A Survey of Parking Functions

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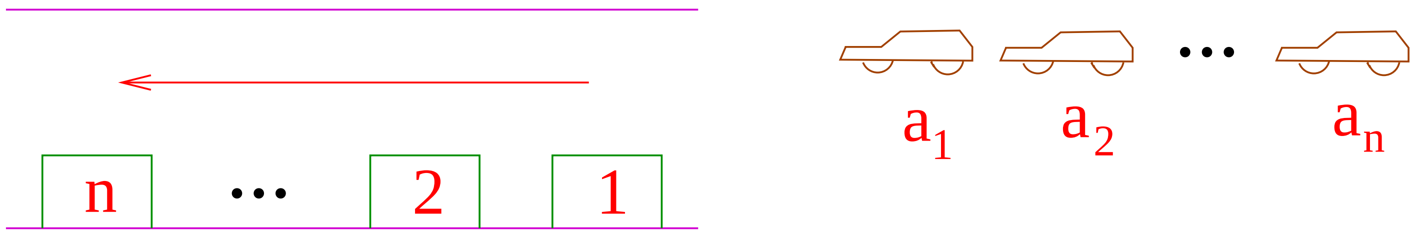
M.I.T.

Parking functions



Car C_i prefers space a_i. If a_i is occupied, then C_i takes the next available space. We call $(a_1, \ldots, a_n)$ a **parking function** (of length n) if all cars can park.

$\boldsymbol{n = 2}$: $11 \quad 12 \quad 21$

$\boldsymbol{n = 3:}$ $111 \quad 112 \quad 121 \quad 211 \quad 113 \quad 131 \quad 311 \quad 122$

$212 \quad 221 \quad 123 \quad 132 \quad 213 \quad 231 \quad 312 \quad 321$

Permutations of parking functions

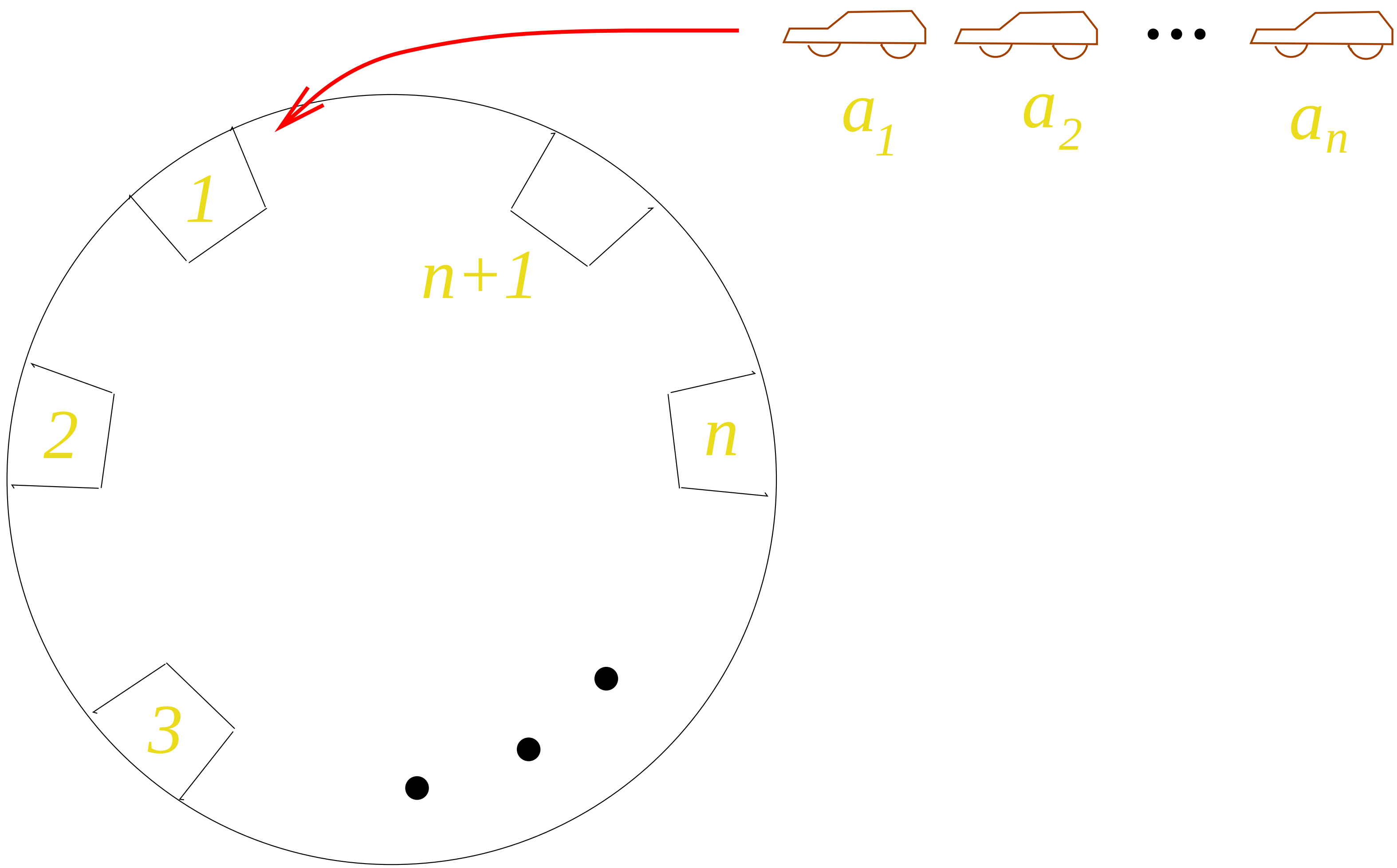
Easy: Let $\alpha = (a_1, \ldots, a_n) \in \mathbb{P}^n$. Let $b_1 \leq b_2 \leq \cdots \leq b_n$ be the increasing rearrangement of α. Then α is a parking function if and only $b_i \leq i$.

