Partition Statistics with Respect to Plancherel Measure

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Basic
Insights
Launched
Longlasting
Exciting
Research
Activity

standard Young tableau (SYT) of shape $\lambda = (4,4,3,1)$:

$<$

$\wedge$

1	3	4	8
2	6	9	11
5	7	12	
10			

f_λ: number of SYT of shape λ

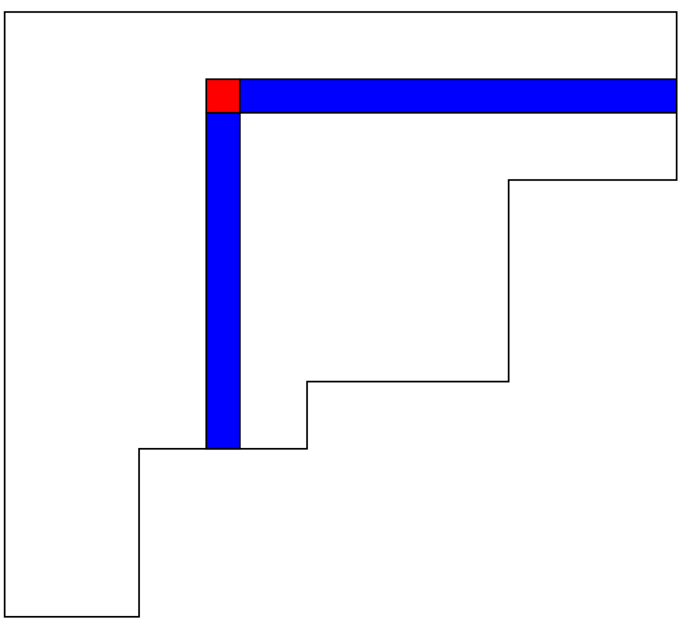
$f_{3,2} = 5$:

$$\begin{array}{lll} 1\,2\,3 & 1\,2\,4 & 1\,2\,5 \\ 4\,5 & 3\,5 & 3\,4 \end{array}$$

$$\begin{array}{ll} 1\,3\,4 & 1\,3\,5 \\ 2\,5 & 2\,4 \end{array}$$

Hook length formula

For $u \in \lambda$, let h_u be the **hook length** at u, i.e., the number of squares directly below or to the right of u (counting u once)



7	5	4	2
6	4	3	1
4	2	1	
1			

Theorem (Frame-Robinson-Thrall). *Let* $\lambda \vdash n$. *Then*

$$f_\lambda = \frac{n!}{\prod_{u\in\lambda} h_u}.$$

$$f^{(4,4,3,1)} = \frac{12!}{7\cdot 6\cdot 5\cdot 4^3\cdot 3\cdot 2^2\cdot 1^3}$$

