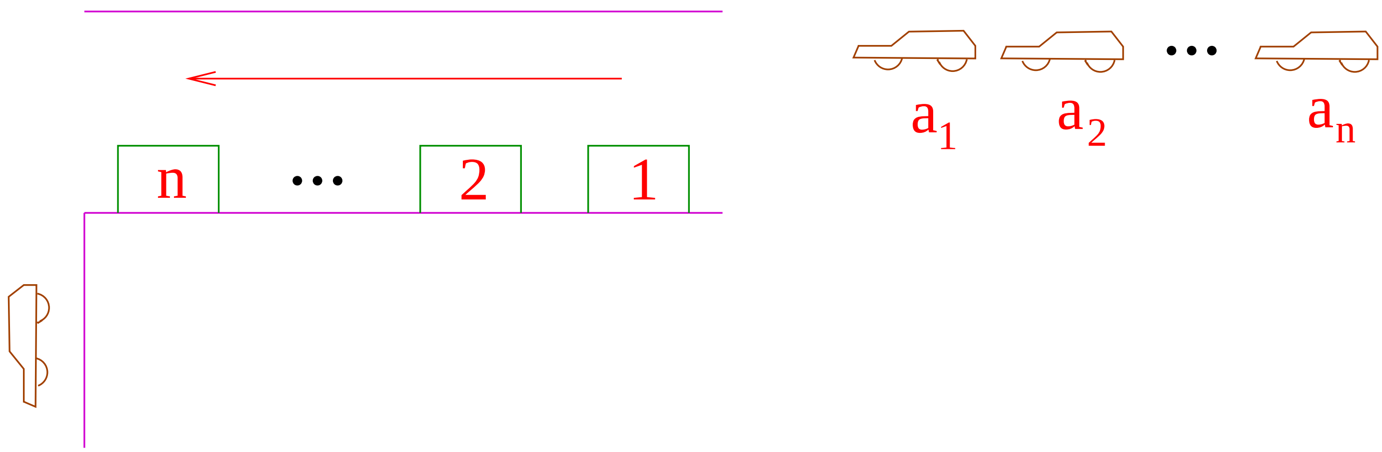
PARKING FUNCTIONS

Richard P. Stanley
Department of Mathematics
M.I.T. 2-375
Cambridge, MA 02139
rstan@math.mit.edu
http://www-math.mit.edu/~rstan

Transparencies available at:

http://www-math.mit.edu/~rstan/trans.html

ENUMERATION OF PARKING FUNCTIONS



Car C_i prefers space a_i. If a_i is occupied, then C_i takes the next available space. We call $(a_1, \ldots, a_n)$ a **parking function** (of length n) if all cars can park.

$\boldsymbol{n=2}$: 11 12 21

$\boldsymbol{n=3}$: 111 112 121 211
113 131 311 122
212 221 123 132
213 231 312 321

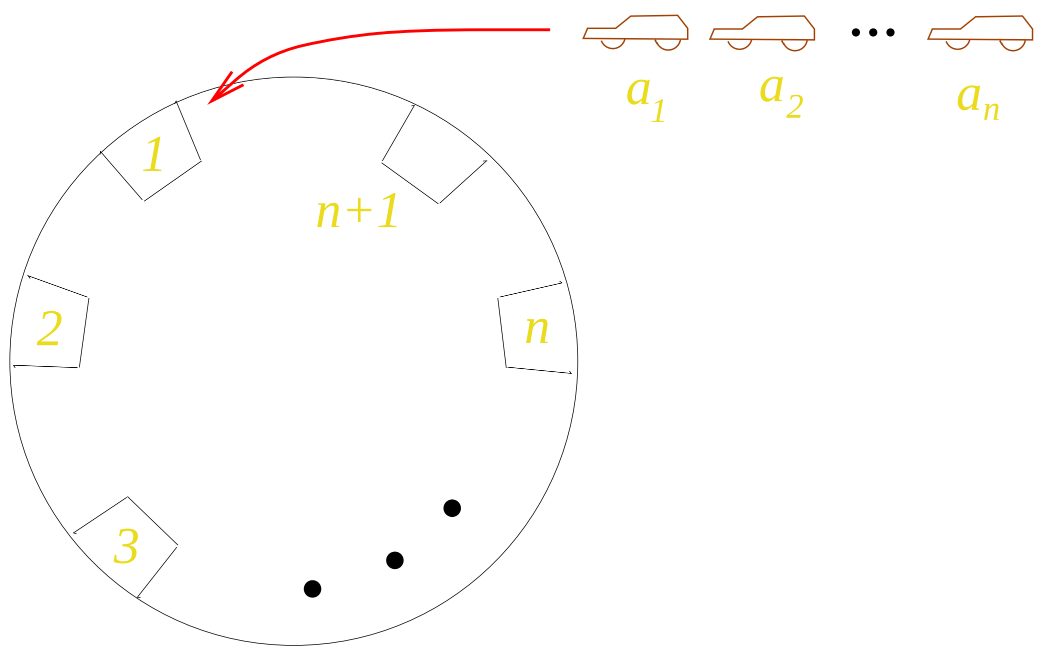
Easy: Let $\alpha = (a_1, \ldots, a_n) \in \mathbb{P}^n$. Let $b_1 \le b_2 \le \cdots \le b_n$ be the increasing rearrangement of α. Then α is a parking function if and only $b_i \le i$.

Corollary. *Every permutation of the entries of a parking function is also a parking function.*

Theorem (Konheim and Weiss, 1966). *Let $f(n)$ be the number of parking functions of length n. Then*

$$f(n) = (n+1)^{n-1}.$$

Proof (Pollak, c. 1974). Add an additional space $n+1$, and arrange the spaces in a circle. Allow $n+1$ also as a preferred space.