Corollary. *Every permutation of the entries of a parking function is also a parking function.*

Enumeration of parking functions

Theorem (**Pyke**, 1959; **Konheim and Weiss**, 1966). *Let $f(n)$ be the number of parking functions of length n. Then* $\boldsymbol{f(n) = (n+1)^{n-1}}$.

Proof (**Pollak**, c. 1974). Add an additional space $n+1$, and arrange the spaces in a circle. Allow $n+1$ also as a preferred space.


1
n+1
2
n
3
a_1
a_2
a_n

Conclusion of Pollak's proof

Now all cars can park, and there will be one empty space. α is a parking function $\Leftrightarrow$ if the empty space is $n+1$. If $\alpha = (a_1, \ldots, a_n)$ leads to car C_i parking at space p_i, then $(a_1 + j, \ldots, a_n + j)$ (modulo $n+1$) will lead to car C_i parking at space $p_i + j$. Hence exactly one of the vectors

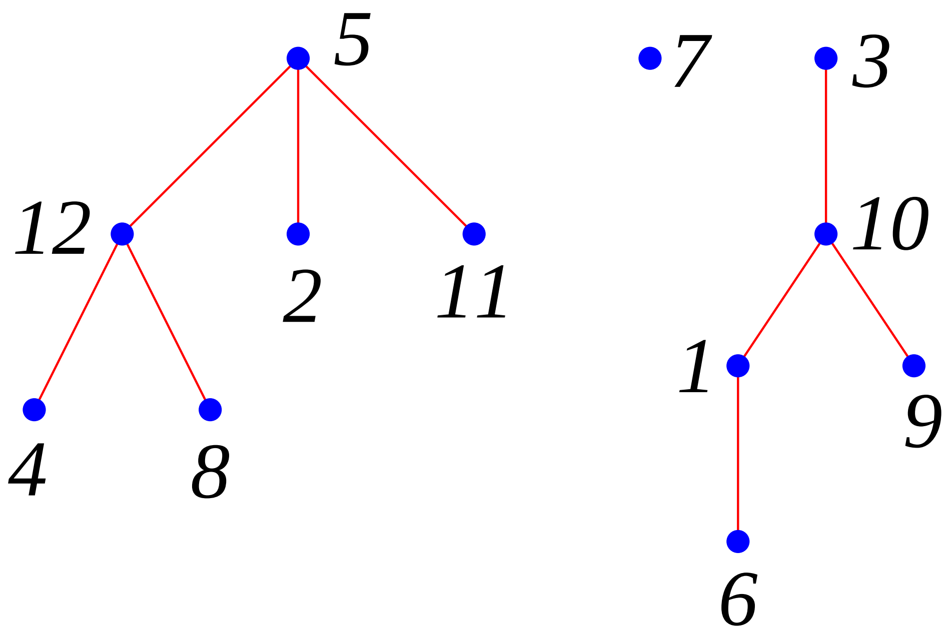
$$(a_1 + i, a_2 + i, \ldots, a_n + i) \text{ (modulo } n+1\text{)}$$

is a parking function, so

$$f(n) = \frac{(n+1)^n}{n+1} = (n+1)^{n-1}.$$

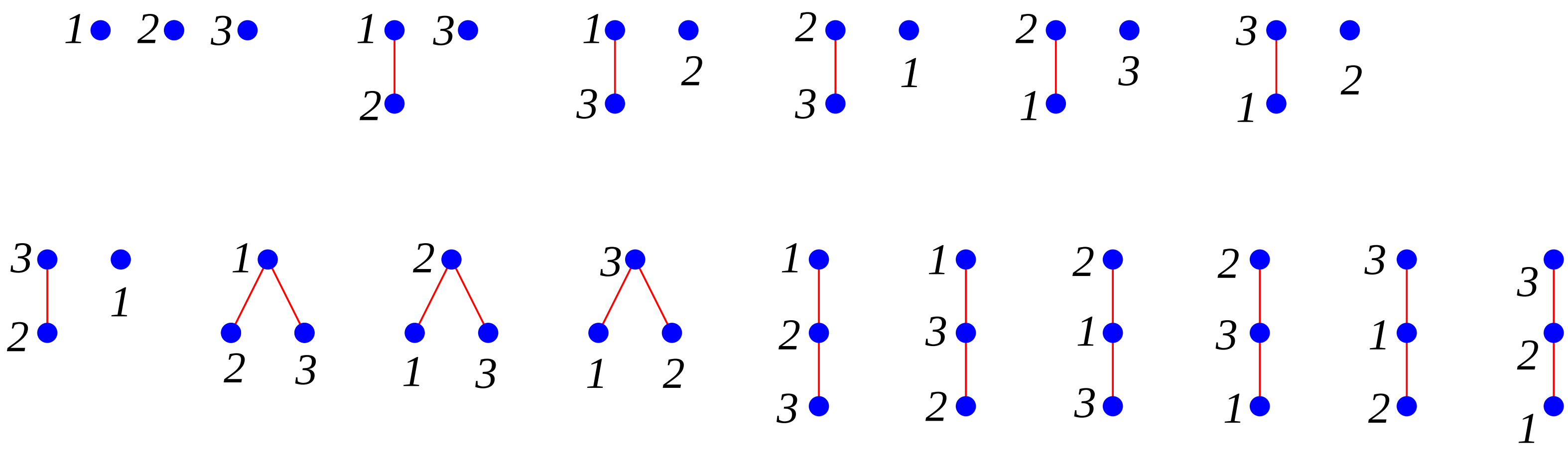
Forest inversions

Let F be a rooted forest on the vertex set $\{1, \ldots, n\}$.



Theorem (**Sylvester-Borchardt-Cayley**). *The number of such forests is* $(n+1)^{n-1}$.

