2n}{n}$, $\displaystyle\sum_{n\geq 0} f_2(n)x^n = \frac{1}{\sqrt{1-4x}}$ (algebraic, not rational)
- $f_3(n) = ??$, $\displaystyle\sum_{n\geq 0} f_3(n)x^n$ is D-finite, not algebraic

Young's lattice



Young diagrams (integer partitions), ordered by diagram containment.

Young's lattice



Label λ by the number f^λ of saturated chains from $\hat{0}$ to λ.

Properties of Young's lattice Y

- Y is a **distributive lattice**, the lattice of finite order ideals of $\mathbb{N} \times \mathbb{N}$.

Properties of Young's lattice Y

- Y is a **distributive lattice**, the lattice of finite order ideals of $\mathbb{N} \times \mathbb{N}$.
- $\sum_{\lambda \vdash n} (f^{\lambda})^2 = n!$

Properties of Young's lattice Y

- Y is a **distributive lattice**, the lattice of finite order ideals of $\mathbb{N} \times \mathbb{N}$.
- $\sum_{\lambda \vdash n} (f^\lambda)^2 = n!$
- $\sum_{\lambda \vdash n} f^\lambda = \boldsymbol{t_n} := \#\{w \in \mathfrak{S}_n \,:\, w^2 = 1\}$

$$\sum_{n \geq 0} t_n \frac{x^n}{n!} = \exp\left(x + \frac{x^2}{2}\right)$$

Properties of Young's lattice Y

- Y is a **distributive lattice**, the lattice of finite order ideals of $\mathbb{N} \times \mathbb{N}$.
- $\sum_{\lambda \vdash n} (f^\lambda)^2 = n!$
- $\sum_{\lambda \vdash n} f^\lambda = \boldsymbol{t_n} := \#\{w \in \mathfrak{S}_n \,:\, w^2 = 1\}$

$$\sum_{n \geq 0} t_n \frac{x^n}{n!} = \exp\left(x + \frac{x^2}{2}\right)$$

- Number of paths with $2n$ steps from $\hat{0}$ to $\hat{0}$ (**oscillating tableaux**) is $(2n-1)!! = 1 \cdot 3 \cdot 5 \cdot (2n-1) = \frac{(2n)!}{2^n n!}$.

Properties of Young's lattice Y

- Y is a **distributive lattice**, the lattice of finite order ideals of $\mathbb{N} \times \mathbb{N}$.
- $\sum_{\lambda \vdash n} (f^\lambda)^2 = n!$
- $\sum_{\lambda \vdash n} f^\lambda = \mathbf{t_n} := \#\{w \in \mathfrak{S}_n : w^2 = 1\}$

$$\sum_{n \geq 0} t_n \frac{x^n}{n!} = \exp\left(x + \frac{x^2}{2}\right)$$

- Number of paths with $2n$ steps from $\hat{0}$ to $\hat{0}$ (**oscillating tableaux**) is $(2n-1)!! = 1 \cdot 3 \cdot 5 \cdot (2n-1) = \frac{(2n)!}{2^n n!}$.
- Y is a **differential poset**. Implies previous three properties (and much more).

Another property

Bratteli diagram of a sequence $\mathfrak{A}_0 \subset \mathfrak{A}_1 \subset \cdots$ of finite-dimensional semisimple algebras (over a field): elements t of rank n are indexed by irreps V_t of $\mathfrak{A}_n$. There is an edge weighted m from t of rank $n-1$ to u of rank n if in the restriction of V_u to $\mathfrak{A}_{n-1}$, V_t has multiplicity m.

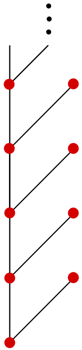
Another property

Bratteli diagram of a sequence $\mathfrak{A}_0 \subset \mathfrak{A}_1 \subset \cdots$ of finite-dimensional semisimple algebras (over a field): elements t of rank n are indexed by irreps V_t of $\mathfrak{A}_n$. There is an edge weighted m from t of rank $n-1$ to u of rank n if in the restriction of V_u to $\mathfrak{A}_{n-1}$, V_t has multiplicity m.