Now all cars can park, and there will be one empty space. α is a parking function if and only if the empty space is $n+1$. If $\alpha = (a_1, \ldots, a_n)$ leads to car C_i parking at space p_i, then $(a_1 + j, \ldots, a_n + j)$ (modulo $n+1$) will lead to car C_i parking at space $p_i + j$. Hence exactly one of the vectors

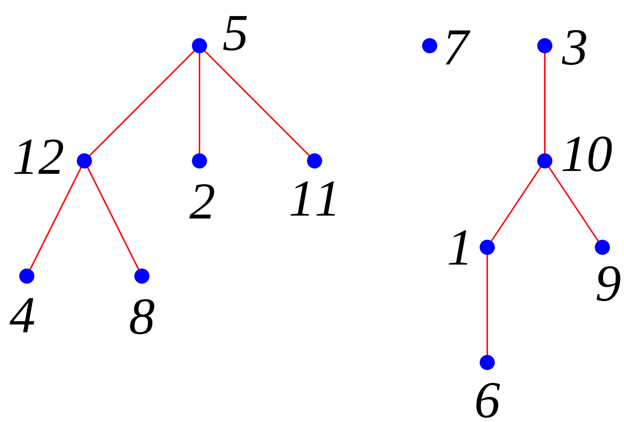
$$(a_1+i, a_2+i, \ldots, a_n+i) \text{ (modulo } n+1)$$

is a parking function, so

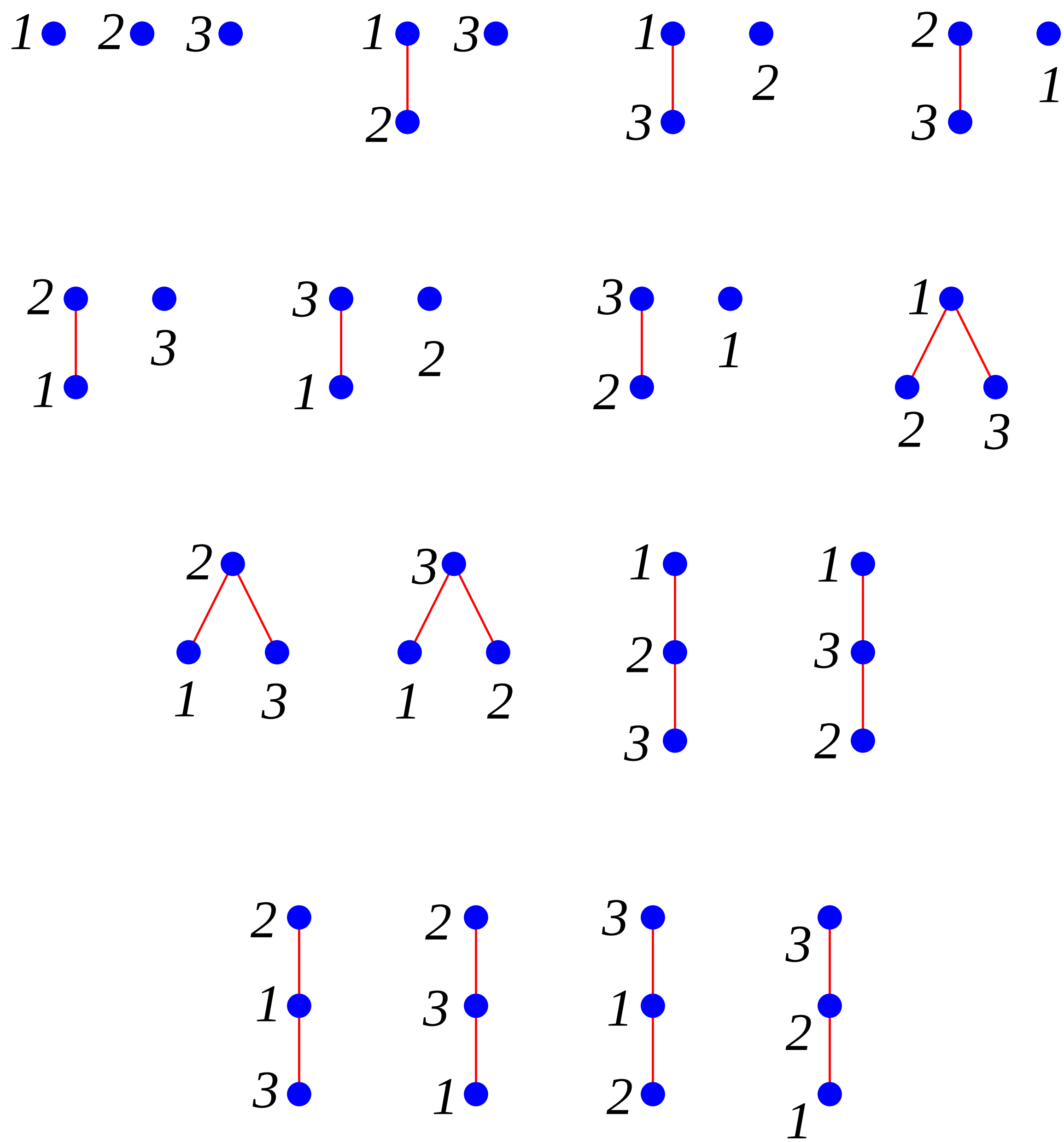
$$f(n) = \frac{(n+1)^n}{n+1} = (n+1)^{n-1}.$$