The case $n = 3$

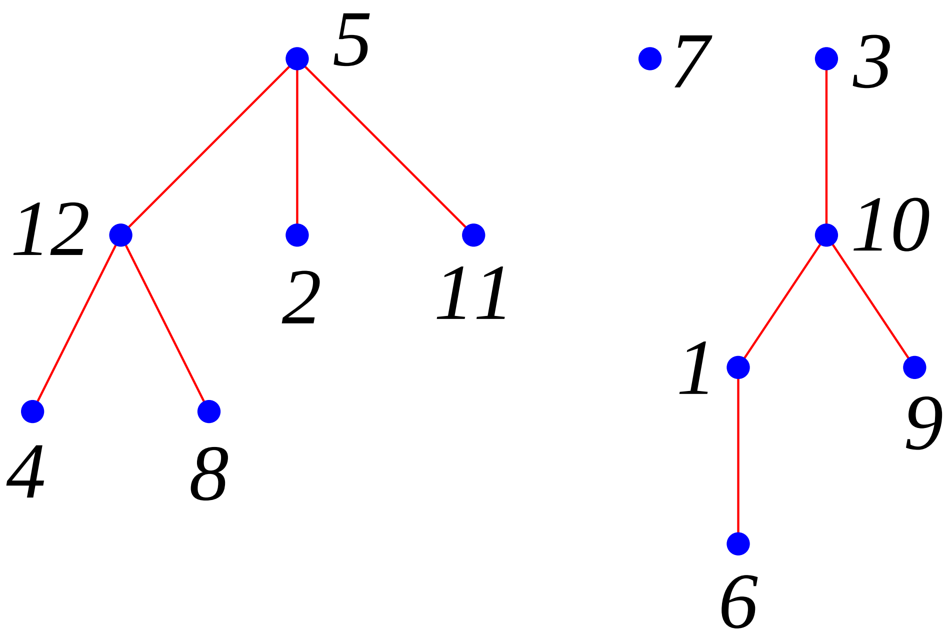


Forest inversions

An **inversion** in F is a pair (i, j) so that $i > j$ and i lies on the path from j to the root.

$$\mathbf{inv}(\boldsymbol{F}) = \#(\text{inversions of } F)$$

Example of forest inversions



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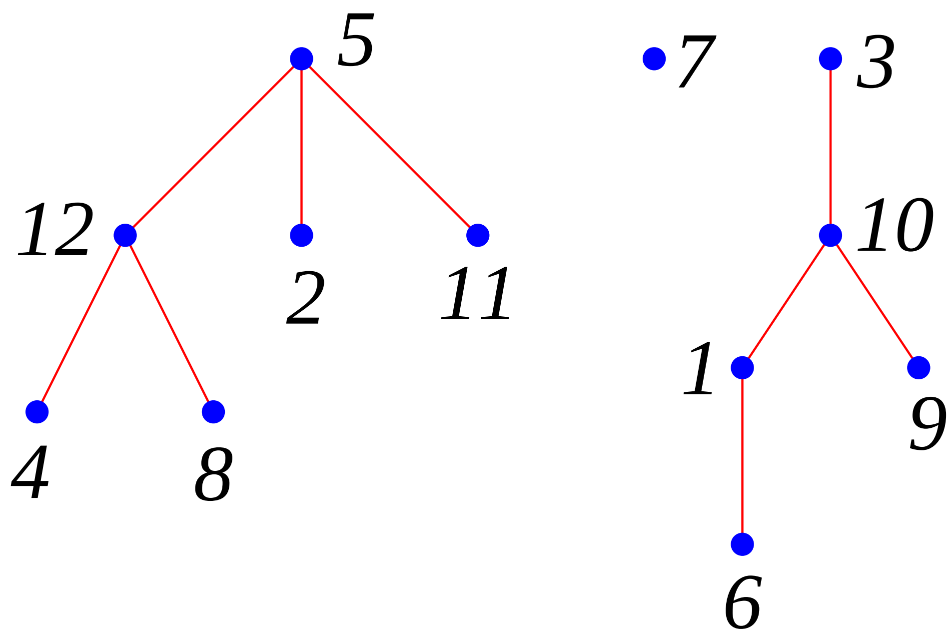


Inversions: $(5,4),\ (5,2),\ (12,4),\ (12,8)$

$(3,1),\ (10,1),\ (10,6),\ (10,9)$

$$\mathrm{inv}(F) = 8$$

The inversion enumerator

Let

$$I_n(q) = \sum_F q^{\mathrm{inv}(F)},$$

summed over all forests F with vertex set $\{1, \ldots, n\}$. E.g.,

$$\begin{aligned} I_1(q) &= 1 \\ I_2(q) &= 2 + q \\ I_3(q) &= 6 + 6q + 3q^2 + q^3 \end{aligned}$$

Relation to connected graphs

Theorem (**Mallows-Riordan** 1968, **Gessel-Wang** 1979) *We have*

$$I_n(1+q) = \sum_G q^{e(G)-n},$$

where G ranges over all connected graphs (without loops or multiple edges) on $n+1$ labelled vertices, and where $e(G)$ denotes the number of edges of G.