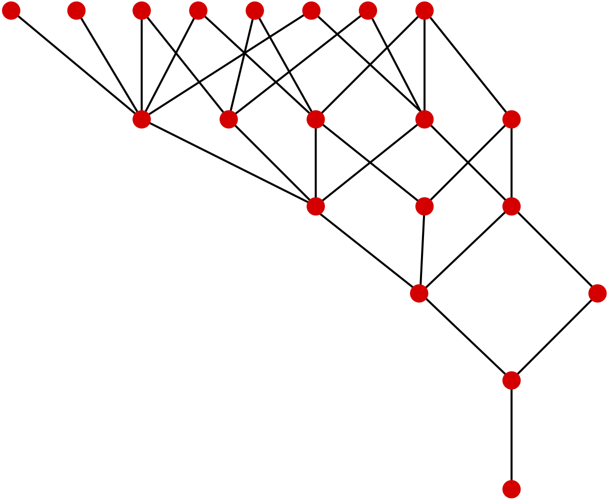
- Y is the **Bratteli diagram** of the sequence $\mathbb{Q}\mathfrak{S}_0 \subset \mathbb{Q}\mathfrak{S}_1 \subset \mathbb{Q}\mathfrak{S}_2 \subset \cdots$ (obvious embeddings). Implies that for $\lambda \vdash n$, f^λ is the dimension of an irreducible representation of $\mathfrak{S}_n$.

Fibonacci fun

Fibonacci distributive lattice FDL: lattice of finite order ideals of the **comb**:



The Fibonacci distributive lattice FDL

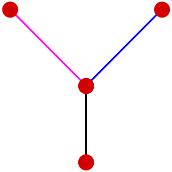


The Fibonacci differential poset Z

Reflection-extension construction:

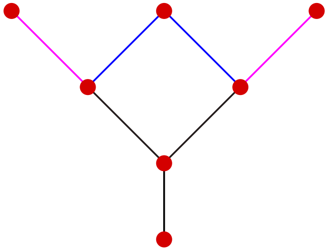
The Fibonacci differential poset Z

Reflection-extension construction:



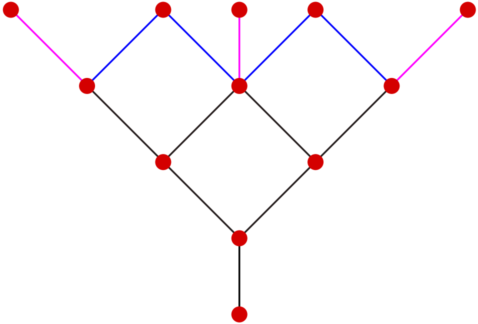
The Fibonacci differential poset Z

Reflection-extension construction:



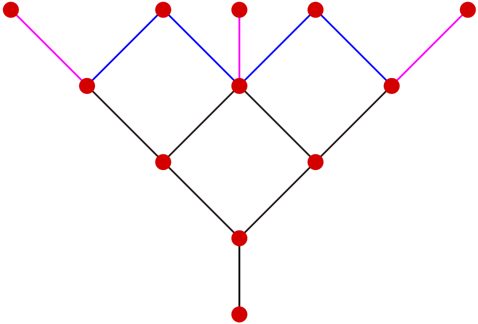
The Fibonacci differential poset Z

Reflection-extension construction:



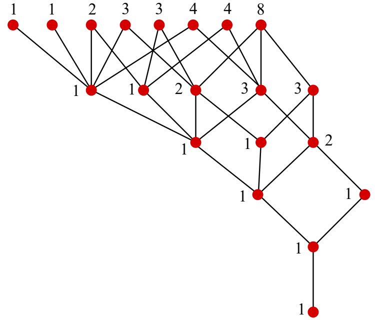
The Fibonacci differential poset Z

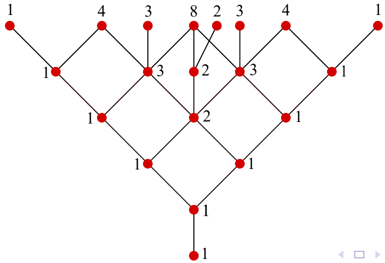
Reflection-extension construction:



Z is the Bratteli diagram of the **Okada algebras** $\mathcal{O}_0 \subset \mathcal{O}_1 \subset \cdots$. Is FDL the Bratteli diagram of a **nice** sequence of algebras?