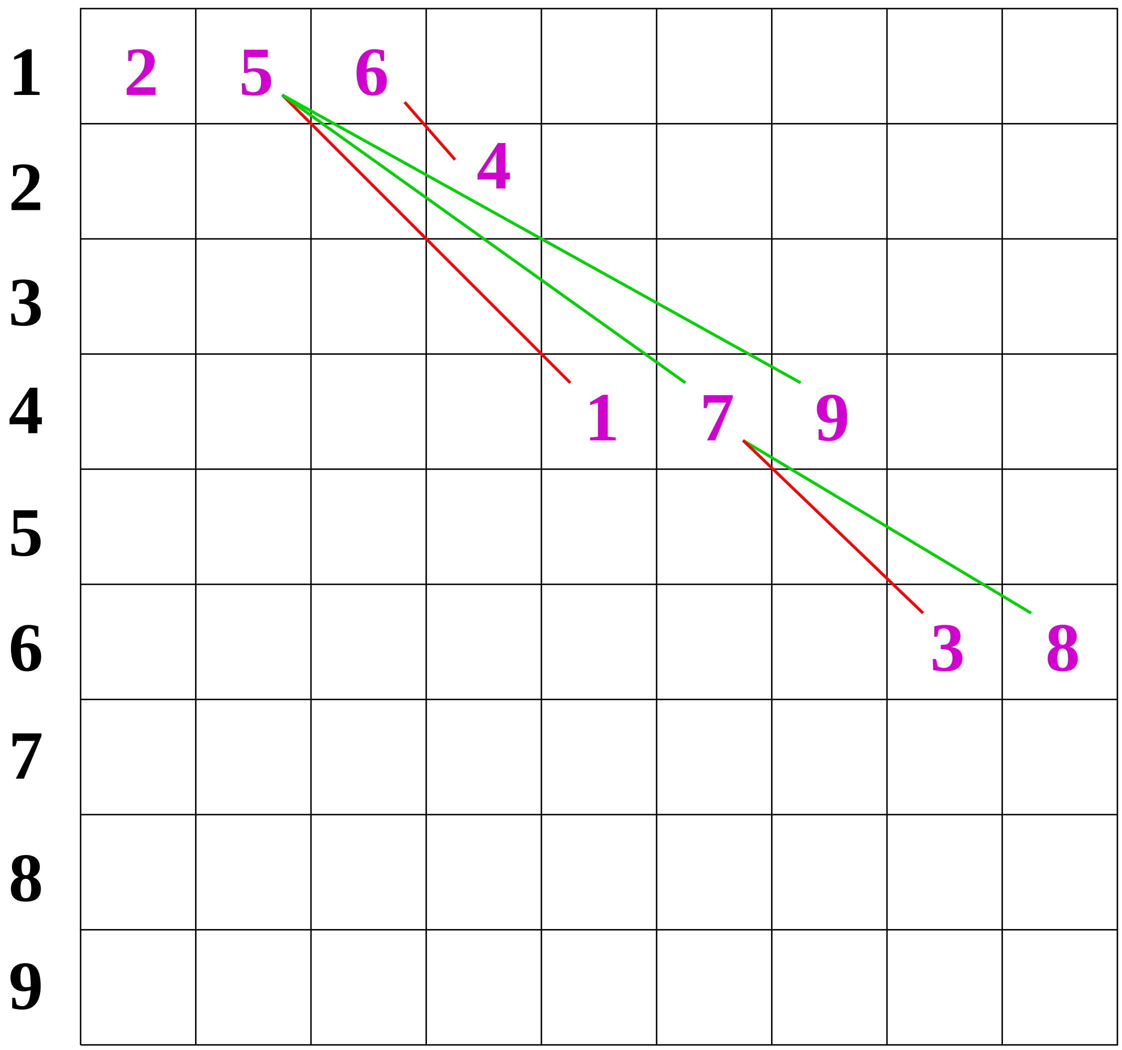
FORESTS

Let F be a rooted forest on the vertex set $\{1, \ldots, n\}$.



THEOREM (Sylvester-Borchardt-Cayley). *The number of such forests is* $(n+1)^{n-1}$.


1 2 3
1 3 2
1 2 3
2 1 3
2 3 1
3 2 1
3 1 2
1 2 3
2 1 3
3 1 2
1 2 3
1 3 2
2 1 3
2 3 1
3 1 2
3 2 1



1	2	3	4	5	6	7	8	9
4	1	6	2	1	1	4	6	4

Theorem. *The number of parking functions of length n with k 1's is the number of rooted forests on the vertex set $\{1, 2, \ldots, n\}$ with exactly k components (trees).*

Note: This number is equal to $\binom{n-1}{k-1} n^{n-k}$.

Exercise. Find a combinatorial proof analogous to Pollak's proof.

NONCROSSING PARTITIONS

A **partition** of a finite set S is a collection $\{B_1, \ldots B_k\}$ of subsets $\emptyset \neq B_i \subseteq S$ satisfying:

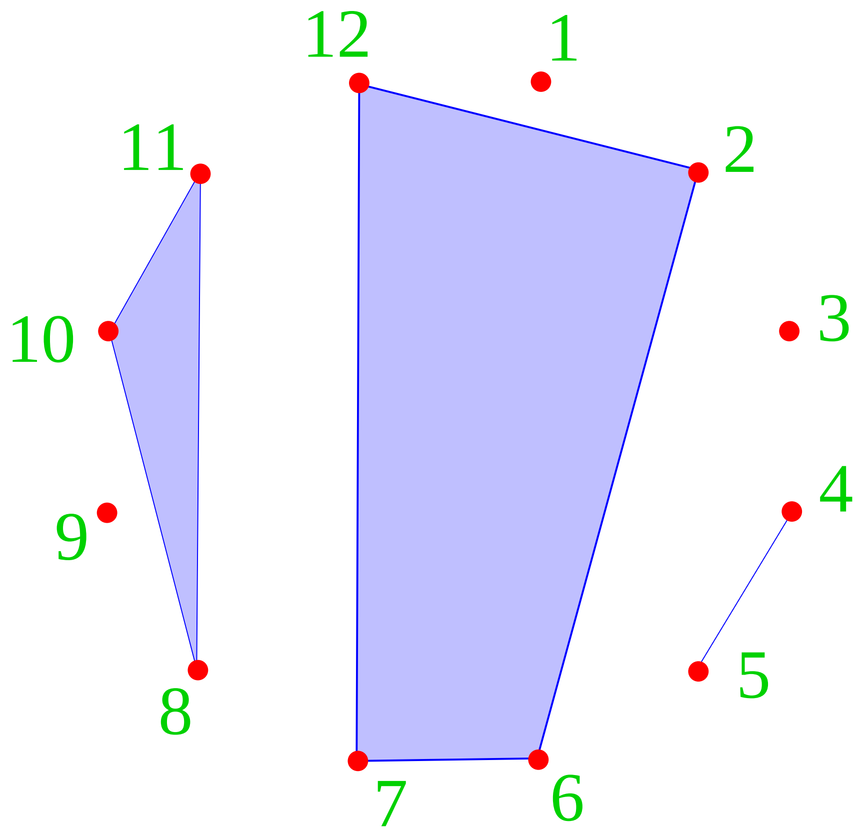
- $B_1 \cup B_2 \cup \cdots \cup B_k = S$
- $B_i \cap B_j = \emptyset$ if $i \neq j$

$n = 3$: 1-2-3, 12-3, 13-2, 1-23, 123 (five in all)

$n = 4$: 1-2-3-4 [1], ab-c-d [6], ab-cd [3], abc-d [4], 1234 [1] (15 in all)

A **noncrossing partition** of $\{1, 2, \ldots, n\}$ is a partition $\{B_1, \ldots, B_k\}$ of $\{1, \ldots, n\}$ such that

$$a < b < c < d,\ a, c \in B_i,\ b, d \in B_j \Rightarrow i = j.$$



Theorem (H. W. Becker, 1948–49) *The number of noncrossing partitions of* $\{1,\ldots,n\}$ *is the* **Catalan number**

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Example. Of the 15 partitions of $\{1,2,3,4\}$, only 13-24 is not noncrossing. Hence $C_4 = 15 - 1 = 14$.

