Increasing and Decreasing Subsequences

Richard P. Stanley

M.I.T.

Definitions

$$\mathbf{3}\ 1\ 8\ \mathbf{4}\ 9\ \mathbf{6}\ \mathbf{7}\ 2\ 5 \quad \text{(i.s)}$$

$$3\ 1\ \mathbf{8}\ \mathbf{4}\ 9\ 6\ 7\ \mathbf{2}\ 5 \quad \text{(d.s)}$$

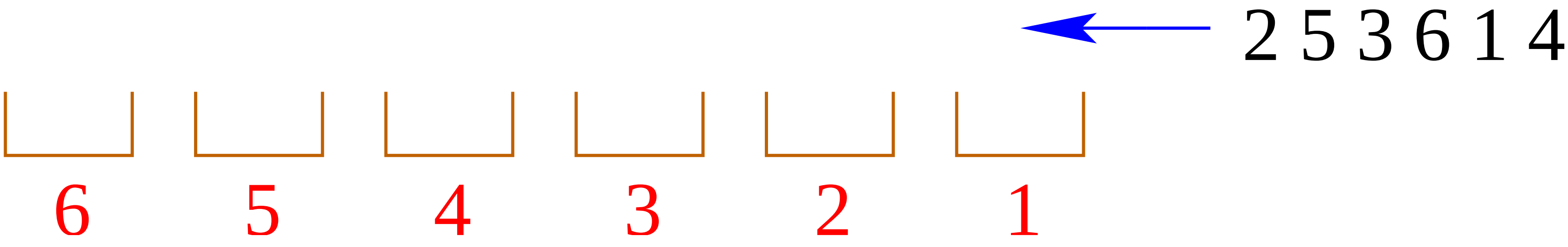
$$\mathbf{is}(w) = |\text{longest i.s.}| = 4$$

$$\mathbf{ds}(w) = |\text{longest d.s.}| = 3$$

Application: airplane boarding

Naive model: passengers board in order $w = a_1 a_2 \cdots a_n$ for seats $1, 2, \ldots, n$. Each passenger takes one time unit to be seated after arriving at his seat.

Boarding process



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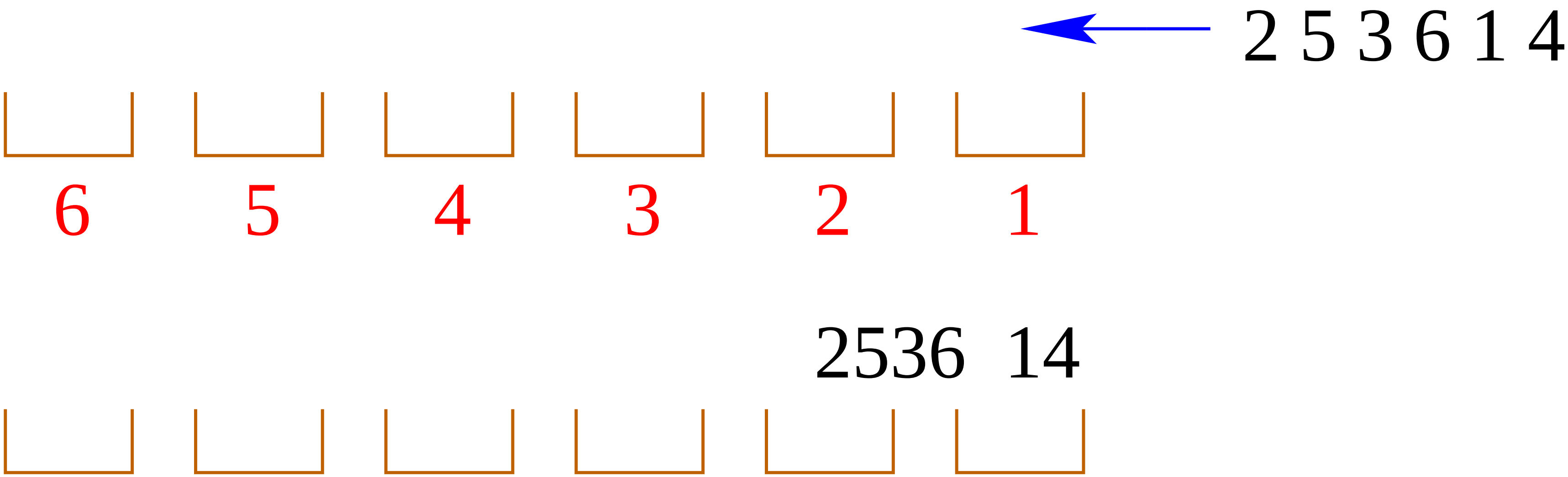

