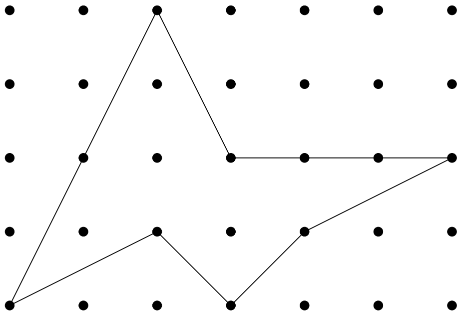
Lattice Points in Polytopes

Richard P. Stanley
U. Miami & M.I.T.

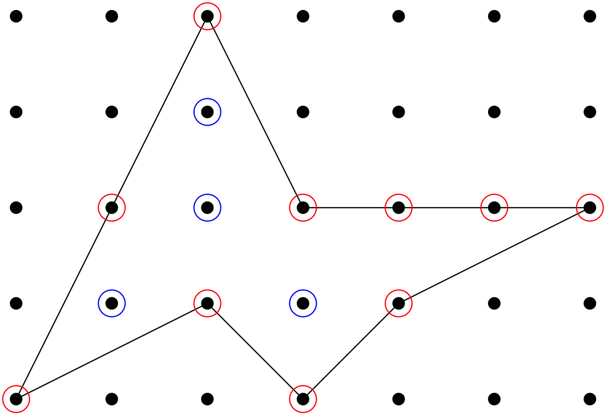
A lattice polygon

Georg Alexander Pick (1859–1942)

P: lattice polygon in $\mathbb{R}^2$
(vertices $\in \mathbb{Z}^2$, no self-intersections)



Boundary and interior lattice points



Pick's theorem

$$\begin{aligned} A &= \text{area of } P \\ I &= \#\text{ interior points of } P \ (= 4) \\ B &= \#\text{boundary points of } P \ (= 10) \end{aligned}$$

Then

$$A = \frac{2I + B - 2}{2}.$$

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Example on previous slide:

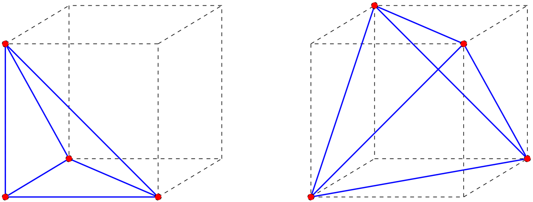
$$\frac{2 \cdot 4 + 10 - 2}{2} = 9.$$

Two tetrahedra

Pick's theorem (seemingly) fails in higher dimensions. For example, let T_1 and T_2 be the tetrahedra with vertices

$$\begin{aligned} v(T_1) &= \{(0,0,0),(1,0,0),(0,1,0),(0,0,1)\} \\ v(T_2) &= \{(0,0,0),(1,1,0),(1,0,1),(0,1,1)\}. \end{aligned}$$

Failure of Pick's theorem in dim 3



Then

$$I(T_1) = I(T_2) = 0$$

$$B(T_1) = B(T_2) = 4$$

$$A(T_1) = 1/6, \quad A(T_2) = 1/3.$$

Polytope dilation

Let $\mathcal{P}$ be a convex polytope (convex hull of a finite set of points) in $\mathbb{R}^m$. For $n \geq 1$, let

$$n\mathcal{P} = \{n\alpha : \alpha \in \mathcal{P}\}.$$

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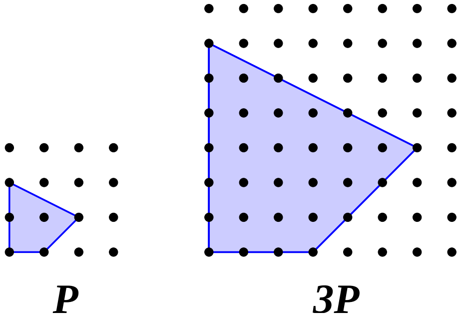

$i(\mathcal{P}, n)$

Let

$$\begin{aligned} i(\mathcal{P}, n) &= \#(n\mathcal{P} \cap \mathbb{Z}^m) \\ &= \#\{\alpha \in \mathcal{P} : n\alpha \in \mathbb{Z}^m\}, \end{aligned}$$

the number of lattice points in $n\mathcal{P}$.

