Sprout Symmetric Functions

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Symmetric functions

$\boldsymbol{K}$: a field of characteristic 0

$\boldsymbol{\Lambda_K} = \boldsymbol{\Lambda_K(x)}$: ring of symmetric functions over K in the variables $\boldsymbol{x} = (x_1, x_2, \dots)$

bases m_λ (monomial symmetric functions), p_λ (power sums), h_λ (complete), e_λ (elementary), s_λ (Schur): knowledge assumed

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If $K \subseteq \mathbb{R}$ and $B = \{b_\lambda\}$ is a K-basis for Λ_K, then $f \in \Lambda_K$ is ***b*-positive** if the expansion of f in the basis B has nonnegative coefficients.

Sprout sequences and their seeds

Definition. A sequence $\mathfrak{R} = (R_0 = 1, R_1, R_2, \dots)$ of symmetric functions is a **sprout sequence** if there exists a power series

$$\boldsymbol{F(t)} = \sum_{j \geq 0} a_j t^j \in K[[t]],\ \ a_0 = 1$$

such that

$$\mathcal{F}\boldsymbol{(t)} := \prod_i F(x_i t) = \sum_{n \geq 0} R_n t^n.$$

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$F(t)$ is the **seed** of the sprout sequence $\mathfrak{R}$.

We also call $R_0, R_1, \dots$ **sprout symmetric functions** (with respect to the seed $F(t)$). Note $R_0 = 1, R_1 = a_1 \sum x_i = a_1 p_1$.

Simple examples

1. $F(t) = e^t$. Then

$$\begin{aligned} \mathcal{F}(t) &= F(x_1 t)F(x_2 t)\cdots = \exp(x_1 t + x_2 t + \cdots) \\ &= \exp(p_1 t) = \sum_{n\geq 0} p_1^n \frac{t^n}{n!}, \end{aligned}$$

whence $R_n = \frac{p_1^n}{n!} = \frac{e_1^n}{n!} = \frac{h_1^n}{n!}$.

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2. $F(t) = 1 + t$, so $\mathcal{F}(t) = (1 + x_1 t)(1 + x_2 t)\cdots = \sum_{n\geq 0} e_n t^n$, whence $R_n = e_n = s_{1^n}$

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3. $F(t) = 1/(1 - t)$, so $\mathcal{F}(t) = 1/(1 - x_1 t)(1 - x_2 t)\cdots = \sum_{n\geq 0} h_n t^n$, whence $R_n = h_n = s_n$

Other occurrences (culture)

- symmetric function generalization of the Tutte polynomial of a graph:
$$F(t) = \sum_{n \geq 0} (1+v)^{\binom{n}{2}} \frac{t^n}{n!}.$$

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- A-genus of spin manifolds:

$$F(t) = \frac{\frac{1}{2}\sqrt{t}}{\sinh(\frac{1}{2}\sqrt{t})}$$

Other occurrences (continued)

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- zeta polynomials of intervals of binomial posets with factorial function $B(n)$:

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Five characterizations of sprout sequence

We state one here to give the flavor.

Theorem. *Let* $\mathfrak{R} = (R_0 = 1, R_1, R_2, \dots)$ *be a sequence of symmetric functions. The following two conditions are equivalent.*

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$$R_n = \sum_{\lambda \vdash n} z_\lambda^{-1} b_{\lambda_1} b_{\lambda_2} \cdots p_\lambda.$$

In fact, $\log F(t) = \sum_{n \geq 1} b_n \frac{t^n}{n}$.

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Proof is straightforward.

The involution ω

Recall $\boldsymbol{\omega}\colon \Lambda_K \to \Lambda_K$ is the linear transformation defined by $\omega(h_\lambda) = e_\lambda$. Then ω is a K-algebra automorphism, $\omega^2 = 1$, $\omega(s_\lambda) = s_{\lambda'}$, and $\omega(p_n) = (-1)^{n-1} p_n$.

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Theorem. *Let* $\mathfrak{R} = (1, R_1, R_2, \dots)$ *be a sprout sequence with seed* $F(t)$*. Then* $(1, \omega R_1, \omega R_2, \dots)$ *is a sprout sequence with seed* $1/F(-t)$*.*

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Example. $F(t) = 1 + t$ and $R_n = e_n$. Then $1/F(-t) = 1/(1 - t)$ and $R_n = h_n$.