The numbers $e(t)$ for FDL and Z





Further properties

- There is a "nice" rank-preserving bijection $\varphi: \mathrm{FDL} \to Z$ such that $e(t) = e(\varphi(t))$.

Further properties

- There is a "nice" rank-preserving bijection $\varphi\colon \mathrm{FDL} \to Z$ such that $e(t) = e(\varphi(t))$.
- $\displaystyle\sum_{\substack{t \in Z \\ \mathrm{rank}(t) = n}} e(t)^2 = n!$

Further properties

- There is a “nice” rank-preserving bijection $\varphi\colon \mathrm{FDL} \to Z$ such that $e(t) = e(\varphi(t))$.
- $$\sum_{\substack{t \in Z \\ \mathrm{rank}(t)=n}} e(t)^2 = n!$$
- $$\sum_{\substack{t \in Z \\ \mathrm{rank}(t)=n}} e(t) = \#\{w \in \mathfrak{S}_n : w^2 = 1\}$$

Further properties

- There is a "nice" rank-preserving bijection $\varphi\colon \mathrm{FDL} \to Z$ such that $e(t) = e(\varphi(t))$.
- $$\sum_{\substack{t \in Z \\ \mathrm{rank}(t)=n}} e(t)^2 = n!$$
- $$\sum_{\substack{t \in Z \\ \mathrm{rank}(t)=n}} e(t) = \#\{w \in \mathfrak{S}_n \,:\, w^2 = 1\}$$
- $\forall t \in Z\ \forall i$ number of chains (or multichains) of length i in the interval $[\hat{0}, t]$ of Z equals number of chains (or multichains) of length i in the interval $[\hat{0}, t]$ of FDL. (Proof is inelegant and nonconceptual.)

The final slide

The final slide

ENCORE

ENCORE

Further Fibonacci Fun

The posets P_{ib}

Let $i, b \geq 2$. Define the poset (partially ordered set) $\boldsymbol{P_{ib}}$ by

- There is a unique minimal element $\hat{0}$

The posets P_{ib}

Let $i, b \geq 2$. Define the poset (partially ordered set) $\boldsymbol{P_{ib}}$ by

- There is a unique minimal element $\hat{0}$
- Each element is covered by exactly i elements.

The posets P_{ib}

Let $i, b \geq 2$. Define the poset (partially ordered set)$\boldsymbol{P_{ib}}$ by

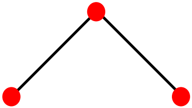
- There is a unique minimal element $\hat{0}$
- Each element is covered by exactly i elements.
- The Hasse diagram is planar. We draw the Hasse diagram upside-down (with $\hat{0}$ at the top).

The posets P_{ib}

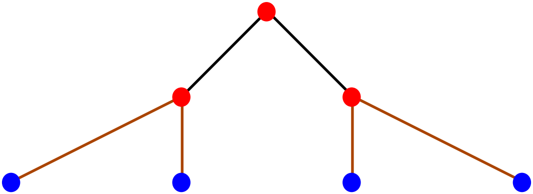
Let $i, b \geq 2$. Define the poset (partially ordered set) $\boldsymbol{P_{ib}}$ by

- There is a unique minimal element $\hat{0}$
- Each element is covered by exactly i elements.
- The Hasse diagram is planar. We draw the Hasse diagram upside-down (with $\hat{0}$ at the top).
- Every  extends to a $2b$-gon (b edges on each side)