For over 100 combinatorial interpretations of C_n, see

`www-math.mit.edu/~rstan/ec.html`

A **maximal chain** $\mathfrak{m}$ of noncrossing partitions of $\{1, \ldots, n+1\}$ is a sequence

$$\pi_0, \pi_1, \pi_2, \ldots, \pi_n$$

of noncrossing partitions of the set

$$\{1, \ldots, n+1\}$$

such that π_i is obtained from π_{i-1} by merging two blocks into one. (Hence π_i has exactly $n+1-i$ blocks.)

$$\mathbf{1-2-3-4-5}$$

$$\mathbf{1-25-3-4}$$

$$\mathbf{1-25-34}$$

$$\mathbf{125-34}$$

$$\mathbf{12345}$$

Define:

$$\mathbf{min}\, \boldsymbol{B} = \text{least element of } B$$

$$\boldsymbol{j < B} \;:\; j < k \;\; \forall k \in B.$$

Suppose π_i is obtained from π_{i-1} by merging together blocks B and B', with $\min B < \min B'$. Define

$$\begin{aligned} \boldsymbol{\Lambda_{\mathbf{i}}}(\mathfrak{m}) &= \max\{j \in B \;:\; j < B'\} \\ \boldsymbol{\Lambda}(\mathfrak{m}) &= (\Lambda_1(\mathfrak{m}), \ldots, \Lambda_n(\mathfrak{m})). \end{aligned}$$

For above example:

$$\mathbf{1-2-3-4-5,\ \ 1-25-3-4,\ \ 1-25-34,}$$

$$\mathbf{125-34,\ \ 12345}$$

we have

$$\Lambda(\mathfrak{m}) = (2, 3, 1, 2).$$

Theorem. *Λ is a bijection between the maximal chains of noncrossing partitions of $\{1,\ldots,n+1\}$ and parking functions of length n.*

Corollary (Kreweras, 1972) *The number of maximal chains of noncrossing partitions of $\{1,\ldots,n+1\}$ is*

$$(n+1)^{n-1}.$$

$$\mathbf{1-2-3,\quad 12-3,\quad 123:\quad (1,2)}$$

$$\mathbf{1-2-3,\quad 13-2,\quad 123:\quad (1,1)}$$

$$\mathbf{1-2-3,\quad 23-1,\quad 123:\quad (2,1)}$$

THE SHI ARRANGEMENT

Braid arrangement $\mathcal{B}_n$: the set of hyperplanes

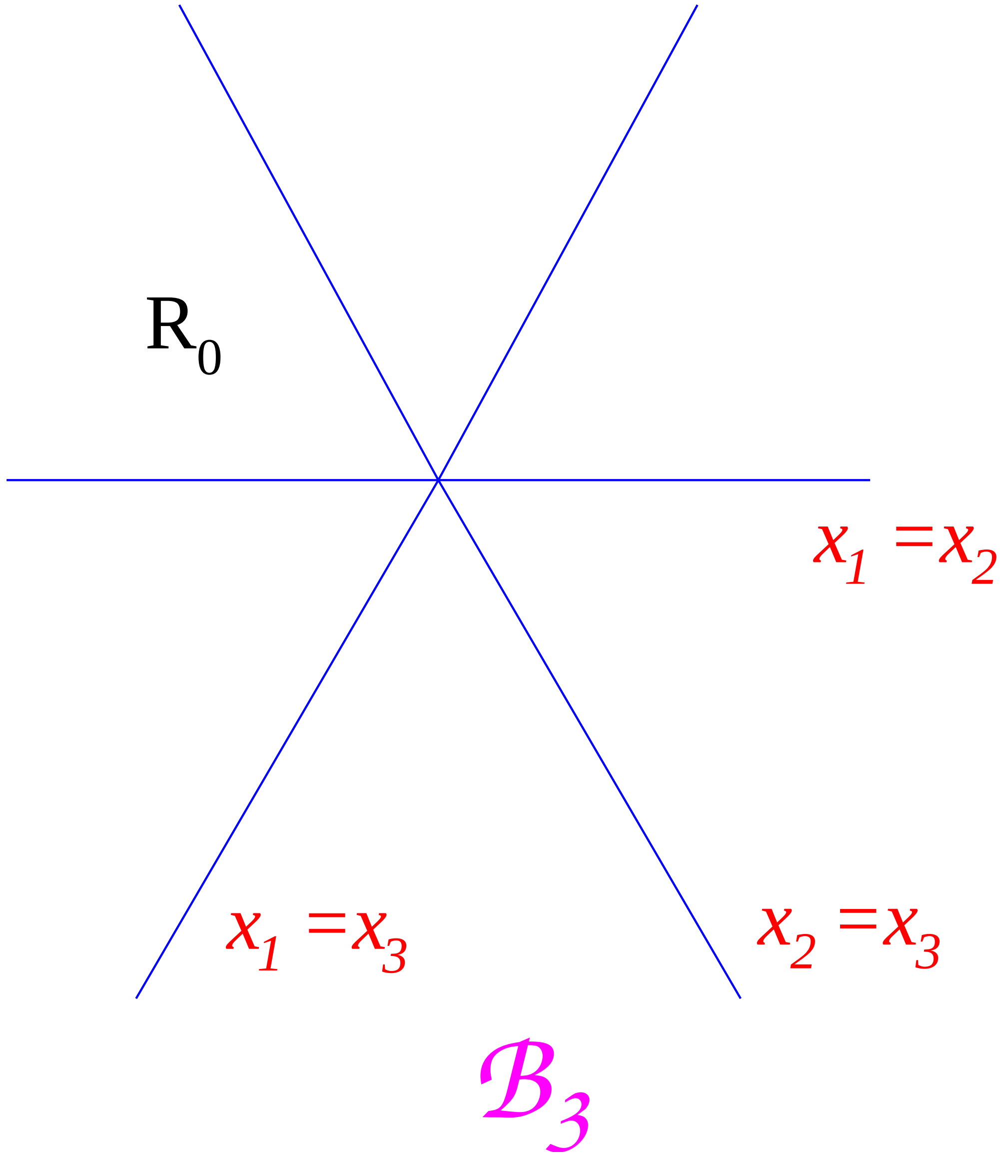
$$x_i - x_j = 0, \quad 1 \le i < j \le n,$$

in $\mathbb{R}^n$.

$$\begin{aligned} \mathcal{R} &= \text{set of regions of } \mathcal{B}_n \\ \#\mathcal{R} &= n! \end{aligned}$$