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(b) *We can write*

$$F(t) = e^{\gamma t} \prod_{k \geq 1} \frac{1 + \alpha_k t}{1 - \beta_k t},$$

where $\gamma \geq 0$ and the α_k's and β_k's are nonnegative real numbers such that $\sum_j (\alpha_k + \beta_k)$ is convergent. (This is an analytic, not a formal or combinatorial, statement.)

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(c) *The matrix $[a_{j-i}]_{i,j \geq 0}$ (where $a_n = 0$ if $n < 0$) is* **totally nonnegative**, *i.e., every minor is nonnegative.*

Proof

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(a) $\Leftrightarrow$ (c): how does the matrix $[a_{j-i}]_{i,j\geq 0}$ enter the picture? Based on the matrix $[h_{j-i}]$.

A corollary

Corollary. *Let* $d \geq 1$. *If the seed* $F(t) = \sum a_i t^i$ *generates a Schur positive sprout sequence* $\mathfrak{R}$, *then* $F_d(t) := \sum a_{di} t^i$ *generates a Schur positive sprout sequence* $\mathfrak{R}_d$.

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Proof. Let $M_d = [a_{d(j-i)}]_{i,j \geq 0}$. Every minor of M_1 is nonnegative since $\mathfrak{R}$ is Schur positive. But M_d is a submatrix of M_1, so every minor of M_d is Schur positive. Hence $\mathfrak{R}_d$ is Schur positive. □.

e and *h*-positivity

Recall: *e*-positivity $\Rightarrow$ Schur positivity and *h*-positivity $\Rightarrow$ Schur positivity.

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Proposition.

(a) *If all* $\beta_j = 0$*, then each* R_n *is e-positive.*

(b) *If all* $\alpha_j = 0$*, then each* R_n *is h-positive.*

Easy proof

Proposition (repeated).

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Proof. (a) Assume all $\beta_j = 0$. Then

$$\begin{aligned}
\sum R_n t^n &= \prod_i e^{\gamma x_i t} \prod_{j\geq 1} (1 + \alpha_j x_i t) \\
&= e^{\gamma e_1 t} \prod_j \prod_i (1 + \alpha_j x_i t) \\
&= e^{\gamma e_1 t} \prod_j \left(\sum_{n\geq 0} \alpha_j^n e_n t^n \right), \text{ etc.}
\end{aligned}$$

(b) is completely analogous. □

The converse

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Theorem. *Let* $F(t) = e^{\gamma t} \prod_{j \geq 1} \frac{1+\alpha_j t}{1-\beta_j t}$.

Suppose that there exists $\alpha > 0$ *such that the multiplicity of* $1 + \alpha t$ *in the numerator of* $F(t)$ *exceeds the multiplicity of* $1 - \alpha t$ *in the denominator. Then some* R_n *is not h-positive.* [dually for *e*-positivity]

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Proof uses complex analysis (**Vivanti-Pringsheim** theorem).


INTERMISSION

The function $\phi(\lambda)$

Amdeberhan-Ono-Singh (2024):

$$\phi(\lambda) := (2n)! \cdot \prod_{k=1}^{n} \frac{1}{m_k!} \left(\frac{4^k(4^k - 1)B_{2k}}{(2k)(2k)!} \right)^{m_k},$$

where $\lambda = \langle 1^{m_1}, \ldots, n^{m_n} \rangle \vdash n = \sum i m_i$ (λ is a partition of n with m_i i's) and B_{2k} is a Bernoulli number.

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Original motivation. Express a certain theta function of Ramanujan in terms of Eisenstein series (not explained here).

Euler numbers E_{2n}

Our motivation. Not hard to see that

$$\phi(\lambda) \in \mathbb{Z}, \quad \sum_{\lambda \vdash n} |\phi(\lambda)| = E_{2n},$$

an **Euler number** or **secant number**, defined by

$$\sec x = \sum_{n \geq 0} E_{2n} \frac{x^{2n}}{(2n)!}.$$

Well-known: E_{2n} is equal to the number of **alternating permutations** $a_1 a_2 \cdots a_{2n} \in \mathfrak{S}_{2n}$, i.e.,

$$a_1 > a_2 < a_3 > a_4 < \cdots > a_{2n}.$$

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Question: what does $|\phi(\lambda)|$ count?

Record partitions

$$\mathfrak{A}_{2n} := \{w \in \mathfrak{S}_{2n} \,:\, w \text{ alternating}\}$$

Recall $\sum_{\lambda \vdash n} |\phi(\lambda)| = E_{2n} = \#\mathfrak{A}_{2n}$.