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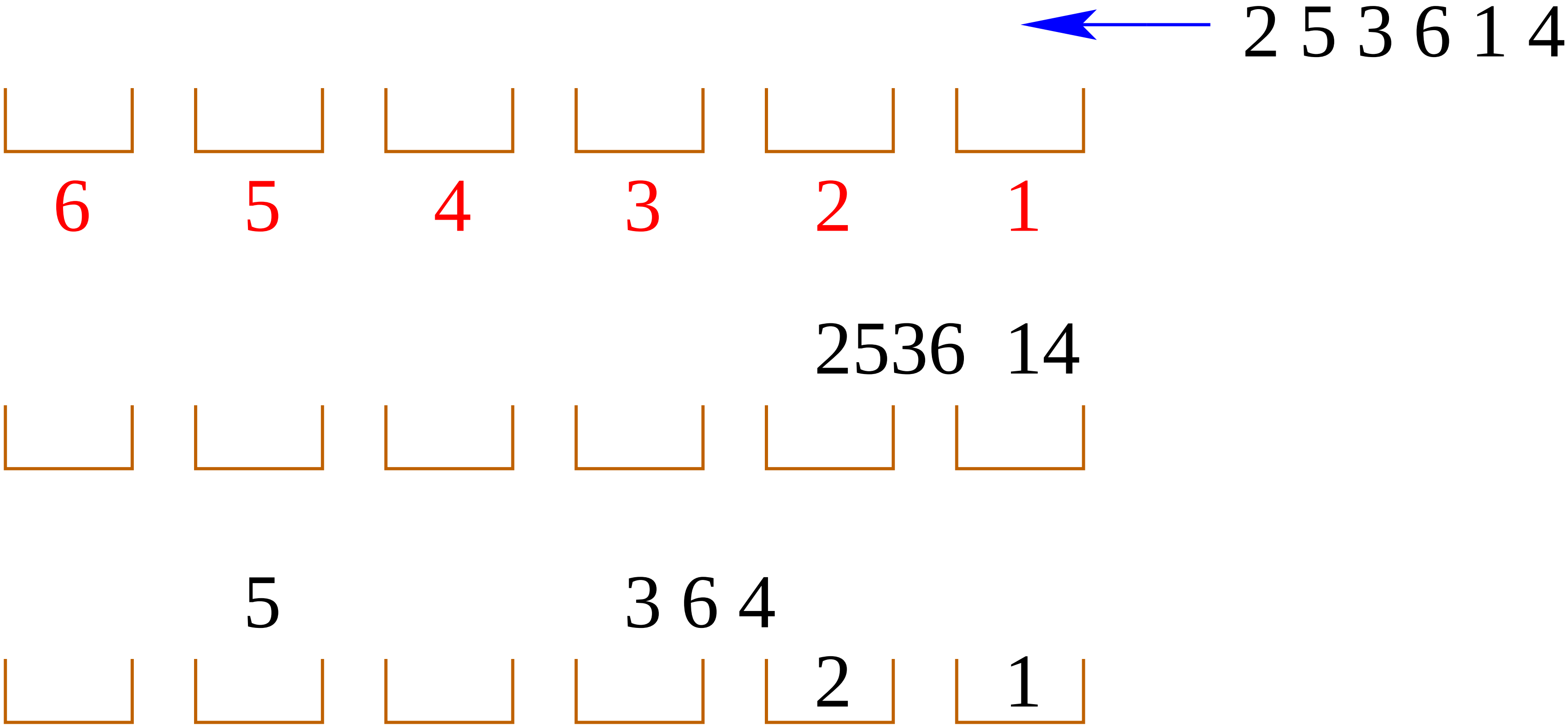