$$= 2970$$

RSK algorithm (or representation theory) $\Rightarrow$

$$\sum_{\lambda \vdash n} f_\lambda^2 = n!$$

Let

Par(n) $= \{\lambda : \lambda \vdash n\}$

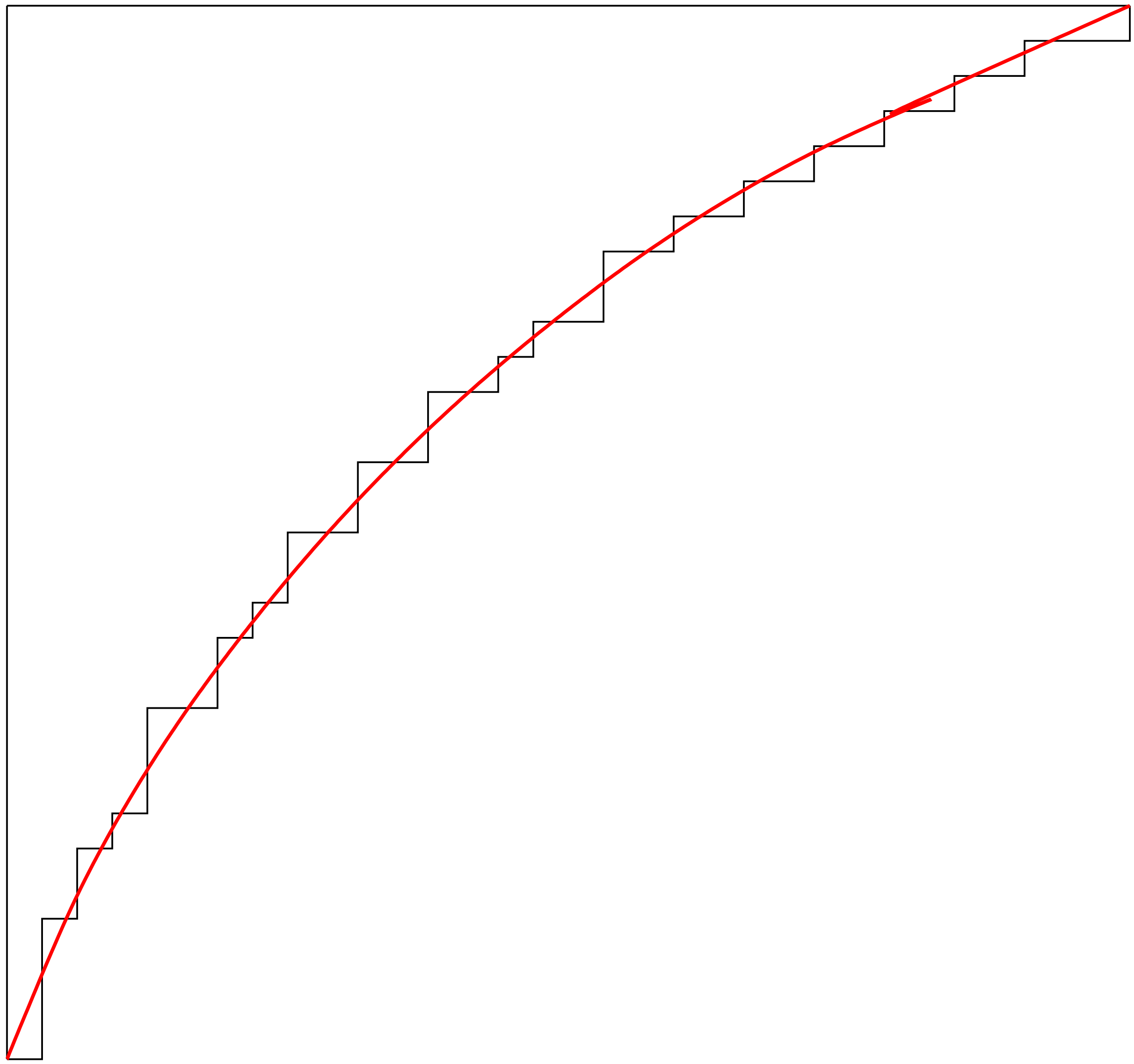
$$\text{Par}(5) = \{5, 41, 32, 311, 221, 2111, 11111\}$$