A generating function

Corollary.

$$\sum_{n\geq 0} I_n(q)(q-1)^n \frac{x^n}{n!} = \frac{\sum_{n\geq 0} q^{\binom{n+1}{2}} \frac{x^n}{n!}}{\sum_{n\geq 0} q^{\binom{n}{2}} \frac{x^n}{n!}}$$

$$\sum_{n\geq 1} I_n(q)(q-1)^{n-1} \frac{x^n}{n!} = \log \sum_{n\geq 0} q^{\binom{n}{2}} \frac{x^n}{n!}$$

Relation to parking functions

Theorem (**Kreweras**, 1980). *We have*

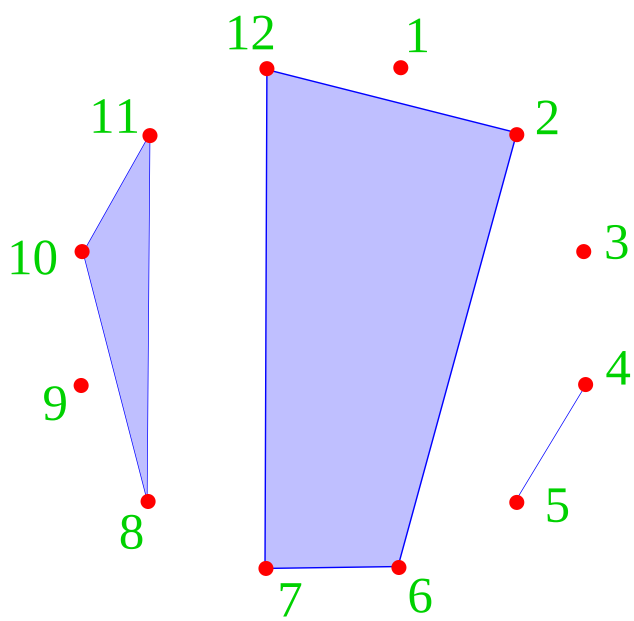
$$q^{\binom{n}{2}} I_n(1/q) = \sum_{(a_1,\ldots,a_n)} q^{a_1+\cdots+a_n},$$

where $(a_1, \ldots, a_n)$ *ranges over all parking functions of length* n.

Noncrossing partitions

A **noncrossing partition** of $\{1, 2, \ldots, n\}$ is a partition $\{B_1, \ldots, B_k\}$ of $\{1, \ldots, n\}$ such that

$$a < b < c < d,\ a, c \in B_i,\ b, d \in B_j \Rightarrow i = j.$$



Enumeration of noncrossing partitio

Theorem (**H. W. Becker**, 1948–49). *The number of noncrossing partitions of* $\{1, \ldots, n\}$ *is the* ***Catalan number***

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Chains of noncrossing partitions

A **maximal chain** $\mathfrak{m}$ of noncrossing partitions of $\{1, \ldots, n+1\}$ is a sequence

$$\pi_0, \ \pi_1, \ \pi_2, \ \ldots, \ \pi_n$$

of noncrossing partitions of $\{1, \ldots, n+1\}$ such that π_i is obtained from π_{i-1} by merging two blocks into one. (Hence π_i has exactly $n+1-i$ blocks.)

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$$\mathbf{1\text{—}2\text{—}3\text{—}4\text{—}5} \quad \mathbf{1\text{—}25\text{—}3\text{—}4} \quad \mathbf{1\text{—}25\text{—}34}$$

$$\mathbf{125\text{—}34} \quad \mathbf{12345}$$

A chain labeling

Define:

$$\mathbf{min}\ \boldsymbol{B} = \text{least element of } B$$

$$\boldsymbol{j < B}\ :\ j < k\ \ \forall k \in B.$$

Suppose π_i is obtained from π_{i-1} by merging together blocks B and B', with $\min B < \min B'$. Define

$$\begin{aligned} \boldsymbol{\Lambda_i(\mathfrak{m})} &= \max\{j \in B\ :\ j < B'\} \\ \boldsymbol{\Lambda(\mathfrak{m})} &= (\Lambda_1(\mathfrak{m}), \ldots, \Lambda_n(\mathfrak{m})). \end{aligned}$$

An example

$$\mathbf{1{-}2{-}3{-}4{-}5}\quad \mathbf{1{-}25{-}3{-}4}\quad \mathbf{1{-}25{-}34}$$

$$\mathbf{125{-}34}\quad \mathbf{12345}$$

we have

$$\Lambda(\mathfrak{m}) = (2, 3, 1, 2).$$

Number of chains

Theorem. Λ *is a bijection between the maximal chains of noncrossing partitions of* $\{1, \ldots, n+1\}$ *and parking functions of length* n.

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Is there a connection with Voiculescu's theory of free probability?

The Shi arrangement: background

Braid arrangement $\mathcal{B}_n$: the set of hyperplanes

$$x_i - x_j = 0, \quad 1 \le i < j \le n,$$

in $\mathbb{R}^n$.

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Let $\boldsymbol{R_0}$ be the **base region**

$$R_0: \ x_1 > x_2 > \cdots > x_n.$$

Labeling the regions

Label R_0 with

$$\boldsymbol{\lambda(R_0)} = (1, 1, \ldots, 1) \in \mathbb{Z}^n.$$

If R is labelled, R' is separated from R only by $x_i - x_j = 0$ $(i < j)$, and R' is unlabelled, then set

$$\boldsymbol{\lambda(R')} = \lambda(R) + e_i,$$

where $\boldsymbol{e_i} = i$th unit coordinate vector.

The labeling rule

R'