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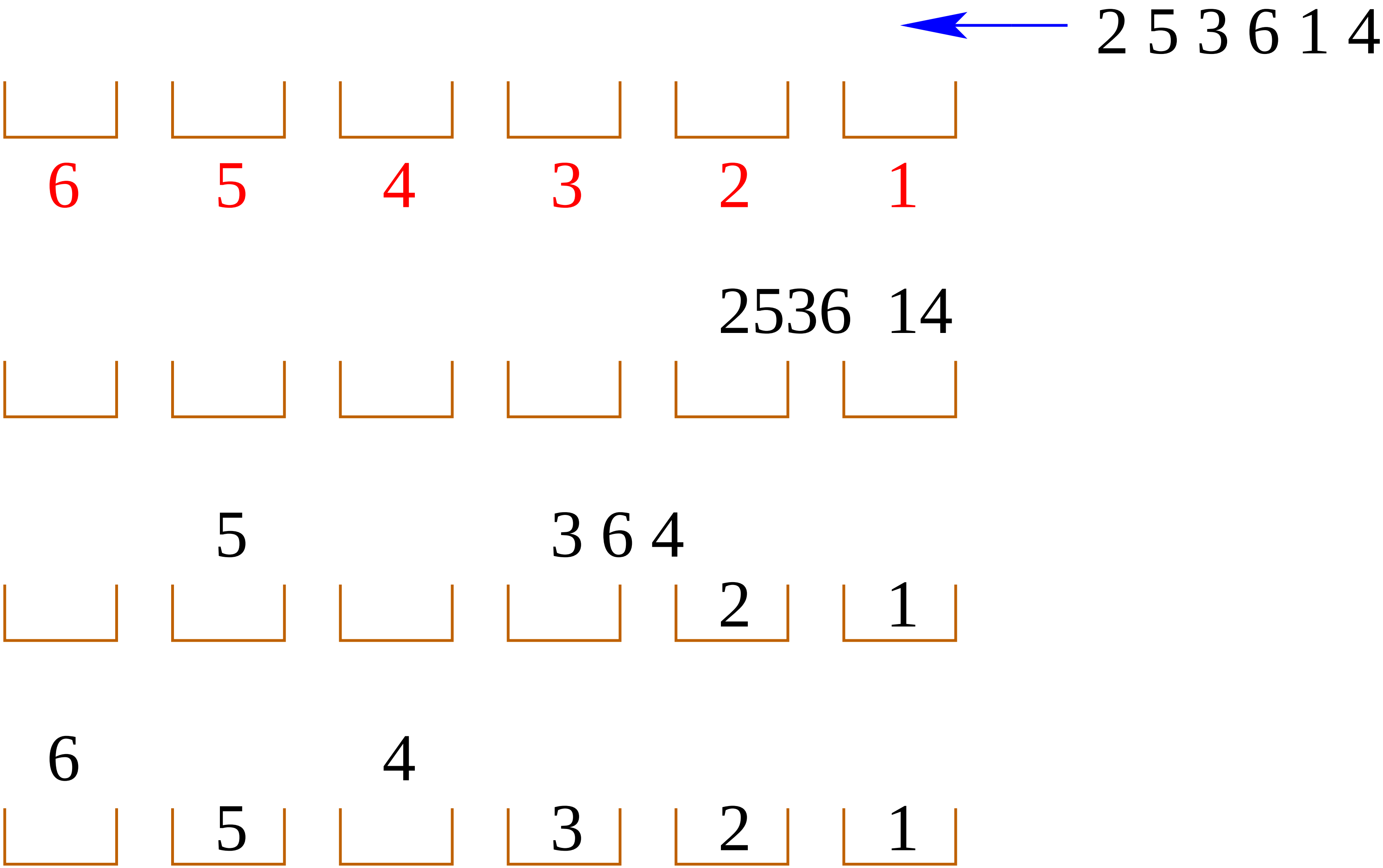

Results

Easy: Total waiting time $= \mathrm{is}(w)$.

Bachmat, et al.: more sophisticated model.

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Two conclusions:

- Usual system (back-to-front) not much better than random.
- Better: first board window seats, then center, then aisle.