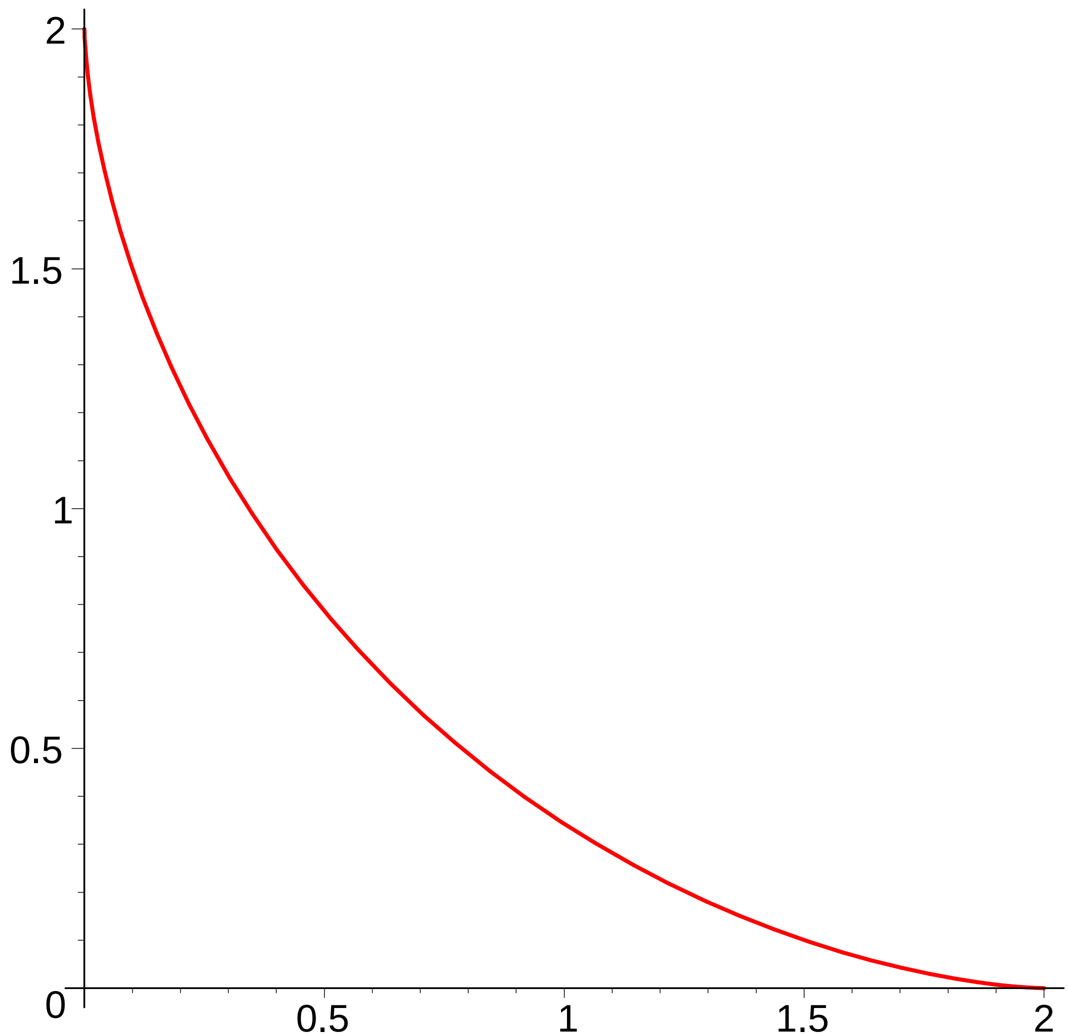
Plancherel measure μ on Par(n):

$$\mu(\lambda) = \frac{f_\lambda^2}{n!}$$

Vershik-Kerov, Logan-Shepp (1977): normalize diagram to have area 1. Then a sequence of random partitions $\lambda^n \vdash n$ will almost surely converge as $n \to \infty$ to the limiting shape $y = \boldsymbol{\psi}(x)$ given by

$$\begin{aligned} x &= y + 2\cos\theta \\ y &= \frac{2}{\pi}(\sin\theta - \theta\cos\theta) \\ & \quad 0 \le \theta \le \pi \end{aligned}$$