Let R_0 be the "base region"

$$R_0: \; x_1 > x_2 > \cdots > x_n.$$



R_0
$x_1 = x_2$
$x_1 = x_3$
$x_2 = x_3$
$\mathcal{B}_3$

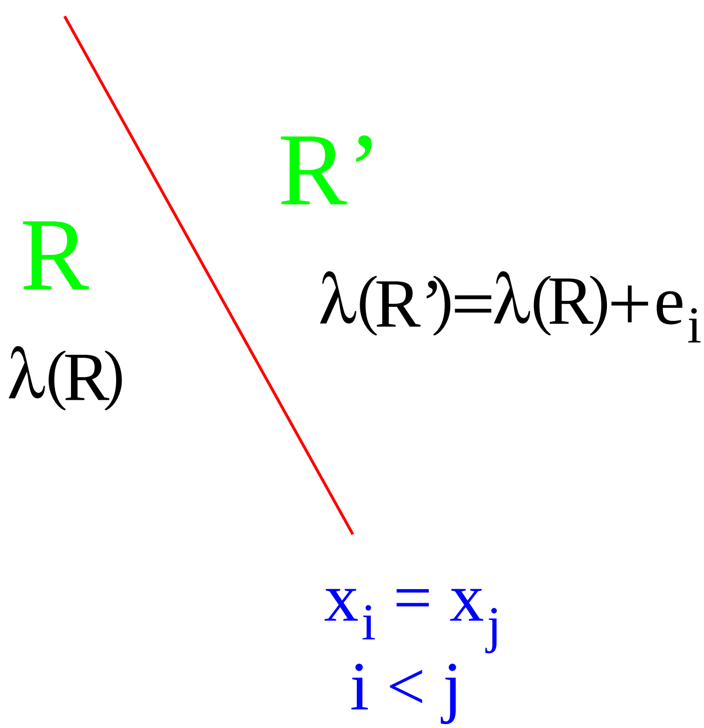
Label R_0 with

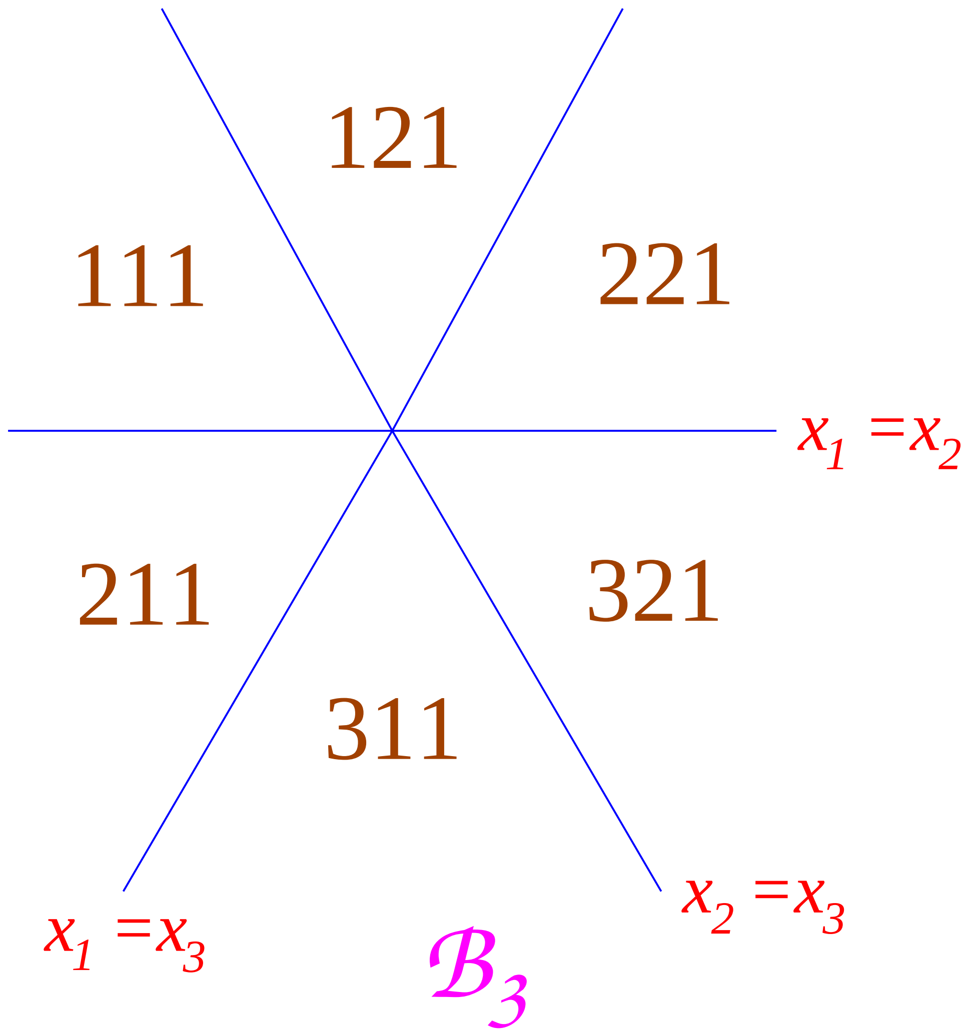
$$\lambda(R_0) = (1, 1, \ldots, 1) \in \mathbb{Z}^n.$$

If R is labelled, R' is separated from R only by $x_i - x_j = 0$ $(i < j)$, and R' is unlabelled, then set

$$\lambda(R') = \lambda(R) + e_i,$$

where $e_i = i$th unit coordinate vector.