Partitions

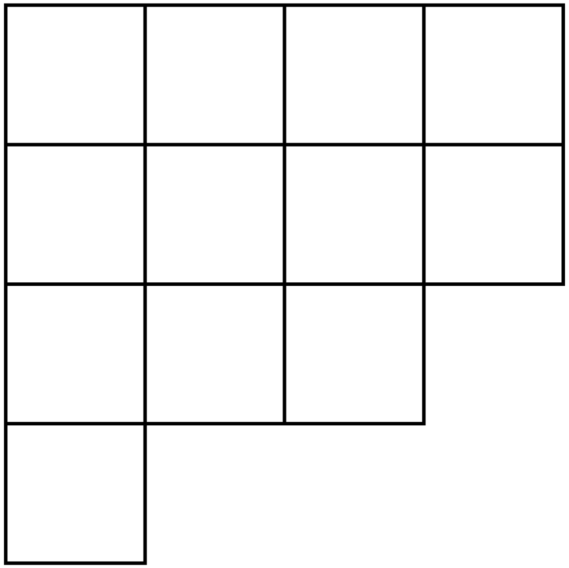
partition $\lambda \vdash n$: $\lambda = (\lambda_1, \lambda_2, \ldots)$

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$$

$$\sum \lambda_i = n$$

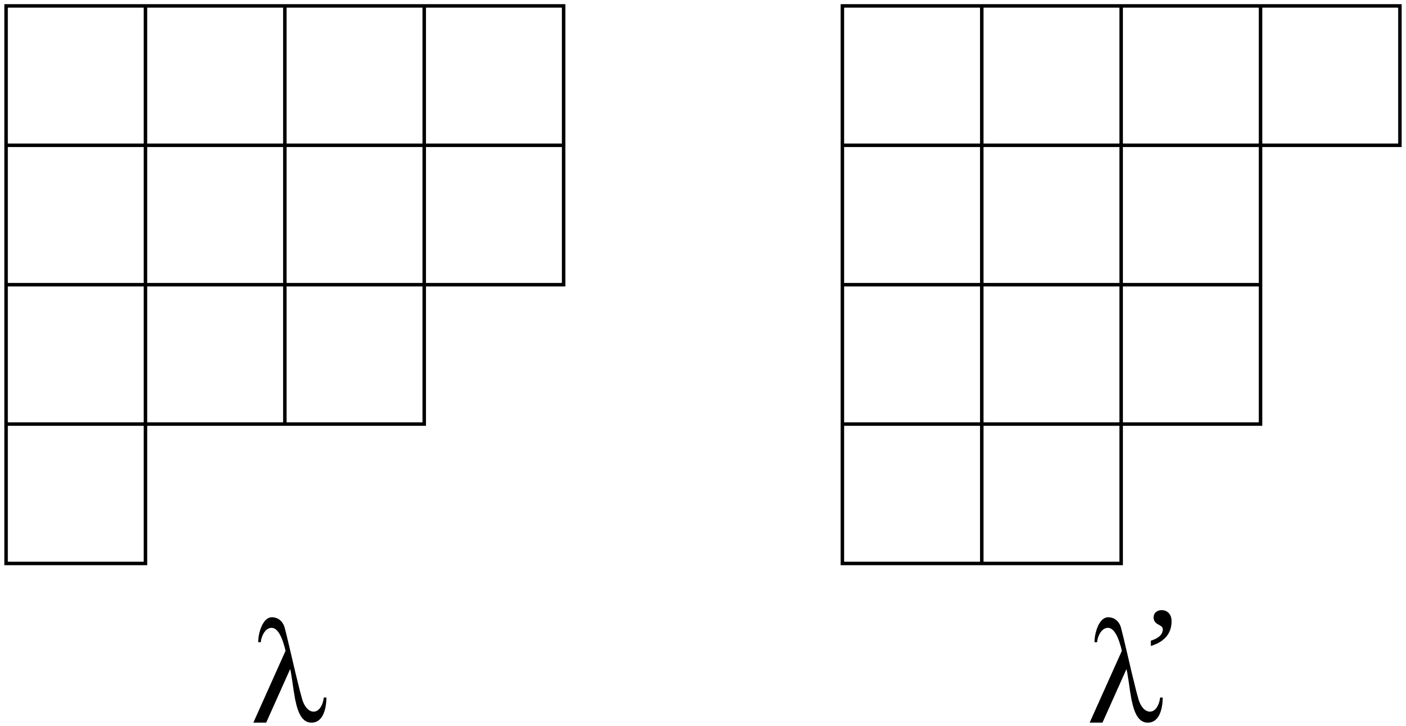
Young diagrams

(Young) diagram of $\lambda = (4, 4, 3, 1)$:



Conjugate partitions

$\lambda' = (4,3,3,2)$, the **conjugate** partition to $\lambda = (4,4,3,2)$



Standard Young tableau

standard Young tableau (SYT) of shape $\lambda \vdash n$, e.g., $\lambda = (4, 4, 3, 1)$:

$<$

$\wedge$

1	2	7	10
3	5	8	12
4	6	11	
9			

f^λ

$f^\lambda = \#$ of SYT of shape λ

E.g., $f^{(3,2)} = 5$:

1 2 3	1 2 4	1 2 5	1 3 4	1 3 5
4 5	3 5	3 4	2 5	2 4

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$\exists$ simple formula for f^λ (Frame-Robinson-Thrall **hook-length formula**)

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Note. $f^\lambda = \dim(\text{irrep. of } \mathfrak{S}_n)$, where $\mathfrak{S}_n$ is the **symmetric group** of all permutations of $1, 2 \ldots, n$.

RSK algorithm

RSK algorithm: a bijection

$$w \xrightarrow{\text{rsk}} (P, Q),$$

where $w \in \mathfrak{S}_n$ and P, Q are SYT of the same shape $\lambda \vdash n$.

Write $\lambda = \mathbf{sh}(w)$, the **shape** of w.

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Example of RSK: $w = 4132$

insert 4, record 1: $\begin{matrix}4\end{matrix} \qquad \begin{matrix}1\end{matrix}$

insert 1, record 2: $\begin{matrix}1\\4\end{matrix} \qquad \begin{matrix}1\\2\end{matrix}$

insert 3, record 3: $\begin{matrix}1&3\\4\end{matrix} \qquad \begin{matrix}1&3\\2\end{matrix}$

insert 2, record 4: $\begin{matrix}1&2\\3\\4\end{matrix} \qquad \begin{matrix}1&3\\2\\4\end{matrix}$

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insert 1, record 2: $\begin{array}{l} 1 \\ 4 \end{array} \qquad \begin{array}{l} 1 \\ 2 \end{array}$

insert 3, record 3: $\begin{array}{ll} 1 & 3 \\ 4 \end{array} \qquad \begin{array}{ll} 1 & 3 \\ 2 \end{array}$

insert 2, record 4: $\begin{array}{ll} 1 & 2 \\ 3 \\ 4 \end{array} \qquad \begin{array}{ll} 1 & 3 \\ 2 \\ 4 \end{array}$

$$(P, Q) = \left(\begin{array}{ll} 1 & 2 \\ 3 \\ 4 \end{array} , \quad \begin{array}{ll} 1 & 3 \\ 2 \\ 4 \end{array} \right)$$

Schensted's theorem

Theorem. *Let* $w \xrightarrow{\text{rsk}} (P, Q)$, *where* *sh*$(P) = \text{sh}(Q) = \lambda$. *Then*

$$\begin{aligned} \text{is}(w) &= \text{longest row length} = \lambda_1 \\ \text{ds}(w) &= \text{longest column length} = \lambda_1'. \end{aligned}$$

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Example. $4132 \xrightarrow{\text{rsk}} \left(\begin{matrix} 1\,2 \\ 3 \\ 4 \end{matrix} \quad , \quad \begin{matrix} 1\,3 \\ 2 \\ 4 \end{matrix} \right)$

$$\text{is}(w) = 2, \quad \text{ds}(w) = 3.$$

Erdős-Szekeres theorem

Corollary (Erdős-Szekeres, Seidenberg). *Let* $w \in \mathfrak{S}_{pq+1}$*. Then either* $\mathrm{is}(w) > p$ *or* $\mathrm{ds}(w) > q$*.*

Erdős-Szekeres theorem

Corollary (Erdős-Szekeres, Seidenberg). *Let* $w \in \mathfrak{S}_{pq+1}$. *Then either* $\mathrm{is}(w) > p$ *or* $\mathrm{ds}(w) > q$.