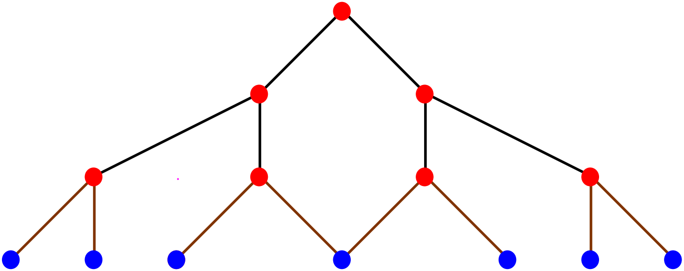
Construction of $\mathcal{F} := P_{23}$



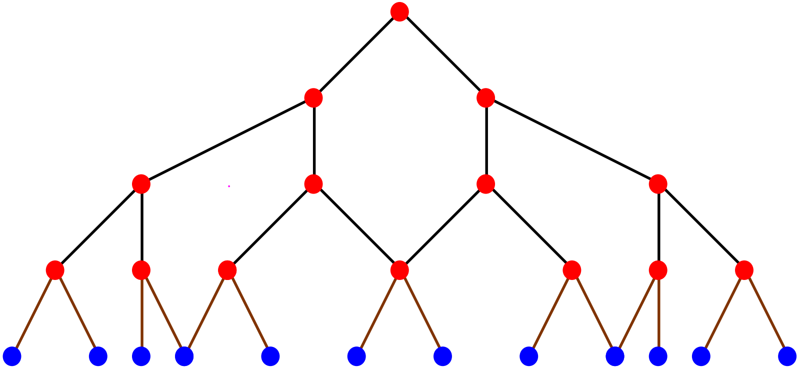
Construction of $\mathcal{F} := P_{23}$



Construction of $\mathcal{F} := P_{23}$



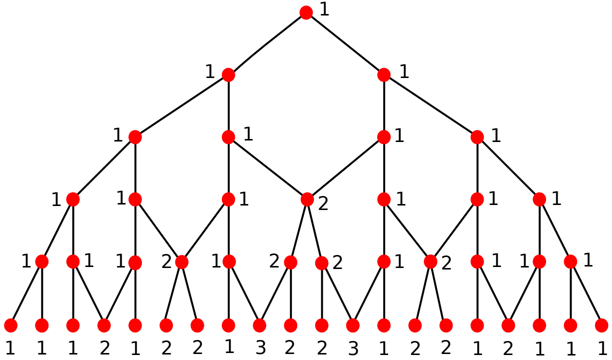
Construction of $\mathcal{F} := P_{23}$



Fibonacci poset

The numbers $\left\langle {n \atop k} \right\rangle$

$\left\langle {n \atop k} \right\rangle$: number of saturated chains from $\hat{0}$ to the kth element of row n, starting with $n = 0$, $k = 0$.



$$\left\langle {5 \atop 0} \right\rangle = \left\langle {5 \atop 1} \right\rangle = \left\langle {5 \atop 2} \right\rangle = 1, \quad \left\langle {5 \atop 2} \right\rangle = 2, \cdots$$

Two theorems

Theorem. $\sum_{k\geq 0} \left\langle {n \atop k} \right\rangle q^k = \prod_{i=1}^{n} \left(1 + q^{F_{i+1}}\right)$

Two theorems

Theorem. $\sum_{k\geq 0} \left\langle {n \atop k} \right\rangle q^k = \prod_{i=1}^{n} \left(1 + q^{F_{i+1}}\right)$

Theorem. *Let* $V_r(x) = \sum_{n\geq 0} \left(\sum_{k\geq 0} \left\langle {n \atop k} \right\rangle^r \right) x^n$. *Then*

$$
\begin{aligned}
V_1(x) &= \frac{1}{1-2x} \text{ (clear)} \\
V_2(x) &= \frac{1-2x^2}{1-2x-2x^2+2x^3} \\
V_3(x) &= \frac{1-4x^2}{1-2x-4x^2+2x^3} \\
V_4(x) &= \frac{1-7x^2-2x^4}{1-2x-7x^2-2x^4+2x^5} \\
V_5(x) &= \frac{1-11x^2-20x^4}{1-2x-11x^2-8x^3-20x^4+10x^5}
\end{aligned}
$$

Open: numerator is "even part" of denominator.

Another open problem

Let $g(n)$ be the number of walks of length $2n$ from $\hat{0}$ to $\hat{0}$ in $\mathfrak{F}$ (Fibonacci analogue of oscillating tableaux).

Another open problem

Let $g(n)$ be the number of walks of length $2n$ from $\hat{0}$ to $\hat{0}$ in $\mathfrak{F}$ (Fibonacci analogue of oscillating tableaux).

$$(g(0), \ldots, g(5)) = (1, 2, 8, 42, 258, 1696, \ldots)$$

Another open problem

Let $g(n)$ be the number of walks of length $2n$ from $\hat{0}$ to $\hat{0}$ in $\mathfrak{F}$ (Fibonacci analogue of oscillating tableaux).

$$(g(0), \dots, g(5)) = (1, 2, 8, 42, 258, 1696, \dots)$$

Is $\sum_{n \geq 0} g(n) x^n$ a rational function?

