HYPERPLANE ARRANGEMENTS AND INTERVAL ORDERS

dedicated to the memory of
Gian-Carlo Rota

Richard P. Stanley
Department of Mathematics
M.I.T. 2-375
Cambridge, MA 02139
rstan@math.mit.edu
http://www-math.mit.edu/~rstan

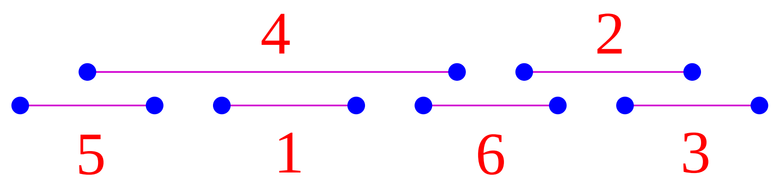
Transparencies available at:

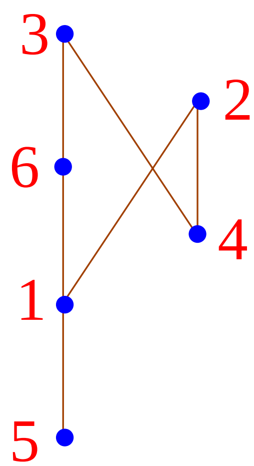
http://www-math.mit.edu/~rstan/trans.html

Let P be a (finite) set of closed intervals $[a, b] \subset \mathbb{R}$, with $a < b$. Define a partial ordering of P by

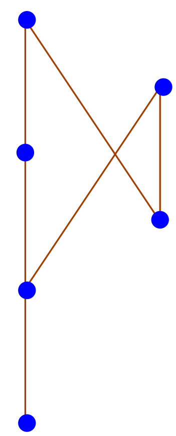
$$[a, b] < [c, d] \quad \text{if} \quad b < c.$$

Any poset (partially ordered set) isomorphic to P is an **interval order**.

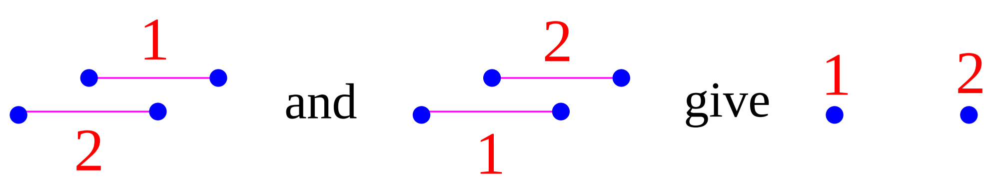




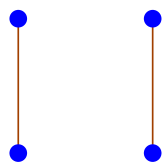
labelled

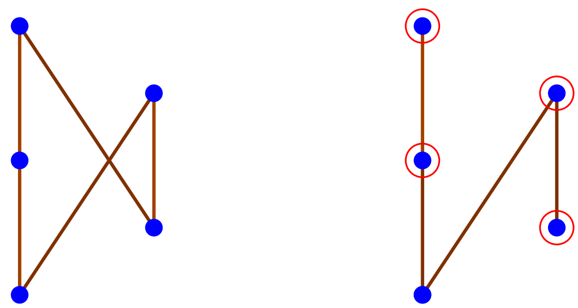


unlabelled



Theorem. (Fishburn, 1970) *A finite poset is an interval order if and only if it contains no induced*





OK not OK

Let $\mathbf{r}(\mathcal{A})$ denote the number of regions of the hyperplane arrangement $\mathcal{A}$.

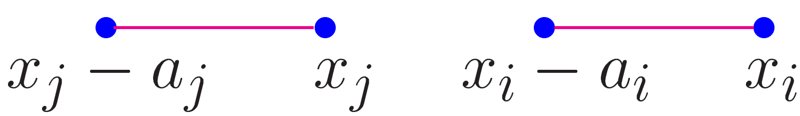
Theorem. *Let* $a_1, \ldots, a_n > 0$. *Let* $I(a_1, \ldots, a_n)$ *be the number of (labelled) interval orders such that interval* I_i *has length* a_i. *Then*

$$I(a_1, \ldots, a_n) = r(\mathcal{I}(a_1, \ldots, a_n)),$$

where $\mathcal{I}(a_1, \ldots, a_n)$ *is the arrangement (in* $\mathbb{R}^n$*)*

$$x_i - x_j = a_i, \ i \neq j, \ 1 \leq i, j \leq n.$$

Proof.