Proof. Let $\lambda = \mathrm{sh}(w)$. If $\mathrm{is}(w) \le p$ and $\mathrm{ds}(w) \le q$ then $\lambda_1 \le p$ and $\lambda_1' \le q$, so $\sum \lambda_i \le pq$. $\square$

An extremal case

Corollary. *Say* $p \leq q$. *Then*

$$\#\{w \in \mathfrak{S}_{pq} \ : \ \mathrm{is}(w) = p, \ \mathrm{ds}(w) = q\}$$

$$= \left(f^{(p^q)}\right)^2$$

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By hook-length formula, this is

$$\left(\frac{(pq)!}{1^1 2^2 \cdots p^p (p+1)^p \cdots q^p (q+1)^{p-1} \cdots (p+q-1)^1}\right)^2.$$

Romik's theorem

Romik: let

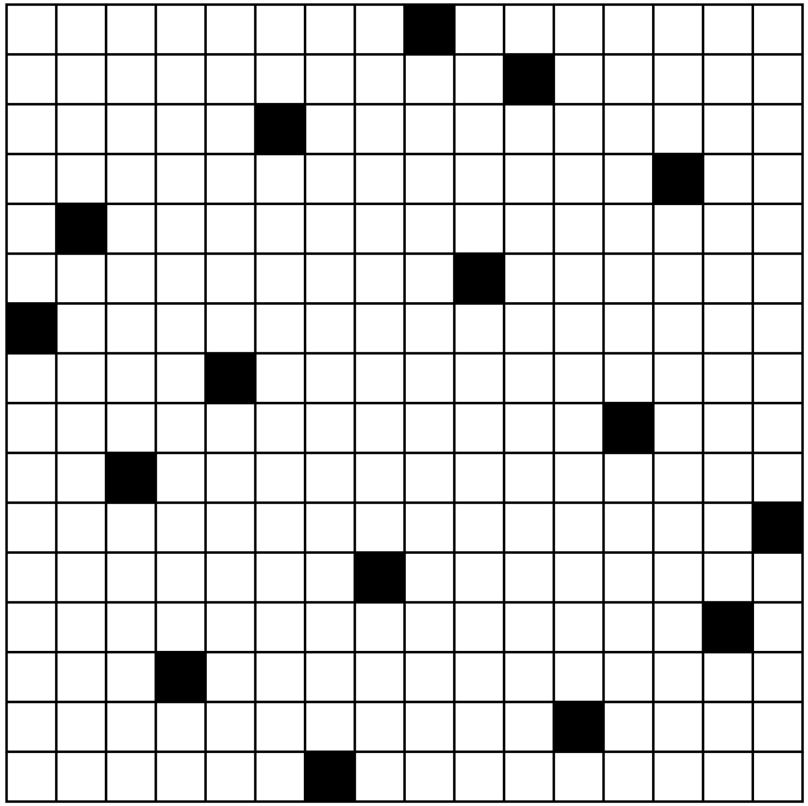
$$w \in \mathfrak{S}_{n^2}, \ \mathrm{is}(w) = \mathrm{ds}(w) = n.$$

Let P_w be the permutation matrix of w with corners $(\pm 1, \pm 1)$. Then (informally) as $n \to \infty$ almost surely the 1's in P_w will become dense in the region bounded by the curve