Another open problem

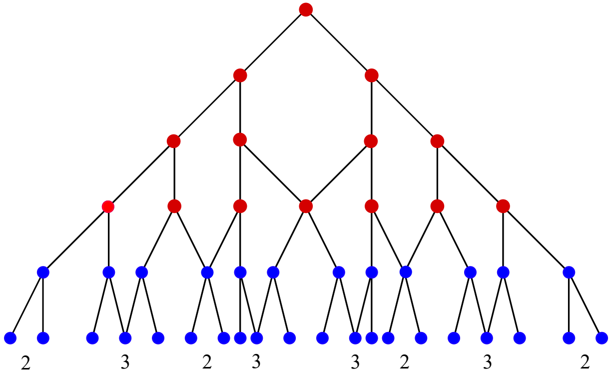
Let $g(n)$ be the number of walks of length $2n$ from $\hat{0}$ to $\hat{0}$ in $\mathfrak{F}$ (Fibonacci analogue of oscillating tableaux).

$$(g(0), \ldots, g(5)) = (1, 2, 8, 42, 258, 1696, \ldots)$$

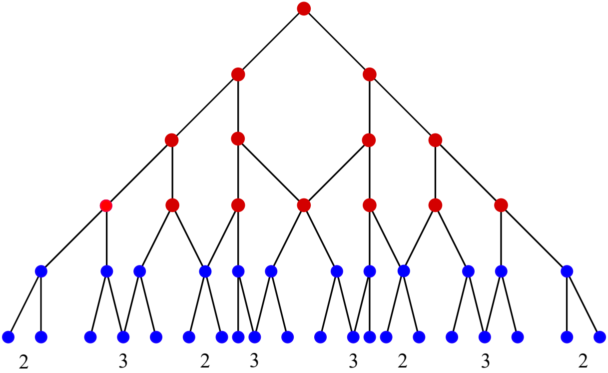
Is $\sum_{n \geq 0} g(n) x^n$ a rational function?

What is $\alpha := \lim_{n \to \infty} g(n)^{1/n}$? If α exists then $5.669 < \alpha < 16$ (very crude bounds).

Two consecutive levels

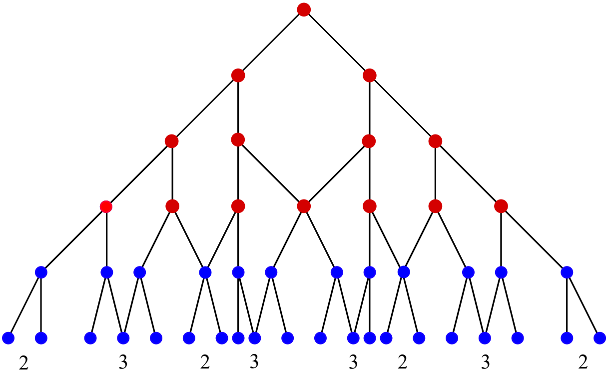


Two consecutive levels



Sequence of components with 2 or 3 minimal elements:
2,3,2,3,3,2,3,2.

Two consecutive levels



Sequence of components with 2 or 3 minimal elements: 2,3,2,3,3,2,3,2.

Approaches a "limiting sequence"

$$(c_1, c_2, \dots) = (2, 3, 2, 3, 3, 2, 3, 2, 3, 3, 2, 3, 3, 2, 3, 2, 3, 3, 2, 3, \dots).$$

Formula for c_n

Let $\phi = (1+\sqrt{5})/2$, the **golden mean**.

Formula for c_n

Let $\phi = (1+\sqrt{5})/2$, the golden mean.

Theorem. *The limiting sequence* $(c_1, c_2, \ldots)$ *is given by*

$$c_n = 1 + \lfloor n\phi \rfloor - \lfloor (n-1)\phi \rfloor.$$

Further properties

- $\gamma = (c_2, c_3, \dots)$ characterized by invariance under $2 \to 3$, $3 \to 32$ (**Fibonacci word** in the letters 2,3).