$$x = y + 2\cos\theta$$

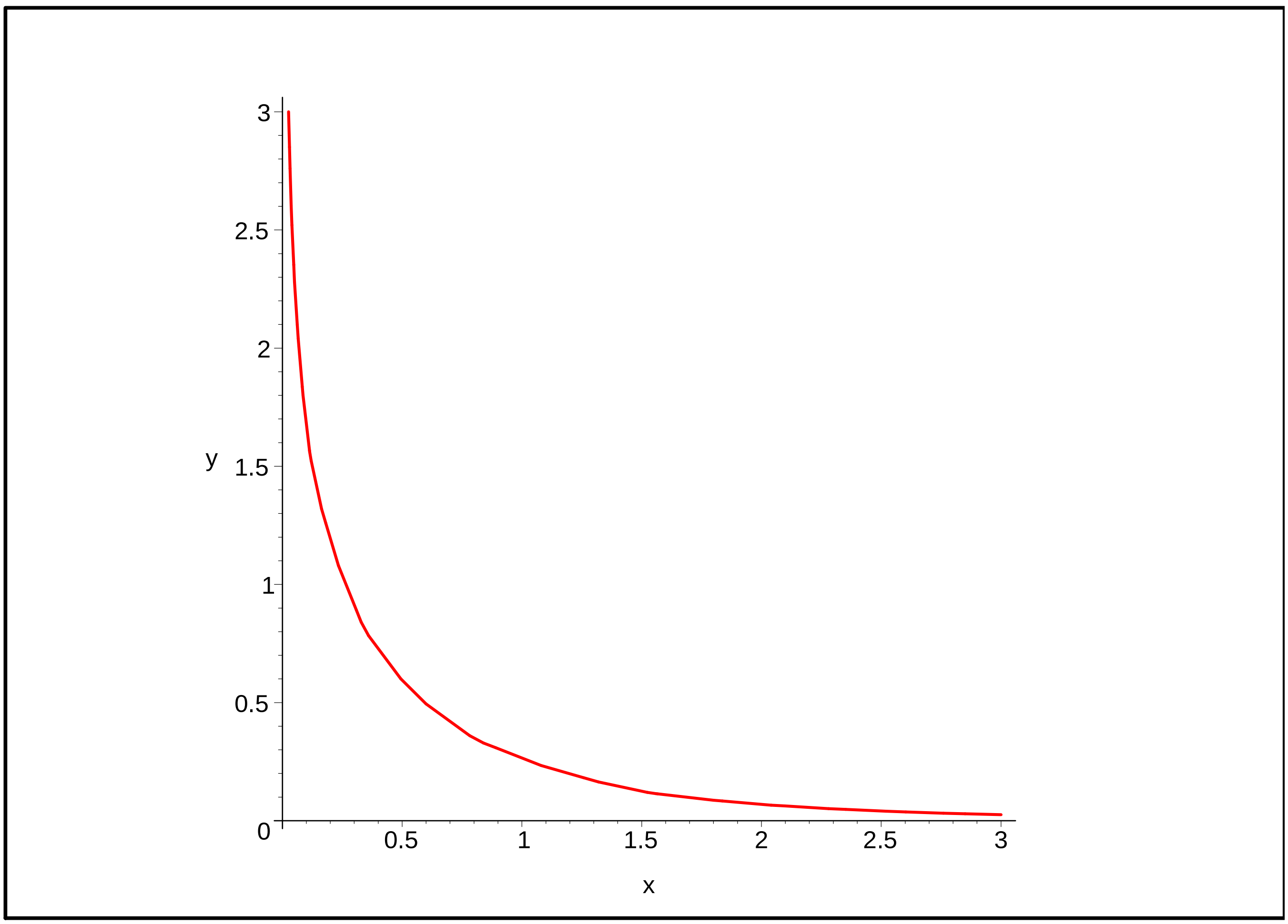
$$y = \frac{2}{\pi}(\sin\theta - \theta\cos\theta)$$

$$0 \leq \theta \leq \pi$$

Digression: What about the **uniform** distribution on partitions?

Vershik (1996): the limiting shape is

$$e^{-(\pi/\sqrt{6})x} + e^{-(\pi/\sqrt{6})y} = 1$$



Think of the boundary of a random partition as a random path of horizontal and vertical steps.

$\psi'(x)$ measures the probability we observe a vertical step at $(x, \psi(x))$ in a random shape.

Borodin-Okounkov-Olshanski (2000): found probability of **any** finite sequence of vertical and horizontal steps at $(x, \psi(x))$ (described by a determinantal point process with discrete sine kernel).

Partition statistics

Recall:

$$\sum_{\lambda \vdash n} f_\lambda^2 = n!$$

Nekrasov-Okounkov (2006), **Guoniu Han** (2008):

$$\sum_{n \geq 0} \left(\sum_{\lambda \vdash n} f_\lambda^2 \prod_{u \in \lambda} (t + h_v^2) \right) \frac{x^n}{n!^2}$$

$$= \prod_{i \geq 1} (1 - x^i)^{-t-1}$$

$\boldsymbol{e_k}$: kth elementary SF

Corollary. *Let*

$$\boldsymbol{g_k(n)} = \frac{1}{n!} \sum_{\lambda \vdash n} f_\lambda^2 e_k(h_u^2 \, : \, u \in \lambda).$$