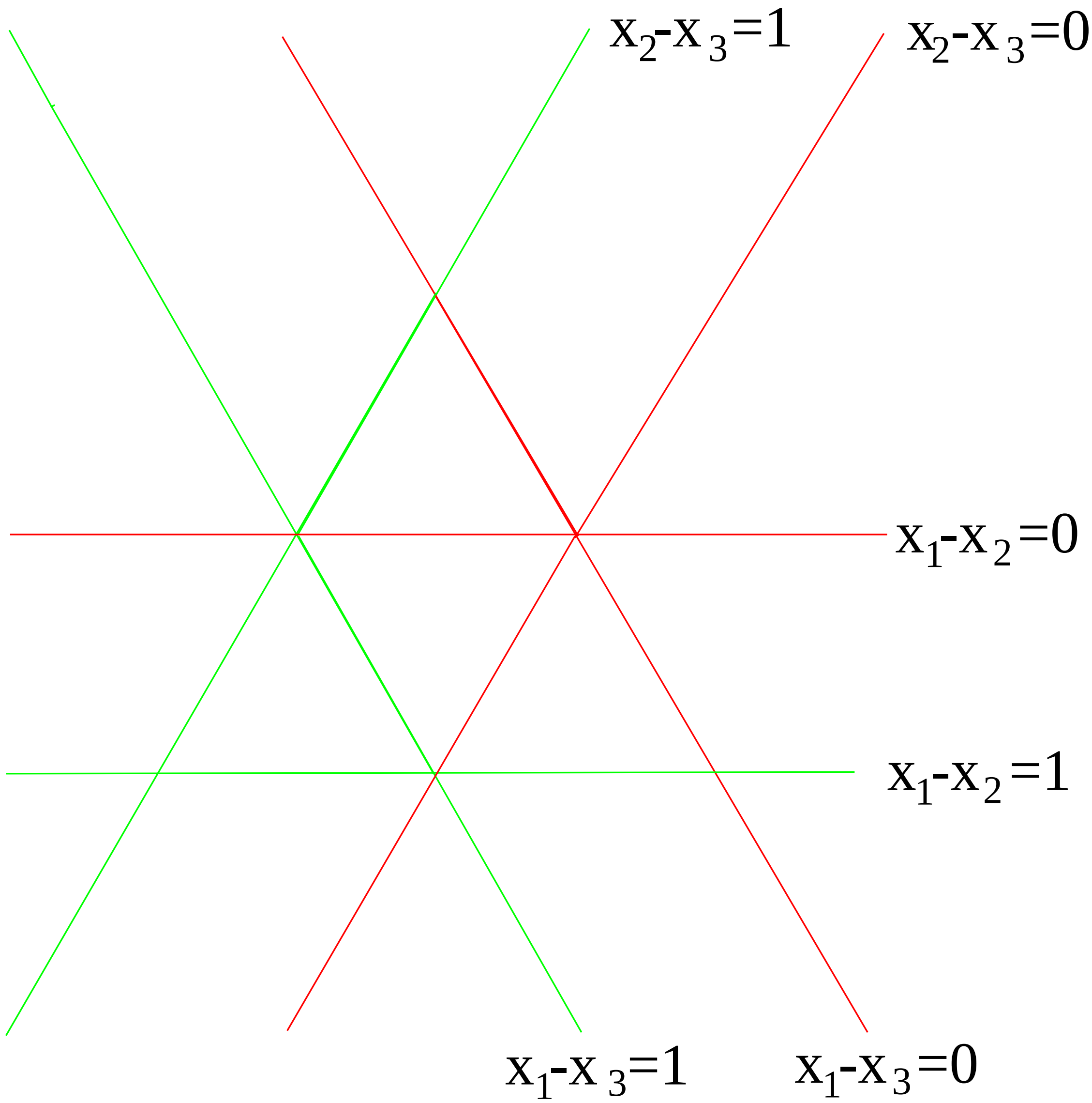




Theorem (easy). *The labels of* $\mathcal{B}_n$ *are the sequences* $(a_1, \ldots, a_n) \in \mathbb{Z}^n$ *such that* $1 \leq a_i \leq n - i + 1$.

Shi arrangement $\mathcal{S}_n$: the set of hyperplanes

$$x_i - x_j = 0, 1;\ 1 \le i < j \le n,\ \text{in } \mathbb{R}^n.$$



base region

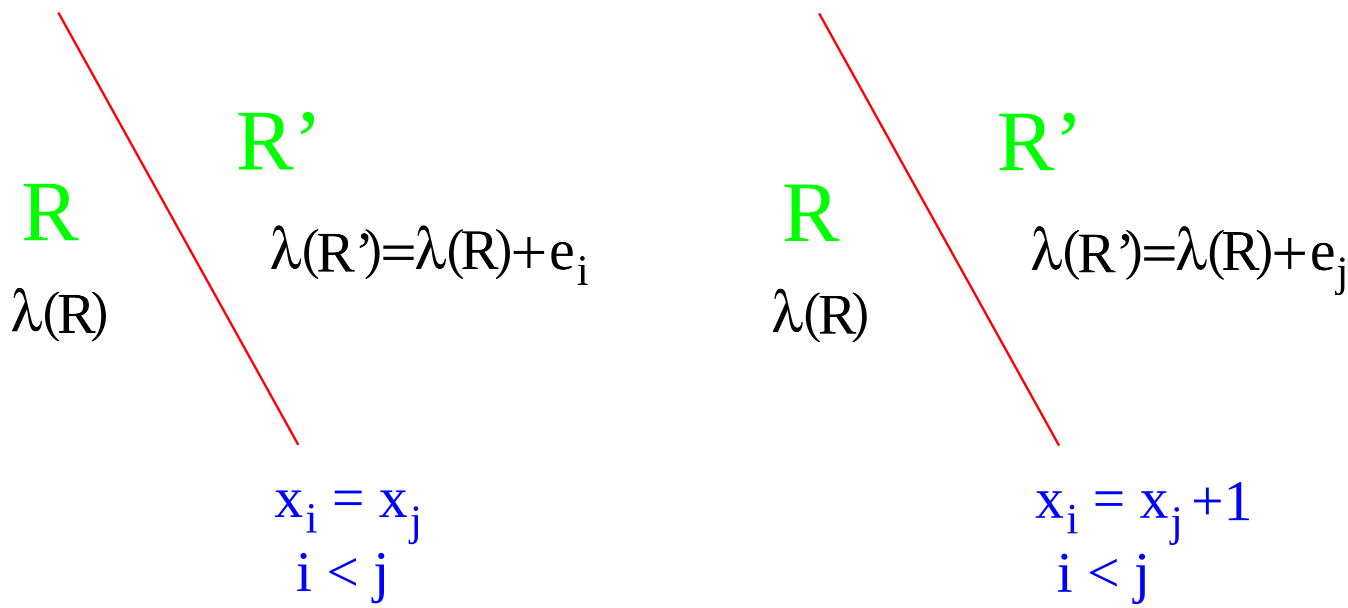
$$R_0 : \quad x_n + 1 > x_1 > \cdots > x_n$$