$\bar{i}(\mathcal{P}, n)$

Similarly let

$$\mathcal{P}^\circ = \text{interior of } \mathcal{P} = \mathcal{P} - \partial\mathcal{P}$$

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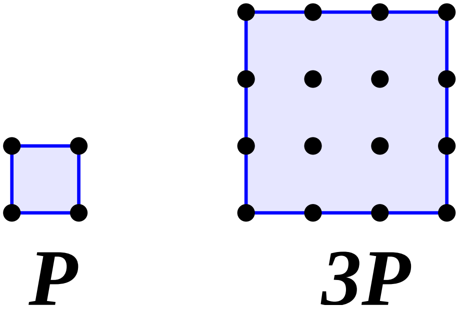
Note. Could use any lattice L instead of $\mathbb{Z}^m$.

An example



$$i(\mathcal{P}, n) = (n+1)^2$$

$$\bar{i}(\mathcal{P}, n) = (n-1)^2 = i(\mathcal{P}, -n).$$

The main result

Theorem (**Ehrhart** 1962, **Macdonald** 1963). *Let*

$$\mathcal{P} = \text{lattice polytope in } \mathbb{R}^m, \ \dim \mathcal{P} = \boldsymbol{d}.$$

Then $i(\mathcal{P}, n)$ is a polynomial (the **Ehrhart polynomial** *of $\mathcal{P}$) in n of degree d.*

Reciprocity and volume

Moreover,

$$\begin{aligned} i(\mathcal{P},0) &= 1 \\ \bar{i}(\mathcal{P},n) &= (-1)^d i(\mathcal{P},-n), \;\; n>0 \\ &\qquad \textbf{(reciprocity)}. \end{aligned}$$

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If $d = N$ then

$$i(\mathcal{P},n) = V(\mathcal{P})n^d + \text{ lower order terms},$$

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(For $d < N$, $V(\mathcal{P})$ is the **relative volume**.)

Eugène Ehrhart

- April 29, 1906: born in Guebwiller, France
- 1932: begins teaching career in lycées
- 1959: Prize of French Sciences Academy
- 1963: begins work on Ph.D. thesis
- 1966: obtains Ph.D. thesis from Univ. of Strasbourg
- 1971: retires from teaching career
- January 17, 2000: dies

Photo of Ehrhart

Self-portrait

Generalized Pick's theorem

Corollary. *Let* $\mathcal{P} \subset \mathbb{R}^d$ *and* $\dim \mathcal{P} = d$. *Knowing any* d *of* $i(\mathcal{P}, n)$ *or* $\bar{i}(\mathcal{P}, n)$ *for* $n > 0$ *determines* $V(\mathcal{P})$.

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Proof. Together with $i(\mathcal{P}, 0) = 1$, this data determines $d + 1$ values of the polynomial $i(\mathcal{P}, n)$ of degree d. This uniquely determines $i(\mathcal{P}, n)$ and hence its leading coefficient $V(\mathcal{P})$. □

Two basic questions

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Note. If $\dim \mathcal{P} = d$ and

$$i(\mathcal{P}, n) = c_d n^d + c_{d-1} n^{d-1} + \cdots + c_0,$$

then $c_d > 0$ (**relative volume**), $c_{d-1} > 0$ (half the **relative surface area**), and $c_0 = 1 > 0$.

Two examples

Example 1. $\mathcal{P}_d$ is the simplex in $\mathbb{R}^d$ with vertices $O, e_1, \ldots, e_d$, where O is the origin, and e_i the ith unit coordinate vector.

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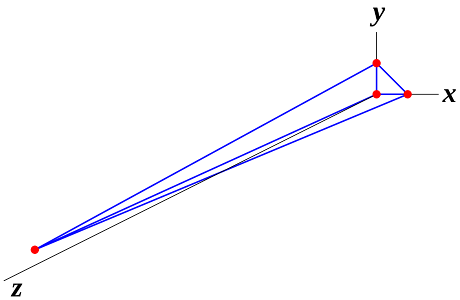
$$i(\mathcal{P}_d, n) = \binom{n+d-1}{d} = \frac{n(n-1)\cdots(n-d+1)}{d!}$$