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If $w = a_1 > a_2 < \cdots > a_{2n} \in \mathfrak{A}_{2n}$ define $\hat{w} = a_1, a_3, \ldots, a_{2n-1}$. Write $\hat{w} = b_1, b_2, , \ldots, b_n$.

record set $\mathrm{rec}(\hat{w})$: set of indices $1 \leq i \leq n$ for which b_i is a left-to-right maximum (or **record**) in $\hat{w}$. (Always $1 \in \mathrm{rec}(\hat{w})$.)

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Example. $w = 7, 2, 5, 1, 8, 3, 10, 6, 9, 4 \in \mathfrak{A}_{10}$, $\hat{w} = 7, 5, 8, 10, 9$;
$r_1 = 1$, $r_2 = 3$, $r_3 = 4$, $r_2 - r_1 = 2$, $r_3 - r_2 = 1$, $6 - r_3 = 2$,
$\mathrm{rp}(\hat{w}) = (2, 2, 1)$

Combinatorial interpretation of $\phi(\lambda)$

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Note on proof. Recall

$$\phi(\lambda) = (2n)! \cdot \prod_{k=1}^{n} \frac{1}{m_k!} \left(\frac{4^k(4^k - 1)B_{2k}}{(2k)(2k)!} \right)^{m_k},$$

where $\lambda = \langle 1^{m_1}, \dots, n^{m_n} \rangle \vdash \sum i m_i$. To get combinatorics into the picture, use

$$E_{2k-1} = 4^k(4^k - 1)\frac{|B_{2k}|}{2k}.$$

Remainder of proof is a bijective argument.

A symmetric function

The general form $\phi(\lambda) = (2n)! \prod \frac{1}{m_k!} f_k^{m_k}$ suggests defining a symmetric function in the variables $\boldsymbol{x} = (x_1, x_2, \ldots)$:

$$A_n = A_n(\boldsymbol{x}) = \frac{1}{(2n)!} \sum_{\lambda \vdash n} |\phi(\lambda)| \cdot p_\lambda,$$

where $\boldsymbol{p_\lambda}$ is a power sum symmetric function.

Examples.

$$
\begin{aligned}
2!\,A_1 &= p_1\\
4!\,A_2 &= 3p_1^2 + 2p_2\\
6!\,A_3 &= 15p_1^3 + 30p_2p_1 + 16p_3\\
8!\,A_4 &= 105p1^4 + 420p_2p_1^2 + 140p_2^2 + 448p_3p_1 + 272p_4
\end{aligned}
$$

$4!\,A_2$:

w	$\hat{w}$	$\mathrm{rp}(\hat{w})$
2143	24	11
3142	34	11
3241	34	11
4132	43	2
4231	43	2

A sprout sequence

Theorem. $\sum A_n t^n = \prod_i \sec(\sqrt{x_i t})$, *i.e.*, $\mathfrak{A} := (A_0, A_1, \dots)$ *is a sprout sequence with seed* $\sec\sqrt{t}$.

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Proof idea. Use earlier characterization of sprout sequences:

There exist elements $b_0 = 1, b_1, b_2, \dots$ in K such that for all $n \geq 1$,

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In fact, $\log F(t) = \sum_{n\geq 1} b_n \frac{t^n}{n}$.

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Here we have $\frac{d}{dt} \log\sec(\sqrt{t}) = \tan(\sqrt{t})/2\sqrt{t}$, so $b_n = E_{2n-1}/(2n)!$, etc. .

h-positivity

Theorem. $A_n(\boldsymbol{x})$ *is h-positive.*

h-positivity

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Very noncombinatorial formula for the coefficients!

Some data

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\begin{aligned}
2!A_1 &= h_1 \\
4!A_2 &= h_1^2 + 4h_2 \\
6!A_3 &= h_1^3 + 12h_2h_1 + 48h_3 \\
8!A_4 &= h_1^4 + 24h_2h_1^2 + 256h_3h_1 + 16h_2^2 + 1088h_4 \\
10!A_5 &= h_1^5 + 40h_2h_1^3 + 800h_3h_1^2 + 80h_2^2h_1 + 9280h_4h_1 \\
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Note. Coefficient of h_n is nE_{2n-1}, the number of "cyclically alternating" permutations in $\mathfrak{S}_{2n}$.