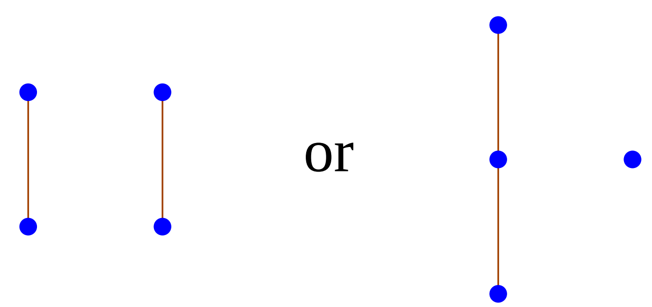
$$\begin{aligned} I_j < I_i &\iff x_j < x_i - a_i \\ &\iff x_i - x_j > a_i. \end{aligned}$$

Special case:

$\mathcal{I}_{\mathbf{n}}: \quad x_i - x_j = 1, \;\; i \neq j, \;\; 1 \leq i, j \leq n$

Definition. A poset is a **semiorder** or **unit interval order** if it is an interval order using intervals of length one.

Theorem. (Scott & Suppes, 1958) *A finite poset is a semiorder if and only if it contains no induced*

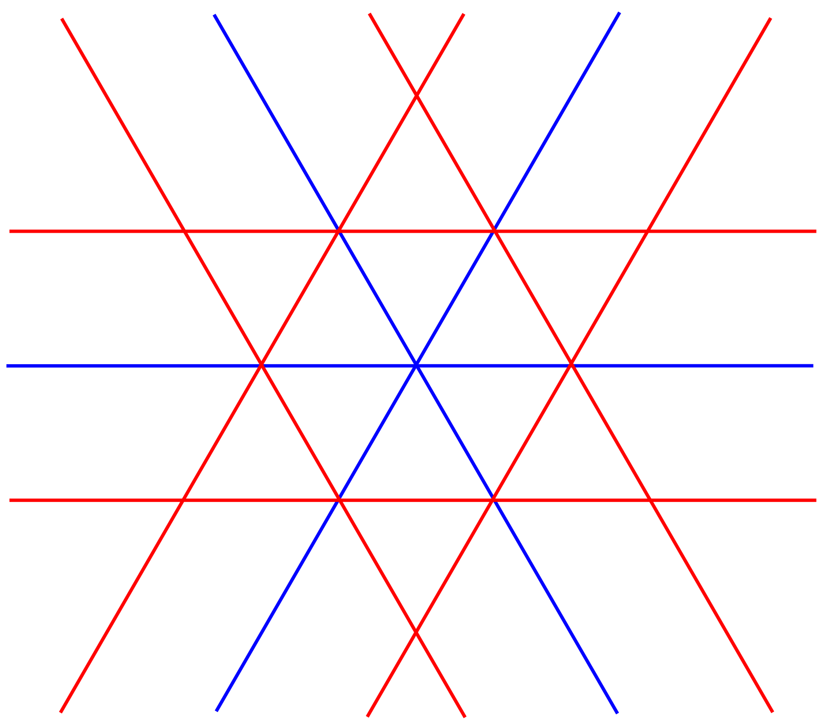


Corollary. *$r(\mathcal{I}_n)$ is the number of semiorders on* $\{1, 2, \ldots, n\}$.

$\mathcal{I}_{\mathbf{n}}^{\mathbf{0}}: \quad x_i - x_j = 0, \pm 1, \;\; 1 \leq i < j \leq n.$

$$
\begin{aligned}
r(\mathcal{I}_n^0) &= n! \cdot \#\ \text{nonisomorphic} \\
&\qquad\qquad \text{semiorders with } n \text{ elements} \\
&= n!\, C_n, \quad C_n = \frac{1}{n+1}\binom{2n}{n}
\end{aligned}
$$

(**Catalan number**).



Let

$$F(x) = \sum_{n \geq 0} C_n x^n = \frac{1 - \sqrt{1 - 4x}}{2x}.$$

Then

$$\sum_{n \geq 0} r(\mathcal{I}_n) \frac{x^n}{n!} = F(1 - e^{-x}).$$

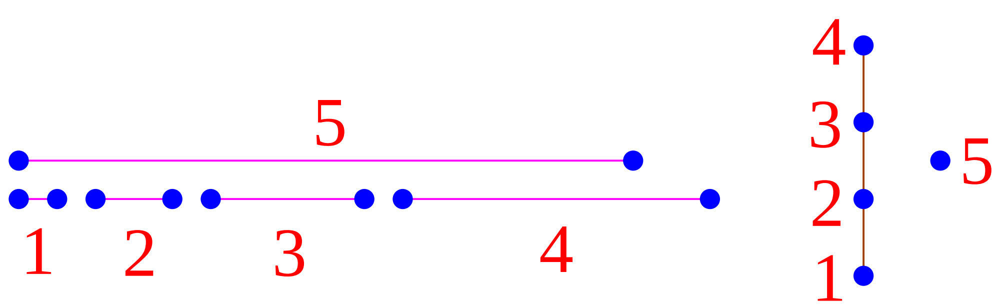
(Chandon, Lemaire, & Pouget, 1978).