Then $g_k(n) \in \mathbb{Q}[n]$.

$$g_1(n) = \frac{1}{2} n(3n-1)$$

$$g_2(n) = \frac{1}{24} n(n-1)(27n^2 - 67n + 74)$$

$$g_3(n) = \frac{1}{48} n(n-1)(n-2) (27n^3 - 174n^2 + 511n - 552).$$

Conjecture (Han). Let $j \in \mathbb{P}$. Then

$$\frac{1}{n!}\sum_{\lambda\vdash n} f_\lambda^2 \sum_{u\in\lambda} h_u^{2j} \in \mathbb{Q}[n].$$

True for $j = 1$ by above.

Stronger conjecture. For **any** symmetric function F,

$$\frac{1}{n!}\sum_{\lambda\vdash n} f_\lambda^2 F(h_u^2 \,:\, u\in\lambda) \in \mathbb{Q}[n].$$

$$\frac{1}{n!}\sum_{\lambda\vdash n} f_\lambda^2 \sum_{u\in\lambda} h_u \;=\; ?$$

$$\frac{1}{n!}\sum_{\lambda\vdash n} f_\lambda \sum_{u\in\lambda} h_u \;=\; ?$$

$$\frac{1}{n!}\sum_{\lambda\vdash n} f_\lambda \sum_{u\in\lambda} h_u^2 \;=\; ?$$

Note. $2\sum_{u\in\lambda} h_u = \sum(\lambda_i^2 + (\lambda_i')^2)$.

For $u \in \lambda$ let $\boldsymbol{c(u)} = i - j$, the **content** of square $u = (i, j)$.

0	1	2	3
−1	0	1	2
−2	−1	0	
−3			

Theorem. *For* ***any*** *symmetric function F,*

$$\frac{1}{n!} \sum_{\lambda \vdash n} f_\lambda^2 F(c_u \, : \, u \in \lambda) \in \mathbb{Q}[n].$$

Idea of proof. By linearity, suffices to take $F = e_\mu$. For any finite group G with conjugacy classes $C_1, \ldots, C_t$ and irreducible character χ, let

$$\widetilde{C}_j = \sum_{w \in C_j} w \in \mathbb{C}G$$

$$E_\chi = \frac{\chi(1)}{\#G} \sum_j \chi(C_j) \widetilde{C}_j \in Z(\mathbb{C}G).$$

Standard result: the E_χ's are a set of primitive orthogonal idempotents for $Z(\mathbb{C}G)$.

Reformulation for $G = \mathfrak{S}_n$: let

$$H_\lambda = \prod_{u \in \lambda} h_u.$$

Then for all $k \geq 1$,

$$\sum_{\lambda \vdash n} H_\lambda^{k-2} s_\lambda(x^{(1)}) \cdots s_\lambda(x^{(k)}) =$$

$$\frac{1}{n!} \sum_{\substack{w_1 \cdots w_k = 1 \\ \text{in } \mathfrak{S}_n}} p_{\rho(w_1)}(x^{(1)}) \cdots p_{\rho(w_k)}(x^{(k)}). \tag{1}$$

Hook-content formula $(q=1)$:

$$s_\lambda(1^t) = H_\lambda^{-1}\prod_{u\in\lambda}(t+c_u)$$

Set $x^{(i)} = 1^{t_i}$ in (1):

$$\sum_{\lambda\vdash n} H_\lambda^{-2}\prod_{u\in\lambda}(t_1+c_u)\cdots(t_k+c_u) =$$

$$\frac{1}{n!}\sum_{\substack{w_1\cdots w_k=1 \\ \text{in } \mathfrak{S}_n}} t_1^{c(w_1)}\cdots t_k^{c(w_k)},$$

where $\boldsymbol{c(w)}$ = # cycles of w.