Chromatic symmetric functions

G: finite simple graph on vertex set $V(G) = \{v_1, v_2, \ldots, v_p\}$

$$X_G = X_G(x) := \sum_{\substack{\kappa\colon V(G)\to\mathbb{P} \\ uv\in E(G)\Rightarrow\kappa(u)\neq\kappa(v)}} x_{\kappa(v_1)} x_{\kappa(v_2)} \cdots x_{\kappa(v_p)}$$

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$$X_G(\underbrace{1, 1, \ldots, 1}_{m\ 1\text{'s}}, 0, 0, \ldots) = \chi_G(m),$$

the **chromatic polynomial** of G.

Interval orders

$\mathcal{I} = \{[a_1, b_1], \ldots, [a_n, b_n]\}$, a collection of closed intervals in $\mathbb{R}$, so $a_i < b_i$.

$G_{\mathcal{I}}$: graph with vertex set $\mathcal{I}$, with $[a_i, b_i]$ adjacent to $[a_j, b_j]$ if $[a_i, b_i] \cap [a_j, b_j] \neq \emptyset$ (incomparability graph of the corresponding interval order: $[a_i, b_i] < [a_j, b_j]$ if $b_i < a_j$]).

M: a complete matching $a_1 b_1, a_2 b_2, \ldots, a_n b_n$ on $[2n] := \{1, 2, \ldots, 2n\}$, with $a_i < b_i$ (so $\{a_1, b_1, \ldots, a_n, b_n\} = [2n]$)

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Theorem. $(2n)!\,\omega(A_n) = \sum_{M \in \mathcal{M}_n} X_{G_{\mathcal{I}(M)}}$, *where* $\mathcal{M}_n$ *is the set of all* $(2n-1)!!$ *complete matchings on* $[2n]$*, and* $X_{G_{\mathcal{I}(M)}}$ *is the chromatic symmetric function of the graph* $G_{\mathcal{I}(M)}$.

The case $n = 2$

matching M	graph $G_{\mathcal{I}(M)}$	$X_{G_{\mathcal{I}(M)}}$
12, 34	• •	e_1^2
13, 24	•——•	$2e_2$
14, 23	•——•	$2e_2$

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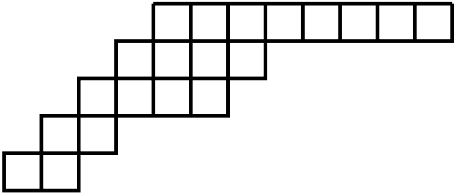
Equivalently, $4!A_2 = h_1^2 + 4h_2$.

Problem. Are there other "nice" examples of sums (or linear combinations) of X_G's being e-positive?

Schur function expansion

Example. To get the coefficient of s_{5311} in $20! \cdot A_{10}$, take the conjugate partition 42211 and double each part: $\boldsymbol{\mu} = 84422$. Form the skew shape $\boldsymbol{\rho(5311)}$:



Row lengths are the parts of μ.
Each row begins one square to the left of the row above.

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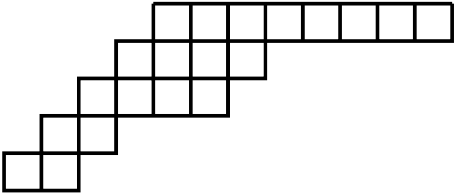

Row lengths are the parts of μ.
Each row begins one square to the left of the row above.

Theorem. *For general $\lambda \vdash n$, the coefficient of s_λ in $(2n)!A_n$ is the number $f^{\rho(\lambda)}$ of standard Young tableaux of (skew) shape $\rho(\lambda)$.*
(Well-known determinantal formula.)

First generalization

Let $c \geq 1$ and

$$F_c(t) = \left(\sum_{n \geq 0} \frac{(-1)^n t^n}{(cn)!} \right)^{-1}.$$

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m, p, s-expansions straightforward generalizations of $c = 2$ case. In particular, there are "natural" skew shapes $\boldsymbol{\rho(\lambda, c)}$ for which

$$(cn)! R_n = \sum_{\lambda \vdash n} f^{\rho(\lambda, c)} s_\lambda.$$

h-expansion of R_n for the seed $F_c(t)$

We don't know poles of $F_c(t)$ (a **Mittag-Leffler function**) explicitly for $c \geq 3$, but can show $F_c(t) = \prod(1 - \beta_j t)^{-1}$ either by a direct analytic argument or the earlier corollary:

Corollary. *Let $d \geq 1$. If the seed $F(t) = \sum a_i t^i$ generates a Schur positive sprout $\mathfrak{R}$, then $F_d(t) := \sum a_{di} t^i$ generates a Schur positive sprout $\mathfrak{R}_d$.*

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Recall coefficients of h-expansion of $(2n)!\, R_n$ for $F_2(t)$ sum to E_{2n}, and a combinatorial interpretation is open. For arbitrary c, the coefficients sum to

$$\#\{w \in \mathfrak{S}_{cn} : \mathrm{Des}(w) = \{c, 2c, 3c, \ldots, (n-1)c\}\},$$

where $\mathrm{Des}(w)$ denotes the descent set of w.