Now let $a_1, \ldots, a_n$ be *generic*, e.g., linearly independent over $\mathbb{Q}$, or

$$a_1 << a_2 << \cdots << a_n.$$

Note. $I(a_1, \ldots, a_n)$ is independent of $a_1, \ldots, a_n$ (though the interval orders themselves depend on $a_1, \ldots, a_n$) (since the intersection posets are the same).

Example. $(a_1, a_2, a_3, a_4) = (1, 2, 4, 8, 16)$



Cannot be achieved by $(a_1, a_2, a_3, a_4) = (1, 1.0001, 1.001, 1.01, 1.1)$.

Let $h_n = I(a_1, \ldots, a_n)$.

Theorem. *Define*

$$y = 1 + x + 5\frac{x^2}{2!} + 46\frac{x^3}{3!} + \cdots$$

by

$$1 = y(2 - e^{xy}).$$

Let

$$\begin{aligned} z &= \sum_{n\geq 0} h_n \frac{x^n}{n!} \\ &= 1 + x + 3\frac{x^2}{2!} + 19\frac{x^3}{3!} + 195\frac{x^4}{4!} + \cdots. \end{aligned}$$

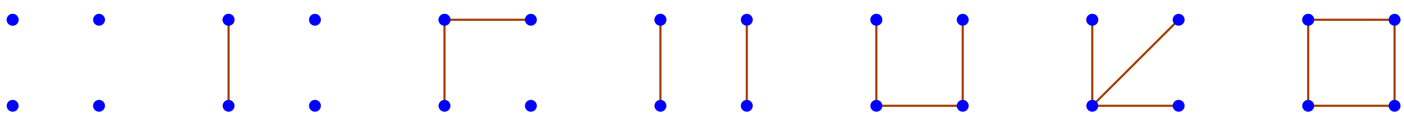
Then

$$\frac{z'}{z} = y^2, \quad z(0) = 1.$$

Proof. Whitney, Zaslavsky $\Longrightarrow$

$$h_n = \sum_B (-1)^{\text{cycle dim}} 2^{\#\text{blocks}},$$

where B ranges over all bipartite graphs with vertices $1, 2, \ldots, n$.



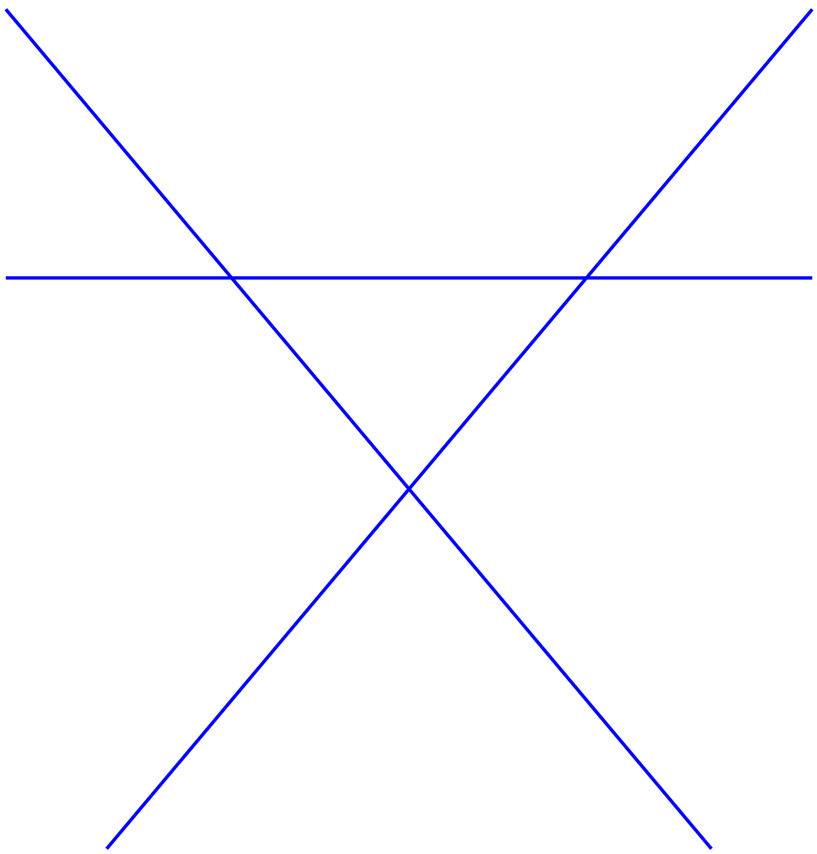
1·1 + 6·2 + 12·4 + 3·4 + 12·8 + 4·8 - 3·2