Example 2. Let $\mathcal{P}$ denote the tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(1,1,13)$. Then

$$i(\mathcal{P}, n) = \frac{13}{6}n^3 + n^2 - \frac{1}{6}n + 1.$$

The “bad” tetrahedron



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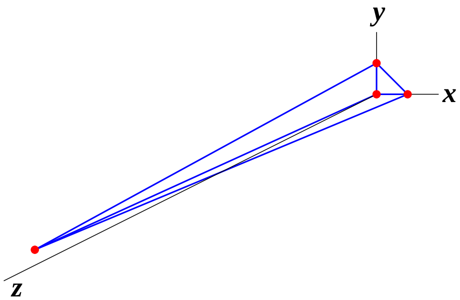

Thus in general the coefficients of Ehrhart polynomials are not "nice." Is there a "better" basis?

The h^*-vector of $i(\mathcal{P}, n)$

Let $\mathcal{P}$ be a lattice polytope of dimension d. Since $i(\mathcal{P}, n)$ is a polynomial of degree d, $\exists$ $\boldsymbol{h_i} \in \mathbb{Z}$ such that

$$\sum_{n \geq 0} i(\mathcal{P}, n) x^n = \frac{h_0 + h_1 x + \cdots + h_d x^d}{(1 - x)^{d+1}}.$$

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Definition. Define

$$h^*(\mathcal{P}) = (h_0, h_1, \ldots, h_d),$$

the ***h**-vector** of $\mathcal{P}$.

Three terms of $h^*(\mathcal{P})$

- $h_0 = 1$
- $h_1 = i(\mathcal{P}, 1) - \dim \mathcal{P} - 1 \geq 0$
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Example. $\mathcal{P} = \mathrm{conv}\{(0,0,0),(1,1,0),(1,0,1),(0,1,1)\}$. Then

$$h^*(\mathcal{P}) = (1,0,1,0).$$

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$$h_i(\mathcal{Q}) \leq h_i(\mathcal{P}) \ \forall i.$$

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B $\Rightarrow$ A: take $\mathcal{Q} = \varnothing$.

Proofs: the Ehrhart ring

$\mathcal{P}$: (convex) lattice polytope in $\mathbb{R}^m$ with vertex set V

$\boldsymbol{x}^{\boldsymbol{\beta}} = x^{\beta_1} \cdots x^{\beta_d}$, $\beta \in \mathbb{Z}^m$

Ehrhart ring (over $\mathbb{Q}$):

$$\boldsymbol{R}_{\mathcal{P}} = \mathbb{Q}\left[x^{\beta} y^{n} \,:\, \beta \in \mathbb{Z}^m,\ n \in \mathbb{P},\ \frac{\beta}{n} \in \mathcal{P}\right]$$

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$$R_\mathcal{P} = (R_\mathcal{P})_0 \oplus (R_\mathcal{P})_1 \oplus \cdots$$

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This means (using finiteness of $R_{\mathcal{P}}$ over $\mathbb{Q}[V]$): if $\dim \mathcal{P} = m$ then there exist algebraically independent $\theta_1, \ldots, \theta_{m+1} \in (R_{\mathcal{P}})_1$ such that $R_{\mathcal{P}}$ is a finitely-generated free $\mathbb{Q}[\theta_1, \ldots, \theta_{m+1}]$-module.

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Thus $R_{\mathcal{P}} = \bigoplus_{j=1}^{r} \eta_j \mathbb{Q}[\theta_1, \ldots, \theta_{m+1}]$, where $\boldsymbol{\eta_j} \in (R_{\mathcal{P}})_{e_j}$.

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Corollary. $\sum_{n \geq 0} \underbrace{H(R_{\mathcal{P}}, n)}_{i(\mathcal{P}, n)} x^n = \dfrac{x^{e_1} + \cdots + x^{e_r}}{(1-x)^{m+1}}$, so $h^*(\mathcal{P}) \geq 0$.

Monotonicity

The result $\mathcal{Q} \subseteq \mathcal{P} \Rightarrow h^*(\mathcal{Q}) \le h^*(\mathcal{P})$ is proved similarly.

We have $R_{\mathcal{Q}} \subset R_{\mathcal{P}}$. The key fact is that we can find an h.s.o.p. $\theta_1, \ldots, \theta_k$ for $R_{\mathcal{Q}}$ that extends to an h.s.o.p. for $R_{\mathcal{P}}$.