R

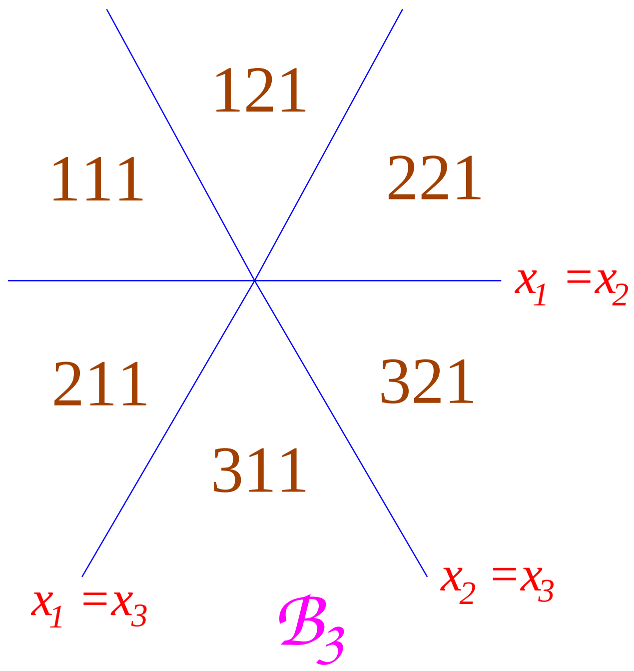
$\lambda(R') = \lambda(R) + e_i$

$\lambda(R)$

$x_i = x_j$

$i < j$

Description of labels



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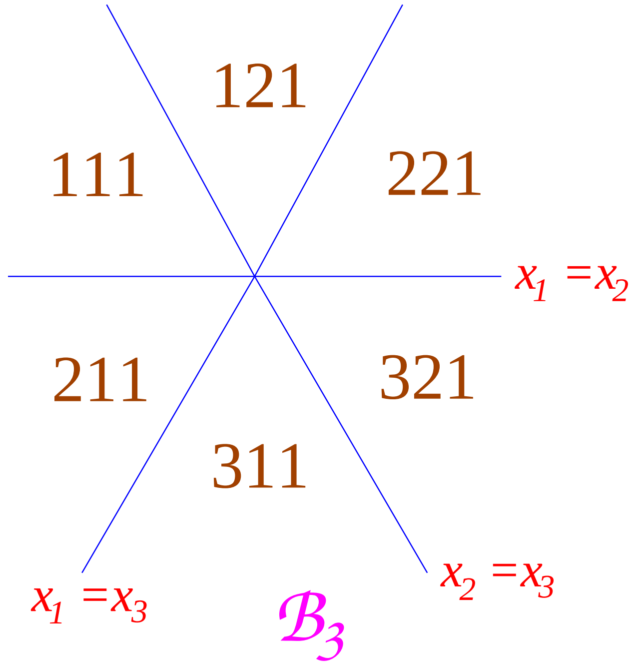


Theorem (easy). *The labels of $\mathcal{B}_n$ are the sequences $(b_1, \ldots, b_n) \in \mathbb{Z}^n$ such that $1 \leq b_i \leq n - i + 1$.*

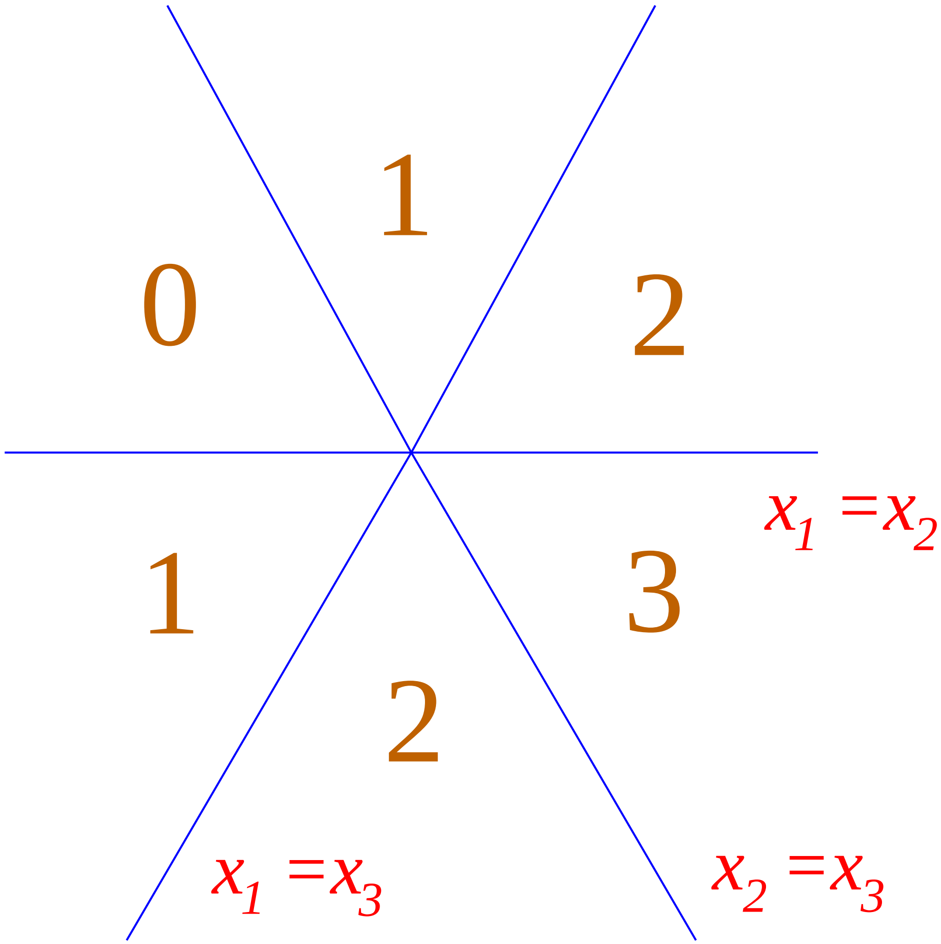
Separating hyperplanes

Recall

$$\boldsymbol{R_0} : \ x_1 > x_2 > \cdots > x_n.$$

Let $\boldsymbol{d(R)}$ be the number of hyperplanes in $\mathcal{B}_n$ separating R_0 from R.

The case $n = 3$



$\mathcal{B}_3$

Generating function for $d(R)$

NOTE: If $\lambda(R) = (b_1, \ldots, b_n)$, then $d(R) = \sum(b_i - 1)$.

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NOTE: If $\lambda(R) = (b_1, \ldots, b_n)$, then $d(R) = \sum(b_i - 1)$.