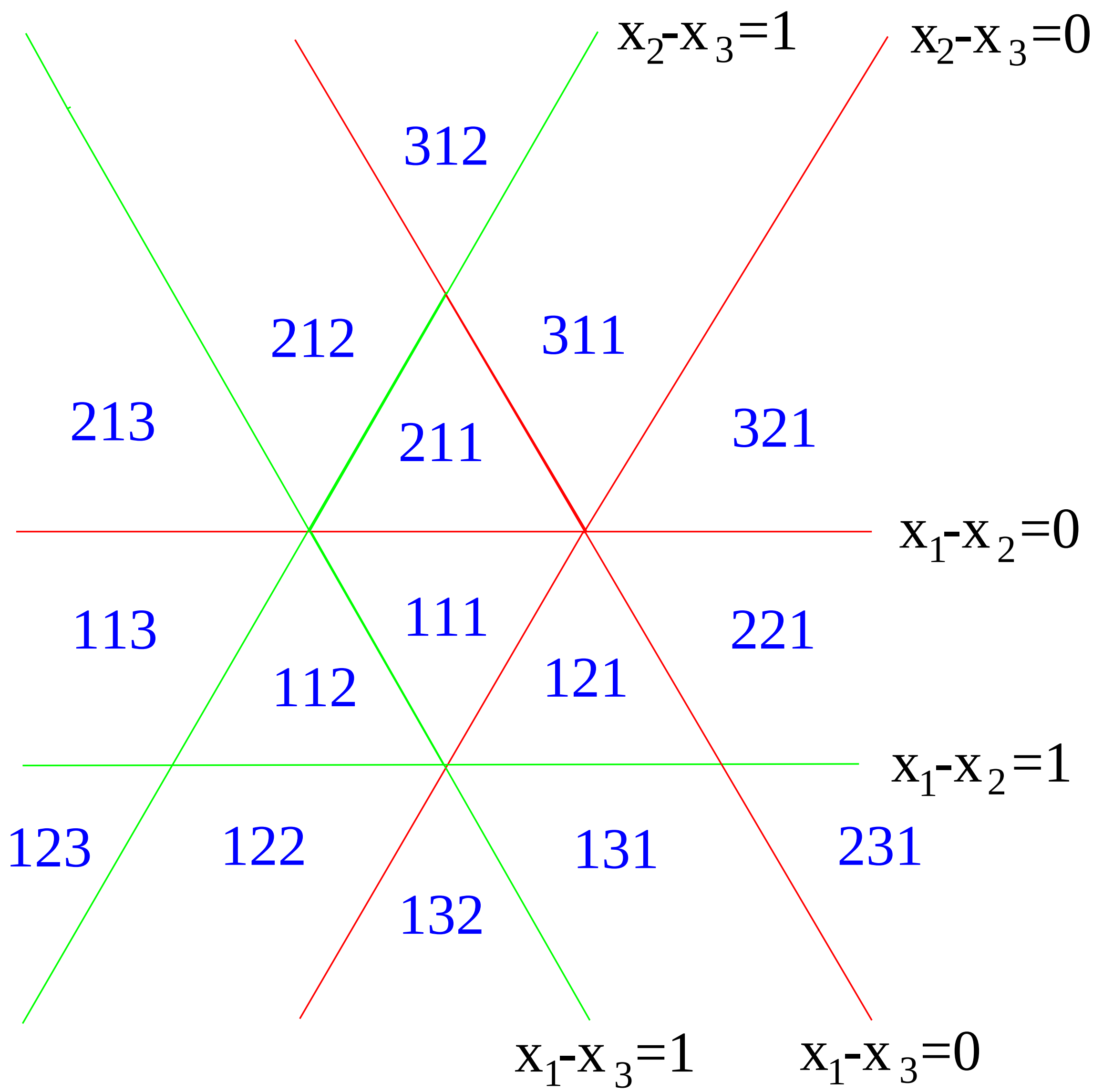
- $\lambda(R_0) = (1, 1, \ldots, 1) \in \mathbb{Z}^n$
- If R is labelled, R' is separated from R only by $x_i - x_j = 0$ $(i < j)$, and R' is unlabelled, then set

$$\lambda(R') = \lambda(R) + e_i.$$

- If R is labelled, R' is separated from R only by $x_i - x_j = 1$ $(i < j)$, and R' is unlabelled, then set

$$\lambda(R') = \lambda(R) + e_j.$$


R
R'
λ(R)
λ(R')=λ(R)+e_i
x_i = x_j
i < j
R
R'
λ(R)
λ(R')=λ(R)+e_j
x_i = x_j +1
i < j



Theorem (Pak, S.). *The labels of* $\mathcal{S}_n$ *are the parking functions of length* n *(each occurring once).*

Corollary (Shi, 1986)

$$r(\mathcal{S}_n) = (n+1)^{n-1}$$

A GENERALIZATION

Let

$$\lambda = (\lambda_1, \ldots, \lambda_n), \quad \lambda_1 \geq \cdots \geq \lambda_n > 0.$$

A **λ-parking function** is a sequence $(a_1, \ldots, a_n) \in \mathbb{P}^n$ whose increasing rearrangement $b_1 \leq \cdots \leq b_n$ satisfies $b_i \leq \lambda_{n-i+1}$.

Ordinary parking functions:

$$\lambda = (n, n-1, \ldots, 1)$$

Number (Steck 1968, Gessel 1996):

$$\mathbf{N}(\lambda) = n! \det \left[\frac{\lambda_{n-i+1}^{j-i+1}}{(j-i+1)!} \right]_{i,j=1}^{n}$$

THE PARKING FUNCTION POLYTOPE

Given $a_1, \ldots, a_n \in \mathbb{R}_{\geq 0}$, define

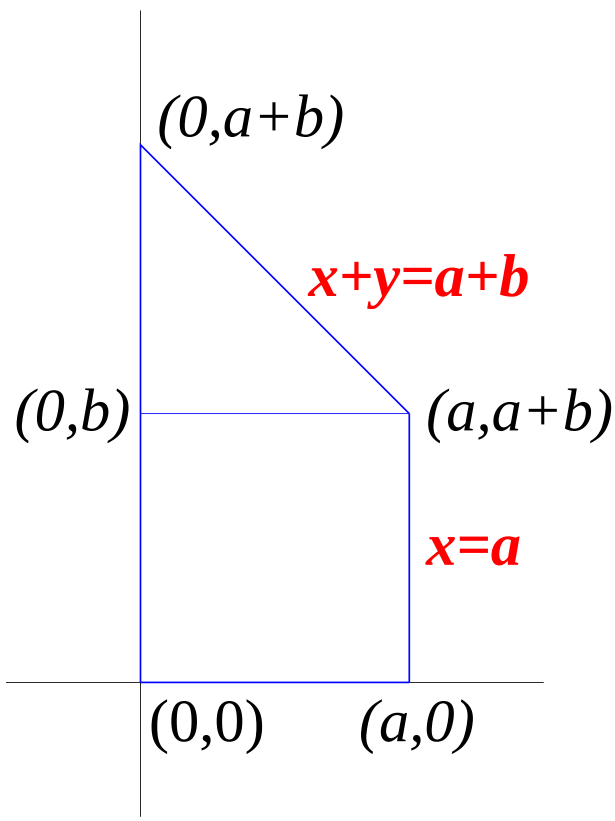
$$\boldsymbol{\mathcal{P} = \mathcal{P}(a_1, \ldots, a_n)} \subset \mathbb{R}^n$$

by: $(x_1, \ldots, x_n) \in \mathcal{P}_n$ if

$$\begin{aligned} x_i &\geq 0 \\ x_1 + \cdots + x_i &\leq a_1 + \cdots + a_i \end{aligned}$$

for $1 \leq i \leq n$.