Further properties

- $\gamma = (c_2, c_3, \dots)$ characterized by invariance under $2 \to 3$, $3 \to 32$ (**Fibonacci word** in the letters 2,3).
- $\gamma = z_1 z_2 \dots$ (concatenation), where $z_1 = 3$, $z_2 = 23$, $z_k = z_{k-2} z_{k-1}$

$$3 \cdot 23 \cdot 323 \cdot 23323 \cdot 32323323 \cdots$$

Further properties

- $\gamma = (c_2, c_3, \ldots)$ characterized by invariance under $2 \to 3$, $3 \to 32$ (**Fibonacci word** in the letters 2,3).
- $\gamma = z_1 z_2 \ldots$ (concatenation), where $z_1 = 3$, $z_2 = 23$, $z_k = z_{k-2} z_{k-1}$

$$3 \cdot 23 \cdot 323 \cdot 23323 \cdot 32323323 \cdots$$

- Sequence of number of 3's between consecutive 2's is the original sequence with 1 subtracted from each term.

$$2 \underbrace{3}_{1} 2 \underbrace{33}_{2} 2 \underbrace{3}_{1} 2 \underbrace{33}_{2} 2 \underbrace{33}_{2} 2 \underbrace{3}_{1} 2 \underbrace{33}_{2} 2 \ldots.$$

An edge labeling of $\mathfrak{F}$

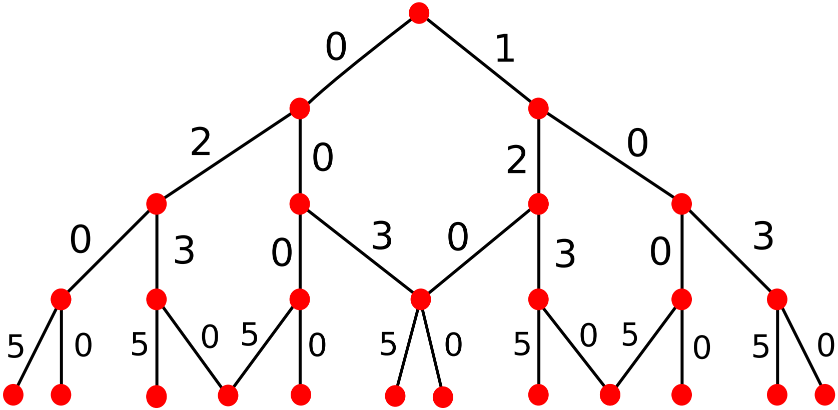
The edges between ranks $2k$ and $2k+1$ are labelled alternately $0, F_{2k+2}, 0, F_{2k+2}, \ldots$ from left to right.

An edge labeling of $\mathfrak{F}$

The edges between ranks $2k$ and $2k+1$ are labelled alternately $0, F_{2k+2}, 0, F_{2k+2}, \ldots$ from left to right.

The edges between ranks $2k-1$ and $2k$ are labelled alternately $F_{2k+1}, 0, F_{2k+1}, 0, \ldots$ from left to right.

Diagram of the edge labeling



Connection with sums of Fibonacci numbers

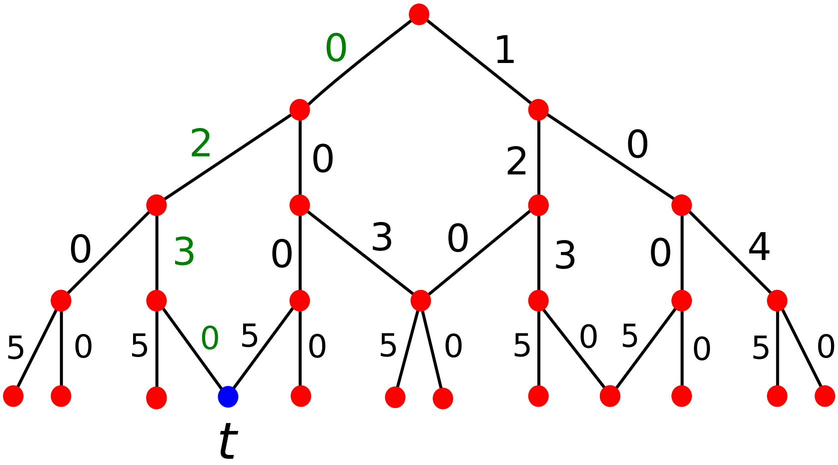
Let $\boldsymbol{t} \in \mathfrak{F}$. All paths (saturated chains) from the top to t have the same sum of their elements $\boldsymbol{\sigma(t)}$.