Valuations

convex body in $\mathbb{R}^m$: a nonempty, compact, convex subset

A **valuation** is a map φ from a family $\mathcal{F}$ of convex bodies in $\mathbb{R}^m$ containing $\emptyset$ into an abelian group G such that $\varphi(\emptyset) = 0$ and

$$\varphi(P \cup Q) = \varphi(P) + \varphi(Q) - \varphi(P \cap Q),$$

for all $P, Q \in \mathcal{F}$ for which $P \cup Q, P \cap Q \in \mathcal{F}$.

Hadwiger's theorem

Theorem (**Hadwiger**, 1957) *The family of continuous, real-valued, rigid-motion invariant valuations on all convex bodies is a* $(d+1)$*-dimensional vector space with basis consisting of the quermassintegrals* $\boldsymbol{W_i}$ *defined by*

$$\mathrm{vol}(tP + \mathcal{B}_d) = \sum_{i=0}^{d} \binom{d}{i} W_i(P) t^i,$$

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where $\boldsymbol{\mathcal{B}_d}$ *is a unit ball of dimension* d.

Note. W_i is monotone (hence nonnegative) and i-homogeneous, i.e., $W_i(\lambda P) = \lambda^i W_i(P)$. $W_0, \ldots, W_d$ is the unique (up to scaling) monotone homogeneous basis.

Lattice point analogue

Theorem (**Betke-Kneser**, 1985) *The family of real-valued, lattice-invariant (i.e., invariant under $GL(m, \mathbb{Z})$) valuations on lattice polytopes in $\mathbb{R}^m$ is an $(m+1)$-dimensional vector space spanned by the coefficients of the Ehrhart polynomial.*

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Note. h_i^* is not a valuation because its definition depends on $d = \dim \mathcal{P}$.

$$\sum_{n\geq 0} i(\mathcal{P}, n)x^n = \frac{h_0 + h_1 x + \cdots + h_d x^d}{(1-x)^{d+1}}.$$

Zonotopes

Let $\mathbf{v_1, \ldots, v_k} \in \mathbb{R}^d$. The **zonotope** $Z(v_1, \ldots, v_k)$ generated by $v_1, \ldots, v_k$:

$$\mathbf{Z(v_1, \ldots, v_k)} = \{\lambda_1 v_1 + \cdots + \lambda_k v_k \; : \; 0 \le \lambda_i \le 1\}$$

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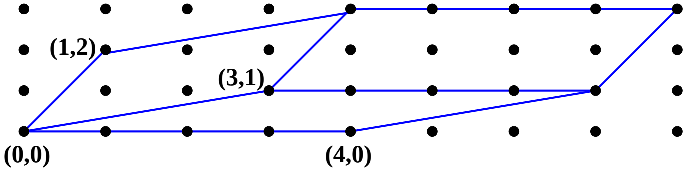
Example. $v_1 = (4, 0)$, $v_2 = (3, 1)$, $v_3 = (1, 2)$



Lattice points in a zonotope

Theorem. *Let*

$$Z = Z(v_1, \ldots, v_k) \subset \mathbb{R}^d,$$

where $v_i \in \mathbb{Z}^d$. *Then*

$$i(Z, 1) = \sum_X h(X),$$

where X *ranges over all linearly independent subsets of* $\{v_1, \ldots, v_k\}$, *and* $h(X)$ *is the gcd of all* $j \times j$ *minors* ($j = \#X$) *of the matrix whose rows are the elements of* X.

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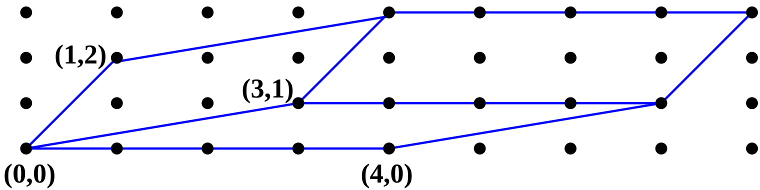

Computation of $i(Z,1)$

$$\begin{aligned} i(Z,1) &= \begin{vmatrix} 4 & 0 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} 4 & 0 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} \\ &\quad + \gcd(4,0) + \gcd(3,1) \\ &\quad + \gcd(1,2) + \det(\varnothing) \\ &= 4+8+5+4+1+1+1 \\ &= 24. \end{aligned}$$