Easy consequence:

Corollary.

$$\sum_{R \in \mathcal{R}} q^{d(R)} =$$

$$(1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1}).$$

The Shi arrangement

Shi Jianyi (时俭益)

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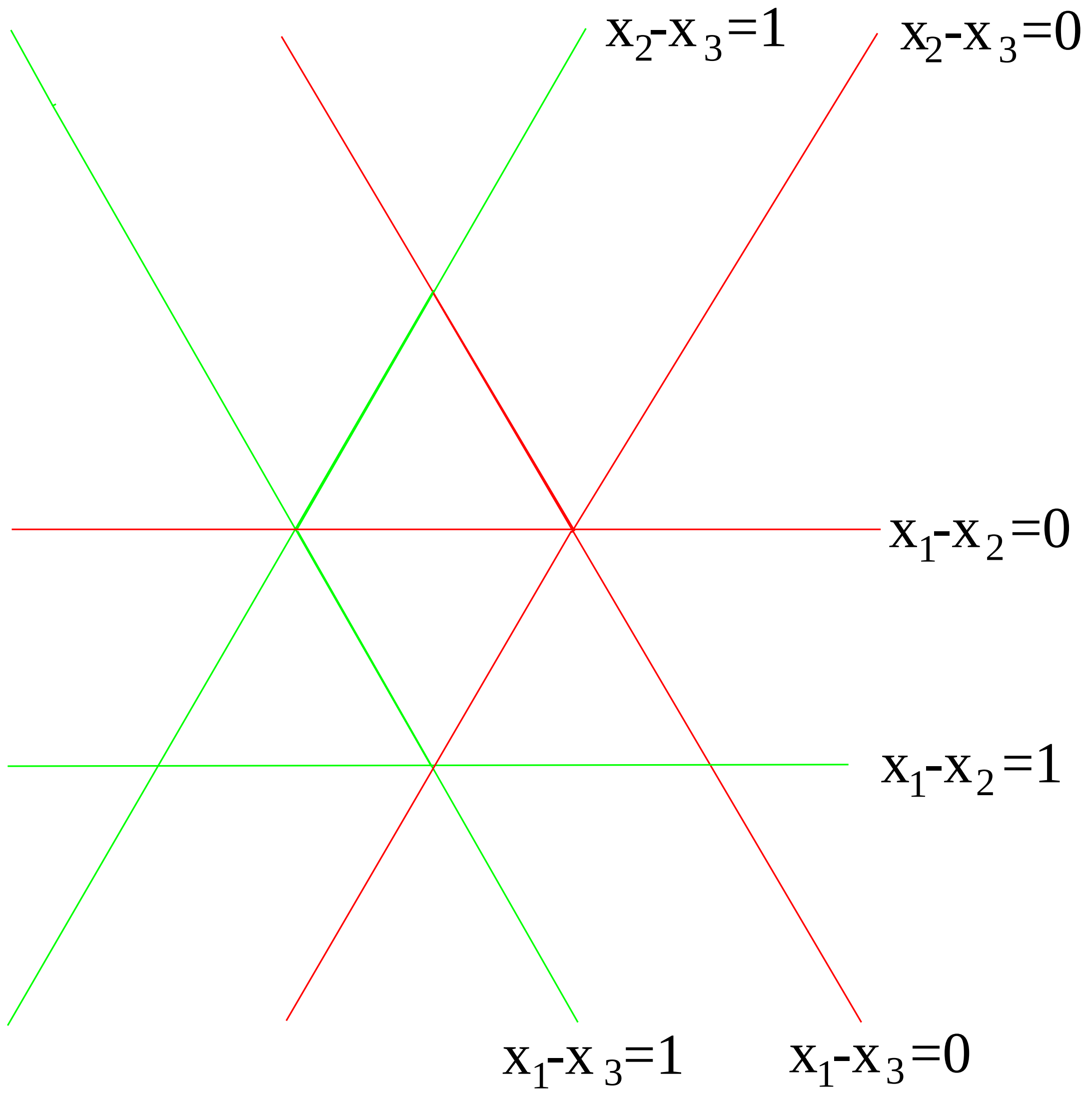
Shi arrangement $\mathcal{S}_n$: the set of hyperplanes

$$x_i - x_j = 0, 1,$$

$1 \le i < j \le n$, in $\mathbb{R}^n$.

The case $n = 3$



Labeling the regions

base region:

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- If R is labelled, R' is separated from R only by $x_i - x_j = 0$ ($i < j$), and R' is unlabelled, then set
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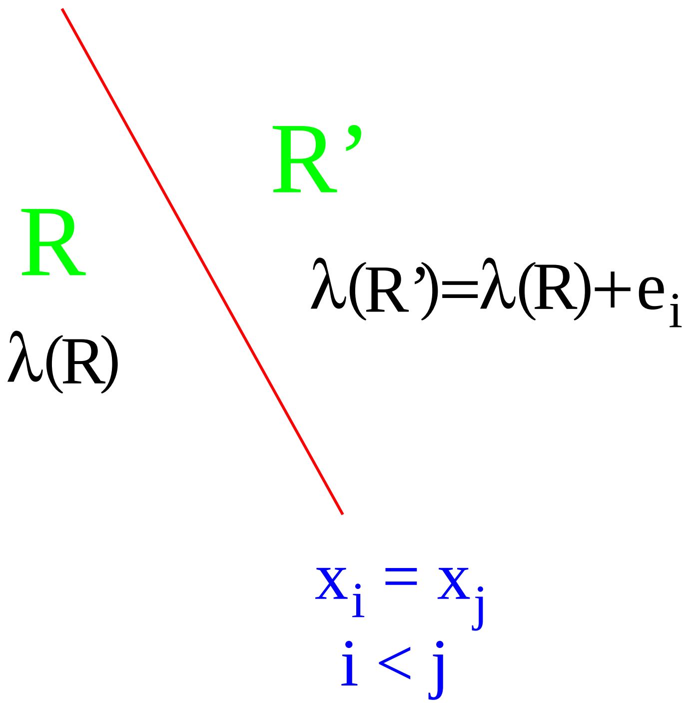
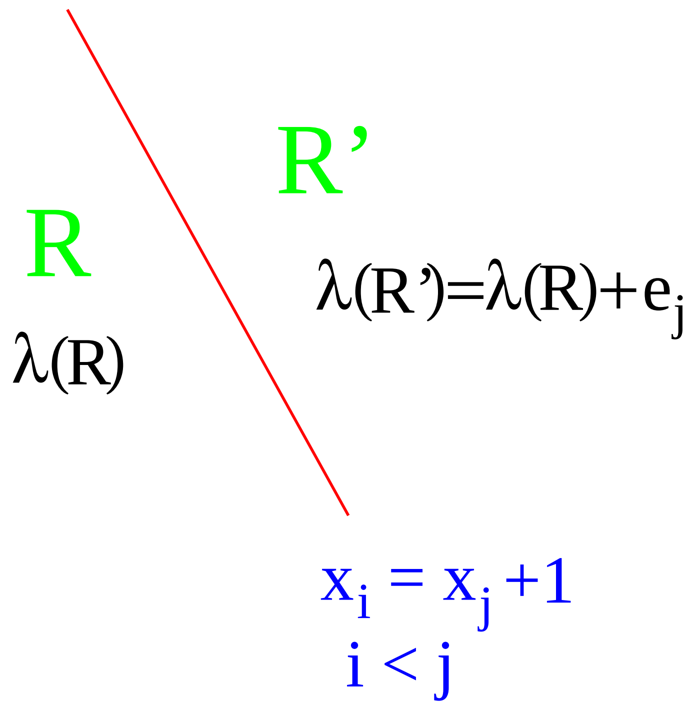
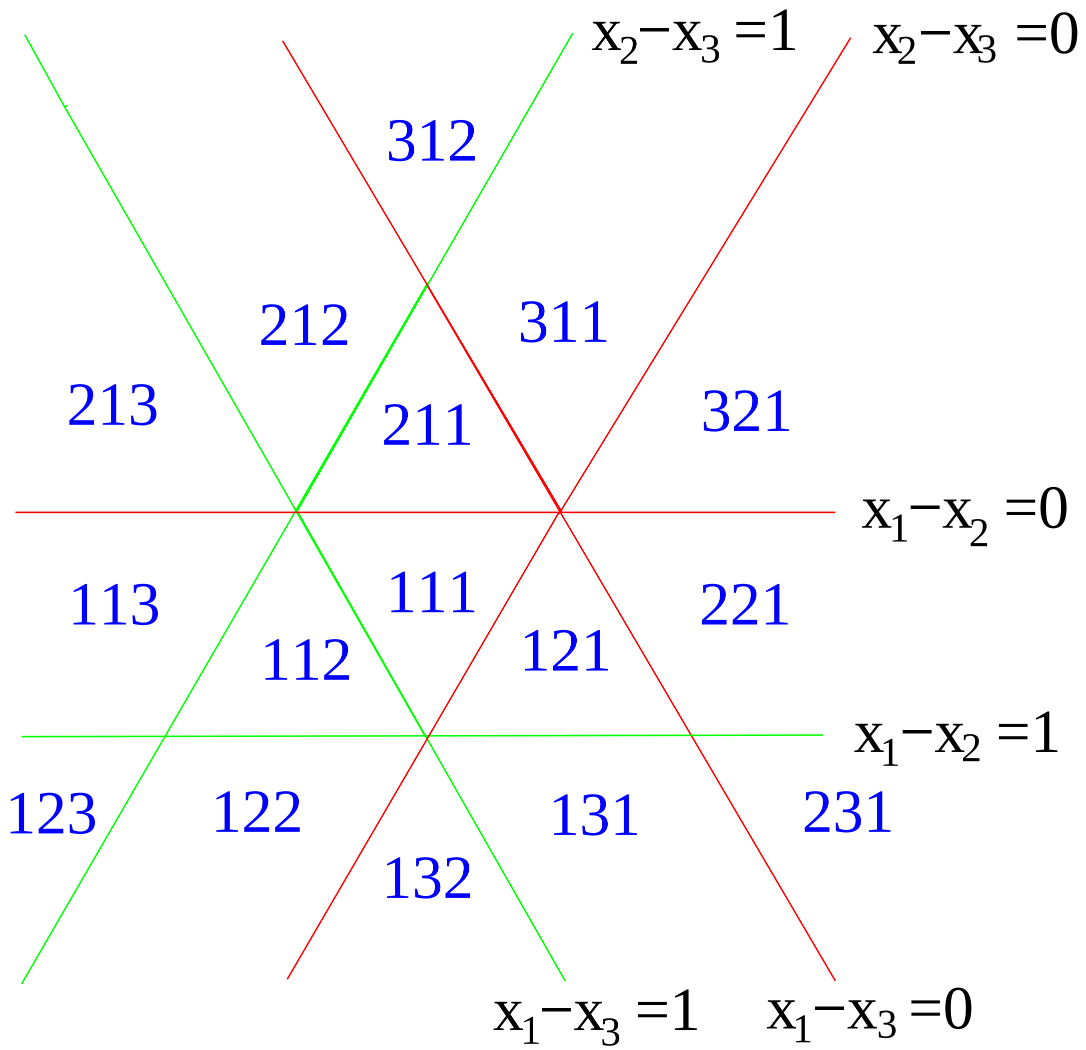
The labeling rule





The labeling for $n = 2$



Description of the labels

Theorem (**Pak, S.**). *The labels of $\mathcal{S}_n$ are the parking functions of length n (each occurring once).*

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Corollary (**Shi**, 1986)

$$r(\mathcal{S}_n) = (n+1)^{n-1}$$

The parking function $\mathfrak{S}_n$-module

The symmetric group $\mathfrak{S}_n$ acts on the set $\mathcal{P}_n$ of all parking functions of length n by permuting coordinates.