Connection with sums of Fibonacci numbers

Let $\boldsymbol{t} \in \mathfrak{F}$. All paths (saturated chains) from the top to t have the same sum of their elements $\boldsymbol{\sigma(t)}$.

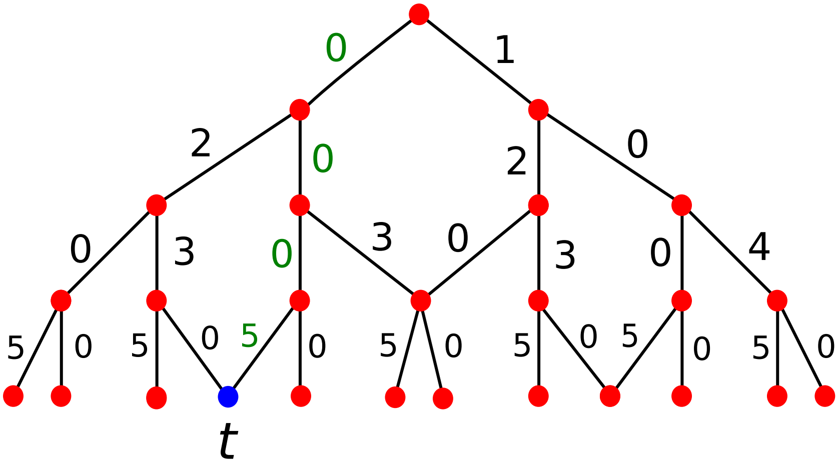
If $\operatorname{rank}(t) = n$, this gives all ways to write $\sigma(t)$ as a sum of distinct Fibonacci numbers from $\{F_2, F_3, \ldots, F_{n+1}\}$.

An example



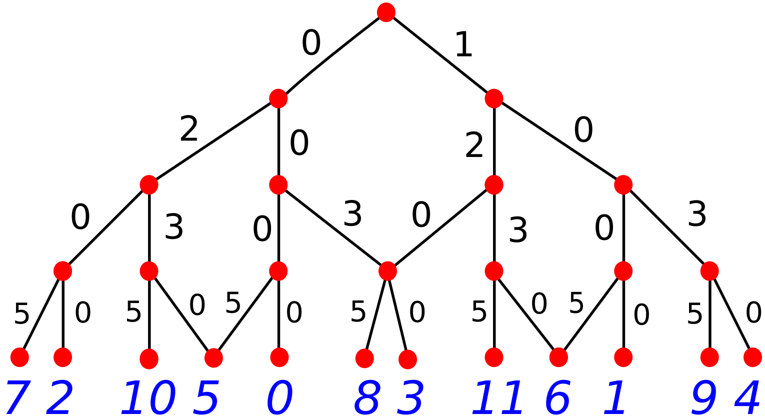
$$2 + 3 = F_3 + F_4$$

An example



$5 = F_5$

An ordering of $\mathbb{N}$



In the limit as rank $\to \infty$, get an interesting dense linear ordering $\prec$ of $\mathbb{N}$.

Special case of $\prec$

Every nonnegative integer has a unique representation as a sum of nonconsecutive Fibonacci numbers, where a summand equal to 1 is always taken to be F_2 (**Zeckendorf's theorem**).

$$n = F_{j_1} + \cdots + F_{j_s}, \quad j_1 < \cdots < j_s$$

Special case of $\prec$

Every nonnegative integer has a unique representation as a sum of nonconsecutive Fibonacci numbers, where a summand equal to 1 is always taken to be F_2 (**Zeckendorf's theorem**).

$$n = F_{j_1} + \cdots + F_{j_s}, \quad j_1 < \cdots < j_s$$

Then $n \prec 0$ if and only if j_1 is odd.

Final curtain call