= 195

Etc.

The Linial Arrangement

$$\mathcal{L}_{\mathbf{n}}: \quad x_i - x_j = 1, \;\; 1 \le i < j \le n$$

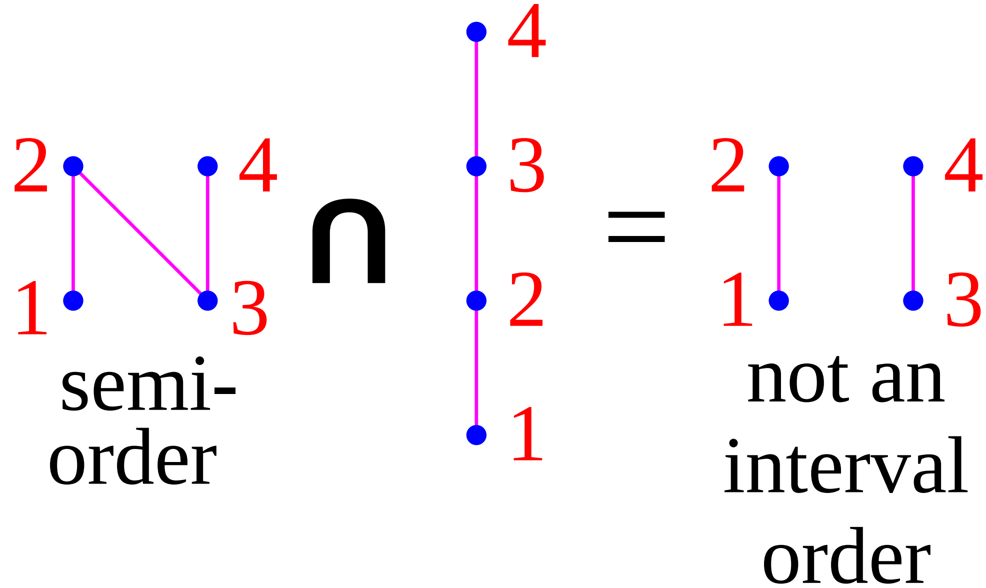


$\mathcal{L}_3$

$$
\begin{aligned}
r(\mathcal{L}_1) &= 1 \\
r(\mathcal{L}_2) &= 2 \\
r(\mathcal{L}_3) &= 7 \\
r(\mathcal{L}_4) &= 36 \\
r(\mathcal{L}_5) &= 246 \\
r(\mathcal{L}_6) &= 2104 \\
r(\mathcal{L}_7) &= 21652 \\
r(\mathcal{L}_8) &= 260720 \\
r(\mathcal{L}_9) &= 3598120 \\
r(\mathcal{L}_{10}) &= 56010096 \\
r(\mathcal{L}_{11}) &= 971055240 \\
r(\mathcal{L}_{12}) &= 18558391936
\end{aligned}
$$

Theorem. *$r(\mathcal{L}_n)$ is the number of posets on $\{1, 2, \ldots, n\}$ obtained by intersecting a semiorder (unit interval order) with the chain*