Take coefficient of $t_1^{n-\mu_1} \cdots t_k^{n-\mu_k}$. Get

$$\frac{1}{n!} \sum_{\lambda \vdash n} f_\lambda^2 e_\mu(c_u : u \in \lambda)$$

$$= \#\{(w_1, \ldots, w_k) \in \mathfrak{S}_n^k : w_1 \cdots w_k = 1, \\ c(w_i) = n - \mu_i\}.$$

Elementary combinatorial argument shows this is a polynomial in n. $\square$

Corollary. *For any* $j \geq 0$,

$$\frac{1}{n!}\sum_{\lambda\vdash n} f_\lambda^2 \left(\sum_{u\in\lambda} h_u^2\right)^{2j} \in \mathbb{Q}[n].$$

Proof. Follows from previous theorem and

$$\sum_{u\in\lambda} h_u^2 = n^2 + \sum_{u\in\lambda} c_u^2.$$

Corollary.

$$\frac{1}{n!}\sum_{\lambda\vdash n} f_\lambda^2 \sum_{u\in\lambda} h_u^4 \in \mathbb{Q}[n].$$

Proof. Previous corollary implies

$$\frac{1}{n!}\sum_{\lambda\vdash n} f_\lambda^2 \left(\sum_{u\in\lambda} h_u^2\right)^2 \in \mathbb{Q}[n].$$

Nekrasov-Okounkov implies

$$\frac{1}{n!}\sum_{\lambda\vdash n} f_\lambda^2 e_2(h_u^2 \,:\, u\in\lambda) \in \mathbb{Q}[n].$$

Proof follows from $p_2 = p_1^2 - 2e_2$.

Recall:

Stronger conjecture. For **any** symmetric function F,

$$\frac{1}{n!}\sum_{\lambda\vdash n} f_\lambda^2 F(h_u^2 : u\in\lambda) \in \mathbb{Q}[n].$$

Theorem. *This conjecture follows from:* $\forall$ symmetric functions F,

$$\frac{1}{n!}\sum_{\lambda=(\lambda_1,\dots,\lambda_n)\vdash n} f_\lambda^2$$

$$F(\lambda_i - i : 1\leq i\leq n) \in \mathbb{Q}[n].$$

Proof. Based on multiset identity

$$\{h_u : u\in\lambda\}\cup\{\lambda_i-\lambda_j-i+j : 1\leq i<j\leq n\}$$

$$=\{n+c_u : u\in\lambda\}\cup\{1^{n-2},2^{n-2},\dots,n-1\}.$$

Recall

$$\sum_{\lambda \vdash n} H_\lambda^{k-2} s_\lambda(x^{(1)}) \cdots s_\lambda(x^{(k)}) =$$

$$\frac{1}{n!} \sum_{\substack{w_1 \cdots w_k = 1 \\ \text{in } \mathfrak{S}_n}} p_{\rho(w_1)}(x^{(1)}) \cdots p_{\rho(w_k)}(x^{(k)}).$$

We earlier set $x^{(i)} = 1^{t_i}$ to prove:

Theorem. *For* ***any*** *symmetric function F,*

$$\frac{1}{n!} \sum_{\lambda \vdash n} f_\lambda^2 F(c_u : u \in \lambda) \in \mathbb{Q}[n].$$

Now apply the **linear** transformation (not a homomorphism)

$$\varphi(s_\lambda(x^{(i)})) = \frac{\prod_{i=1}^{n}(t_i + \lambda_i + n - i)}{H_\lambda}.$$

$$\varphi(g_1(x^{(1)}) \cdots g_k(x^{(k)}) = \varphi(g_1(x^{(1)}) \cdots \varphi(g_k(x^{(k)}).$$

Key Lemma. Let $\mu \vdash n$, $\ell = \ell(\mu)$. Then

$$\varphi(p_\mu) =$$

$$(-1)^{n-\ell} \sum_{i=0}^{m} \binom{m}{i} t(t+1) \cdots (t+i-1),$$

where $\boldsymbol{m} = m_1(\mu)$, the number of parts of μ equal to 1.

Proof of lemma reduces to:

$$(1-p_1)^{-t}\sum_{n\geq 0} e_n =$$

$$\sum_{\lambda} \frac{\prod_{i=1}^{n}(t+\lambda_i+n-i)}{H_\lambda} s_\lambda.$$