A q-analogue of $F_c(t)$

$$F_c(t,q) = \left(\sum_{n \geq 0} \frac{(-1)^n t^n}{(\boldsymbol{cn})!_q} \right)^{-1},$$

where $(m)!_q = 1 \cdot (1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{m-1})$, the standard q-analogue of $m!$.

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If $c = 2$ then

$$\mathbf{(4)!}_q\, R_2 = (q^4+q^3+2q^2+q)h_1^2 + (q^4+q^3+2q^2+q-1)h_2,$$

so (h,q)-positivity fails even for $c = 2$.

Schur expansion of R_n for the seed $F_d(q,t)$

Recall that for $F_c(t) = (\sum(-1)^n t^n/(cn)!)^{-1}$ we have

$$(cn)!R_n = \sum_{\lambda \vdash n} f^{\rho(\lambda,c)} s_\lambda. \qquad (*)$$

for some "natural" skew shape $\rho(\lambda, c)$.

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$$(\boldsymbol{cn})!_q\, R_n = \sum_{\lambda \vdash n} \left(\sum_{\substack{\text{SYT } T \\ \text{sh}(T) = \rho(\lambda,c)}} q^{\text{maj}(T)} \right) s_\lambda,$$

the "nicest" possible q-analogue of (*).

Second special case

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Theorem (**Carlitz-Scoville-Vaughan** (1976) for $d = 2$) *Let $d \geq 1$ and*

$$F(t) = \sum_{n \geq 0} v_d(n) \frac{t^n}{n!^d}.$$

Then

$$v_d(n) = \#\{(w_1, \ldots, w_d) \in \mathfrak{S}_n^d \; : \; \mathrm{Des}(w_1) \cap \cdots \cap \mathrm{Des}(w_d) = \emptyset\}.$$

Schur expansion

Let $F(t) = \left(\sum_{n \geq 0} \frac{(-1)^n t^n}{n!^d} \right)^{-1}$.

E.g.,$d = 2$, $3!^2 R_3 = s_{111} + 8 s_{21} + 19 s_3$.

What statistic on $\mathfrak{S}_3 \times \mathfrak{S}_3$ (or $\mathfrak{S}_n^d$ in general) do the coefficients count? (open)

h-expansion

Analytic methods (**M. Kwaśnicki**, MO 477780) show that

$$F(t) := \left(\sum_{n \geq 0} \frac{(-1)^n t^n}{n!^d} \right)^{-1} = \prod (1 - \beta_i t)^{-1},$$

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$$\begin{aligned} R_1 &= h_1 \\ 2!^2 R_2 &= h_1^2 + 2h_2 \\ 3!^2 R_3 &= h_1^3 + 6h_2 h_1 + 12h_3 \\ 4!^2 R_4 &= h_1^4 + 12h_2 h_1^2 + 60h_3 h_1 + 6h_2^2 + 132h_4 \end{aligned}$$

Coefficients of h-expansion

Open Problem. What do the coefficients count?

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Their sum is

$$v_d(n) = \#\{(w_1, \dots, w_d) \in \mathfrak{S}_n^d : \mathrm{Des}(w_1) \cap \cdots \cap \mathrm{Des}(w_d) = \emptyset\}.$$

The seed $F(t) = \left(\sum_{n\geq 0} t^n/(n)!_q^d\right)^{-1}$

Example. For $d = 2$,

$$\begin{aligned}(1)!_q^2 R_1 &= s_1\\ (2)!_q^2 R_2 &= s_{11} + (q^2 + 2q)s_2\\ (3)!_q^2 R_3 &= s_{111} + (q^4 + 2q^3 + 3q^2 + 2q)s_{21}\\ &\quad +(q^6 + 4q^5 + 6q^4 + 6q^3 + 2q^2)s_3\end{aligned}$$

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Conjecture (can be greatly generalized). For any $d \geq 1$, $(n)!_q^d R_n$ is (q, s)-positive. (It's not (q, h) or (q, e)-positive in general.)

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Note. No nice q-analogue of total positivity or the Edrei-Thoma theorem is known.

The final slide

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