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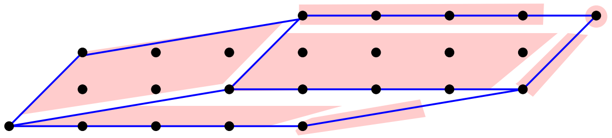

Corollary

Let $n \in \mathbb{P}$. If $Z = Z(v_1, \ldots, v_k)$, then

$$nZ = Z(nv_1, \ldots, nv_k),$$

and

$$i(Z, n) = i(nZ, 1) = \sum_X h(X) n^{\#X}.$$

Corollary. *If Z is an integer zonotope generated by integer vectors, then the coefficients of $i(Z, n)$ are nonnegative integers.*

The permutohedron

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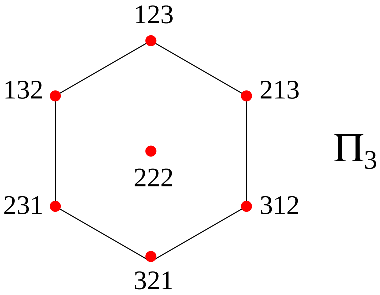
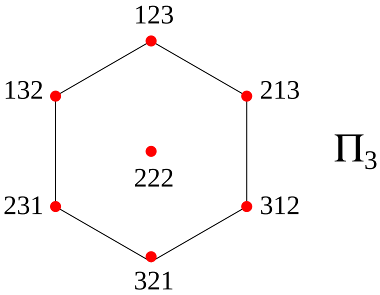
$$\Pi_m \approx Z(e_i - e_j : 1 \le i < j \le m)$$

Π_3

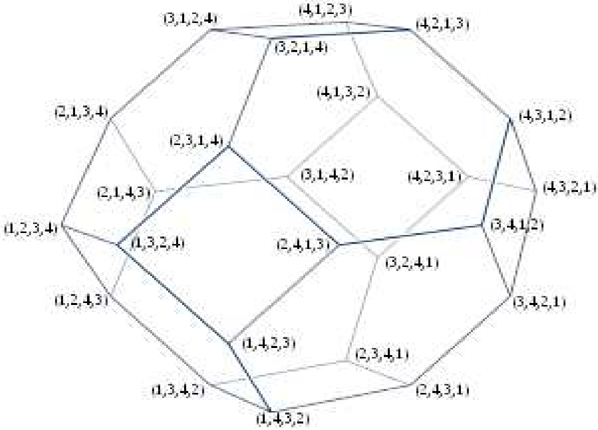


Π_3



$$i(\Pi_3, n) = 3n^2 + 3n + 1$$

Π_4



(truncated octahedron)

$i(\Pi_m, n)$

Theorem. $i(\Pi_m, n) = \sum_{k=0}^{m-1} f_k(m) n^k$, *where*

$$f_k(m) = \#\{\text{forests with } k \text{ edges on vertices } 1, \ldots, m\}$$

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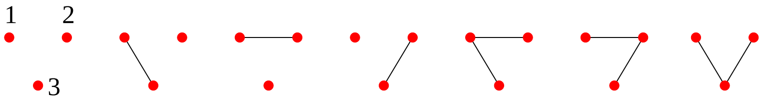

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Generalized permtohedra

Definition (**A. Postnikov**, 2005) A **generalized permutohedron** is a lattice polytope in $\mathbb{R}^m$ for which every edge is parallel to some edge of the permutohedron Π_m, that is, parallel to some vector $e_i - e_j$.

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Example. M: matroid on $E = \{v_1, \dots, v_m\}$

$$\chi_B \in \{0,1\}^n : \text{ characteristic vector of } B \subseteq E$$

$$\mathcal{B} : \text{ set of all bases of } M$$

matroid polytope $\mathcal{P}_M$: $\operatorname{conv}\{\chi_B : B \in \mathcal{B}\}$

Castillo-Liu conjecture

Conjecture (**F. Castillo** and **F. Liu**, 2015). Every integral generalized permutohedron is Ehrhart positive.

Open even for matroid polytopes.