$$1 < 2 < \cdots < n.$$



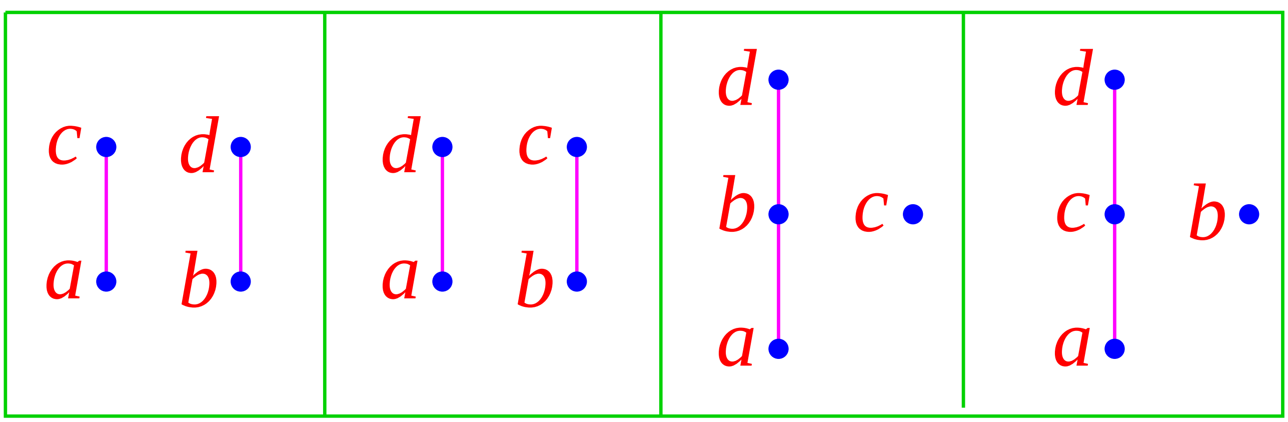
Theorem (A. Postnikov). *Let*

$$a < b < c < d.$$

The obstructions to being an intersection of a semiorder with the chain

$$1 < 2 < \cdots < n$$

are the induced posets



Theorem (Athanasiadis, Postnikov)
Let

$$y = \sum_{n \geq 0} r(\mathcal{L}_n) \frac{x^n}{n!}.$$

Then

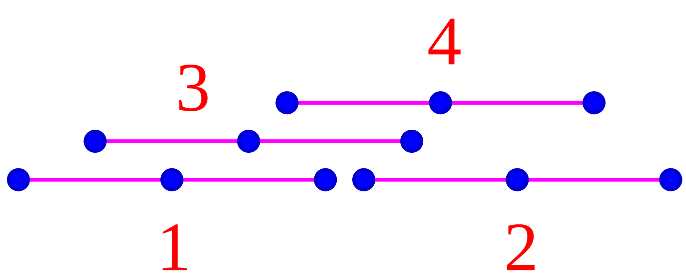
$$y = \exp \frac{x}{2}(1+y)$$

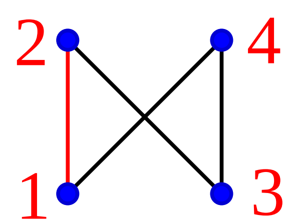
$$r(\mathcal{L}_n) = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} (k+1)^{n-1}.$$

Define

$$[a, a+2] < [b, b+2] \text{ if } a+1 < b$$
$$[a, a+2] < [b, b+2] \text{ if } a+2 < b.$$

A **double semiorder** (easily generalized to k-interval orders and k-semiorders) is the "double poset" obtained from a (finite) set of intervals of length two and the two relations $<$ and $<$.





Theorem. (a) *The double semiorders on* $1, 2 \ldots, n$ *are in one-to-one correspondence with the regions of the arrangement*

$$\mathcal{I}_{\mathbf{n,2}}: \ x_i - x_j = \pm 1, \ \pm 2, \ 1 \le i < j \le n.$$

(b) *Let*

$$\mathcal{I}^{\mathbf{0}}_{\mathbf{n,2}}: \ x_i - x_j = 0, \pm 1, \pm 2, \ 1 \le i < j \le n.$$

Then

$$r(\mathcal{I}^0_{n,2}) = n! \cdot \# \text{ nonisomorphic double semiorders with } n \text{ elements}$$

$$= n! \frac{1}{2n+1} \binom{3n}{n}$$

(c) *Let*

$$G(x) = \sum_{n\geq 0} r(\mathcal{I}^0_{n,2})\frac{x^n}{n!} = \sum_{n\geq 0} \frac{1}{2n+1}\binom{3n}{n}x^n.$$

Then

$$\sum_{n\geq 0} r(\mathcal{I}_{n,2})\frac{x^n}{n!} = G(1-e^{-x}).$$

Compare:

$$F(x) = \sum_{n\geq 0} r(\mathcal{I}^0_n)\frac{x^n}{n!} = \sum_{n\geq 0} \frac{1}{n+1}\binom{2n}{n}x^n$$

$$\sum_{n\geq 0} r(\mathcal{I}_n)\frac{x^n}{n!} = F(1-e^{-x}).$$

Reference:

R. Stanley, Hyperplane arrangements, interval orders, and trees, *Proc. Nat. Acad. Sci.* **93** (1996), 2620–2625.