$$\begin{aligned} n = 2: \qquad x, y &\geq 0 \\ x &\leq a \\ x + y &\leq a + b \end{aligned}$$



$$\text{area} = \frac{1}{2}(a^2 + 2ab)$$

Theorem. (a) *Let* $a_1, \ldots, a_n \in \mathbb{N}$. *Then*

$$n!\,V(\mathcal{P}_n) = N(\lambda),$$

where $\lambda_{n-i+1} = a_1 + \cdots + a_i$.

(b) $$n!\,V(\mathcal{P}_n) = \sum_{\substack{\text{parking functions} \\ (i_1,\ldots,i_n)}} a_{i_1} \cdots a_{i_n}.$$

Example. $n = 2$:

$$\begin{array}{ll} 11 & a^2 \\ 12 & ab \\ 21 & ba \end{array}$$

$$\Rightarrow \text{area} = \frac{1}{2}(a^2 + 2ab)$$

Note: If each $a_i > 0$, then $\mathcal{P}_n$ has the combinatorial type of an n-cube.

ALGEBRAIC ASPECTS OF PARKING FUNCTIONS

The symmetric group acts on the set $\mathcal{P}_{\mathbf{n}}$ of all parking functions of length n by permuting coordinates.

Sample properties:

- Multiplicity of trivial representation (number of orbits) $= C_n = \frac{1}{n+1}\binom{2n}{n}$

  $n = 3:$ **111 211 221 311 321**

- Number of elements of $\mathcal{P}_n$ fixed by $w \in \mathfrak{S}_n$ (character value at w):

  $$\#\mathrm{Fix}(w) = (n+1)^{(\#\,\text{cycles of } w)-1}$$

- Multiplicity of **any** irreducible representation: simple product formula

REFERENCES

1. H. W. Becker, Planar rhyme schemes, *Bull. Amer. Math. Soc.* **58** (1952), 39. *Math. Mag.* **22** (1948–49), 23–26

2. P. H. Edelman, Chain enumeration and non-crossing partitions, *Discrete Math.* **31** (1980), 171–180.

3. P. H. Edelman and R. Simion, Chains in the lattice of non-crossing partitions, *Discrete Math.* **126** (1994), 107–119.

4. D. Foata and J. Riordan, Mappings of acyclic and parking functions, *aequationes math.* **10** (1974), 10–22.

5. J. Françon, Acyclic and parking functions, *J. Combinatorial Theory (A)* **18** (1975), 27–35.

6. I. Gessel and D.-L. Wang, Depth-first search as a combinatorial correspondence, *J. Combinatorial Theory (A)* **26** (1979), 308–313.

7. P. Headley, Reduced expressions in infinite Coxeter groups, Ph.D. thesis, University of Michigan, Ann Arbor, 1994.

8. P. Headley, On reduced expressions in affine Weyl groups, in *Formal Power Series and Algebraic Combinatorics, FPSAC '94, May 23–27, 1994*, DIMACS preprint, pp. 225–232.

9. A. G. Konheim and B. Weiss, An occupancy discipline and applications, *SIAM J. Applied Math.* **14** (1966), 1266–1274.

10. G. Kreweras, Sur les partitions non croisées d'un cycle, *Discrete Math.* **1** (1972), 333–350.

11. G. Kreweras, Une famille de polynômes ayant plusieurs propriétés énumeratives, *Periodica Math. Hung.* **11** (1980), 309–320.

12. J. Lewis, Parking functions and regions of the Shi arrangement, preprint dated 1 August 1996.

13. C. L. Mallows and J. Riordan, The inversion enumerator for labeled trees, *Bull Amer. Math. Soc.* **74** (1968), 92–94.

14. J. Pitman and R. Stanley, A polytope related to empirical distributions, plane trees, parking functions, and the associahedron, *Discrete Comput. Geom.* **27** (2002), 603–634.

15. Y. Poupard, Étude et denombrement paralleles des partitions non croisées d'un cycle et des decoupage d'un polygone convexe, *Discrete Math.* **2** (1972), 279–288.

16. V. Reiner, Non-crossing partitions for classical reflection groups, *Discrete Math.* **177** (1997), 195–222.

17. J.-Y. Shi, The Kazhdan-Lusztig cells in certain affine Weyl groups, *Lecture Note in Mathematics*, no. 1179, Springer, Berlin/Heidelberg/New York, 1986.

18. J.-Y. Shi, Sign types corresponding to an affine Weyl group, *J. London Math. Soc.* **35** (1987), 56–74.

19. R. Simion, Combinatorial statistics on non-crossing partitions, *J. Combinatorial Theory (A)* **66** (1994), 270–301.

20. R. Simion and D. Ullman, On the structure of the lattice of noncrossing partitions, *Discrete Math.* **98** (1991), 193–206.

21. R. Speicher, Multiplicative functions on the lattice of noncrossing partitions and free convolution, *Math. Ann.* **298** (1994), 611–624.

22. R. Stanley, Hyperplane arrangements, interval orders, and trees, *Proc. Nat. Acad. Sci.* **93** (1996), 2620–2625.

23. R. Stanley, Parking functions and noncrossing partitions, *Electronic J. Combinatorics* **4**, R20 (1997), 14 pp.

24. R. Stanley, Hyperplane arrangements, parking functions, and tree inversions, in *Mathematical Essays in Honor of Gian-Carlo Rota* (B. Sagan and R. Stanley, eds.), Birkhäuser, Boston/Basel/Berlin, 1998, pp. 359–375.

25. G. P. Steck, The Smirnov two-sample tests as rank tests, *Ann. Math. Statist.* **40** (1969), 1449-1466.

26. C. H. Yan, Generalized tree inversions and k-parking functions, *J. Combinatorial Theory (A)* **79** (1997), 268–280.