Sample properties

- Multiplicity of trivial representation (number of orbits) $= C_n = \frac{1}{n+1}\binom{2n}{n}$

$$n = 3: \quad \mathbf{111 \ \ 211 \ \ 221 \ \ 311 \ \ 321}$$

- Number of elements of $\mathcal{P}_n$ fixed by $w \in \mathfrak{S}_n$ (character value at w):

$$\#\text{Fix}(w) = (n+1)^{(\#\text{ cycles of } w)-1}$$

Symmetric functions

For symmetric function aficionados: Let **$\mathbf{PF}_n$** $= \mathrm{ch}(\mathcal{P}_n)$.

$$\begin{aligned}\mathrm{PF}_n &= \sum_{\lambda \vdash n} (n+1)^{\ell(\lambda)-1} z_\lambda^{-1} p_\lambda \\ &= \sum_{\lambda \vdash n} \frac{1}{n+1} s_\lambda(1^{n+1}) s_\lambda \\ &= \sum_{\lambda \vdash n} \frac{1}{n+1} \left[\prod_i \binom{\lambda_i + n}{n} \right] m_\lambda\end{aligned}$$

More properties

$$\mathrm{PF}_n = \sum_{\lambda \vdash n} \frac{n(n-1)\cdots(n-\ell(\lambda)+2)}{m_1(\lambda)!\cdots m_n(\lambda)!} h_\lambda.$$

$$\omega \mathrm{PF}_n = \sum_{\lambda \vdash n} \frac{1}{n+1} \left[\prod_i \binom{n+1}{\lambda_i} \right] m_\lambda.$$

Background: invariants of $\mathfrak{S}_n$

The group $\mathfrak{S}_n$ acts on $R = \mathbb{C}[x_1, \ldots, x_n]$ by permuting variables, i.e., $w \cdot x_i = x_{w(i)}$. Let

$$\boldsymbol{R^{\mathfrak{S}_n}} = \{f \in R \,:\, w \cdot f = f \text{ for all } w \in \mathfrak{S}_n\}.$$

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Well-known:

$$R^{\mathfrak{S}_n} = \mathbb{C}[e_1, \ldots, e_n],$$

where

$$\boldsymbol{e_k} = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} x_{i_1} x_{i_2} \cdots x_{i_k}.$$

The coinvariant algebra

Let

$$\boldsymbol{D} = R/\left(R_+^{\mathfrak{S}_n}\right) = R/(e_1, \ldots, e_n).$$

Then $\dim D = n!$, and $\mathfrak{S}_n$ acts on D according to the **regular representation**.

Diagonal action of $\mathfrak{S}_n$

Now let $\mathfrak{S}_n$ act **diagonally** on

$$R = \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n],$$

i.e,

$$w \cdot x_i = x_{w(i)}, \quad w \cdot y_i = y_{w(i)}.$$

As before, let

$$\begin{aligned} R^{\mathfrak{S}_n} &= \{f \in R : w \cdot f = f \text{ for all } w \in \mathfrak{S}_n\} \\ D &= R/\left(R_+^{\mathfrak{S}_n}\right). \end{aligned}$$

Hainman's theorem

Theorem (**Haiman**, 1994, 2001). $\dim D = (n+1)^{n-1}$, and the action of $\mathfrak{S}_n$ on D is isomorphic to the action on $\mathcal{P}_n$, tensored with the sign representation. (Connections with Macdonald polynomials, Hilbert scheme of points in the plane, etc.)

$\boldsymbol{\lambda}$-parking functions

Let

$$\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n), \quad \lambda_1 \geq \cdots \geq \lambda_n > 0.$$

A **$\boldsymbol{\lambda}$-parking function** is a sequence $(a_1, \ldots, a_n) \in \mathbb{P}^n$ whose increasing rearrangement $b_1 \leq \cdots \leq b_n$ satisfies $b_i \leq \lambda_{n-i+1}$.

λ-parking functions

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A **λ-parking function** is a sequence $(a_1, \ldots, a_n) \in \mathbb{P}^n$ whose increasing rearrangement $b_1 \leq \cdots \leq b_n$ satisfies $b_i \leq \lambda_{n-i+1}$.

Ordinary parking functions:

$$\lambda = (n, n-1, \ldots, 1)$$

Enumeration of λ-parking fns.

Theorem (**Steck** 1968, **Gessel** 1996). *The number* $N(\lambda)$ *of* λ*-parking functions is given by*

$$N(\lambda) = n! \det \left[\frac{\lambda_{n-i+1}^{j-i+1}}{(j-i+1)!} \right]_{i,j=1}^{n}$$

The parking function polytope

Given $x_1, \ldots, x_n \in \mathbb{R}_{\geq 0}$, define $\boldsymbol{\mathcal{P} = \mathcal{P}(x_1, \ldots, x_n)} \subset \mathbb{R}^n$ by: $(y_1, \ldots, y_n) \in \mathcal{P}_n$ if

$$0 \leq y_i, \quad y_1 + \cdots + y_i \leq x_1 + \cdots + x_i$$

for $1 \leq i \leq n$.

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for $1 \leq i \leq n$.
(also called **Pitman-Stanley polytope**)

Volume of $\mathcal{P}$

Theorem. (a) *Let* $x_1, \ldots, x_n \in \mathbb{N}$*. Then*

$$n!\,V(\mathcal{P}_n) = N(\lambda),$$

where $\lambda_{n-i+1} = x_1 + \cdots + x_i$.

(b) $n!\,V(\mathcal{P}_n) = \displaystyle\sum_{\substack{\text{parking functions} \\ (i_1,\ldots,i_n)}} x_{i_1} \cdots x_{i_n}$.

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(b) $$n!\,V(\mathcal{P}_n) = \sum_{\substack{\text{parking functions} \\ (i_1,\ldots,i_n)}} x_{i_1} \cdots x_{i_n}.$$

Note. If each $x_i > 0$, then $\mathcal{P}_n$ has the combinatorial type of an n-cube.

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