Cross polytopes

cross polytope $\mathcal{C}_d$: $\operatorname{conv}\{\pm e_1, \ldots, \pm e_d\} \subset \mathbb{R}^d$ (dual to d-cube)

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Theorem. *$\mathcal{C}_d$ is Ehrhart positive.*

Crucial lemma

Lemma. *Let $f(n)$ be polynomial of degree d satisfying*

$$\sum_{n\geq 0} f(n)x^n = \frac{P(x)}{(1-x)^{d+1}},$$

where $P(x) = \prod_{j=1}^{k}(1+\gamma_j x)$, $|\gamma_j| = 1$. *Then* $f(n) = (n+1)(n+2)\cdots(n+d-k)g(n)$, *where*

$$g(\alpha) = 0 \Rightarrow Re(\alpha) = -\frac{1}{2}(d+1-k).$$

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Proof. Exercise.

Proof that $\mathcal{C}_d$ is Ehrhart positive

Apply to $i(\mathcal{C}_d, n)$ to get that all zeros of $i(\mathcal{C}_d, n)$ have real part $-1/2$. Thus $i(\mathcal{C}_d, n)$ is a product of factors $n + \frac{1}{2}$ and

$$\left(n + \frac{1}{2} + \beta i\right)\left(n + \frac{1}{2} - \beta i\right) = n^2 + n + \beta^2 + \frac{1}{4},$$

so $i(\mathcal{C}_d, n)$ has positive coefficients. $\square$

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Not so easy to give a "positive formula" for the coefficients.

Rational polytopes

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Example. $\mathcal{P} = [0, 1/2]$. Then

$$\begin{aligned} i(\mathcal{P}, n) &= \begin{cases} \frac{1}{2}n, & n \text{ even} \\ \frac{1}{2}(n+1), & n \text{ odd} \end{cases} \\ &= \frac{1}{2}n + \frac{1}{4}(1 - (-1)^n). \end{aligned}$$

Theorem. *Let $N\mathcal{P}$ have integer vertices, $N \in \mathbb{P}$. Then there exist polynomials $P_0(n), \ldots, P_{N-1}(n)$ such that*

$$i(\mathcal{P}, n) = P_j(n), \ \ n \equiv j \pmod{N}.$$

Irrational polytopes

Example. $\mathcal{P} = [0, \sqrt{2}]$, then

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which is poorly behaved.

For instance, $\sum_{n \geq 0} \lfloor \sqrt{2} n \rfloor x^n$ has the unit circle as a natural boundary.

Uninteresting irrational polytopes

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Uninteresting, because $\mathcal{P}$ is the translate of a rational (in fact, integer) polytope.

Period collapse

If there are polynomials $P_0(n), \ldots, P_{M-1}(n)$ for which

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then M is called a **period** of $\mathcal{P}$ or $i(\mathcal{P}, n)$.

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If $\mathcal{P}$ has a period smaller than the least $N > 0$ for which $N\mathcal{P}$ has integer vertices, then $\mathcal{P}$ exhibits **period collapse**.

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Special case of period collapse: $\mathcal{P}$ does not have integer vertices, but $i(\mathcal{P}, n)$ is a polynomial.

Poorly understood, but lots of examples, such as **Gelfand-Zetlin polytopes**.

Some curious triangles

For $\alpha > 0$ let $\boldsymbol{T_\alpha}$ be the triangle in $\mathbb{R}^2$ with vertices $(0,0),(0,\alpha),(1/\alpha,0)$, so area$(T_\alpha) = \frac{1}{2}$. Can define

$$\boldsymbol{i(T_\alpha, n)} = \#(nT_\alpha \cap \mathbb{Z}^2), \quad n \geq 1.$$

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Easy. T_1 is a lattice triangle with $i(T_1, n) = \binom{n+2}{2}$.

Theorem (**Cristofaro-Gardiner, Li, S**). *Let* $\alpha > 1$. *We have* $i(T_\alpha, n) = \binom{n+2}{2}$ *for all* $n \geq 1$ *if and only if either:*

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- $\alpha = \frac{F_{2k+1}}{F_{2k-1}}$ (Fibonacci numbers)
- $\alpha = \frac{1}{2}(3 + \sqrt{5})$

Generalizations?

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However: no "interesting" irrational polytope $\mathcal{P}$ is known for which $i(\mathcal{P}, n)$ is a polynomial and some vertex of $\mathcal{P}$ is algebraic of degree at least three.


THE
END
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12