$$(x^2 - y^2)^2 + 2(x^2 + y^2) = 3,$$

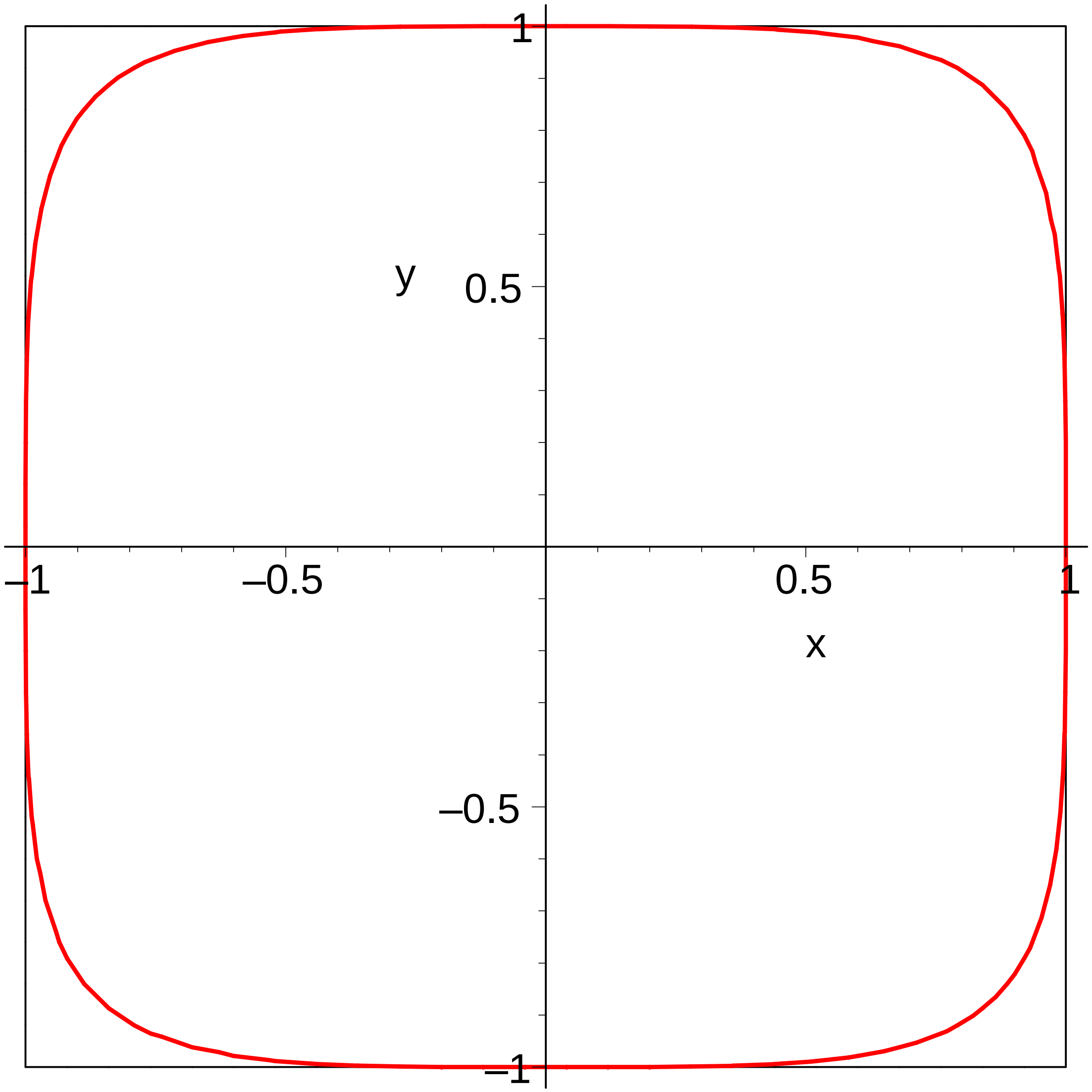
and will remain isolated outside this region.

An example



$$w = 9, 11, 6, 14, 2, 10, 1, 5, 13, 3, 16, 8, 15, 4, 12, 17$$

$(x^2 - y^2)^2 + 2(x^2 + y^2) = 3$



Area enclosed by curve

$$\begin{aligned}\alpha &= 8\int_0^1 \frac{1}{\sqrt{(1-t^2)(1-(t/3)^2)}}dt \\ &\quad -6\int_0^1 \sqrt{\frac{1-(t/3)^2}{1-t^2}}dt \\ &= 4(0.94545962\cdots)\end{aligned}$$

Expectation of $\text{is}(w)$

$$\begin{aligned} \boldsymbol{E(n)} &= \text{expectation of is}(w),\ w \in \mathfrak{S}_n \\ &= \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \text{is}(w) \\ &= \frac{1}{n!} \sum_{\lambda \vdash n} \lambda_1 \left(f^{\lambda}\right)^2 \end{aligned}$$

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Ulam: what is distribution of $\mathrm{is}(w)$? rate of growth of $E(n)$?

Work of Hammersley

Hammersley (1972):

$$\exists\ c = \lim_{n\to\infty} n^{-1/2} E(n),$$

and

$$\frac{\pi}{2} \le c \le e.$$

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Conjectured $c = 2$.

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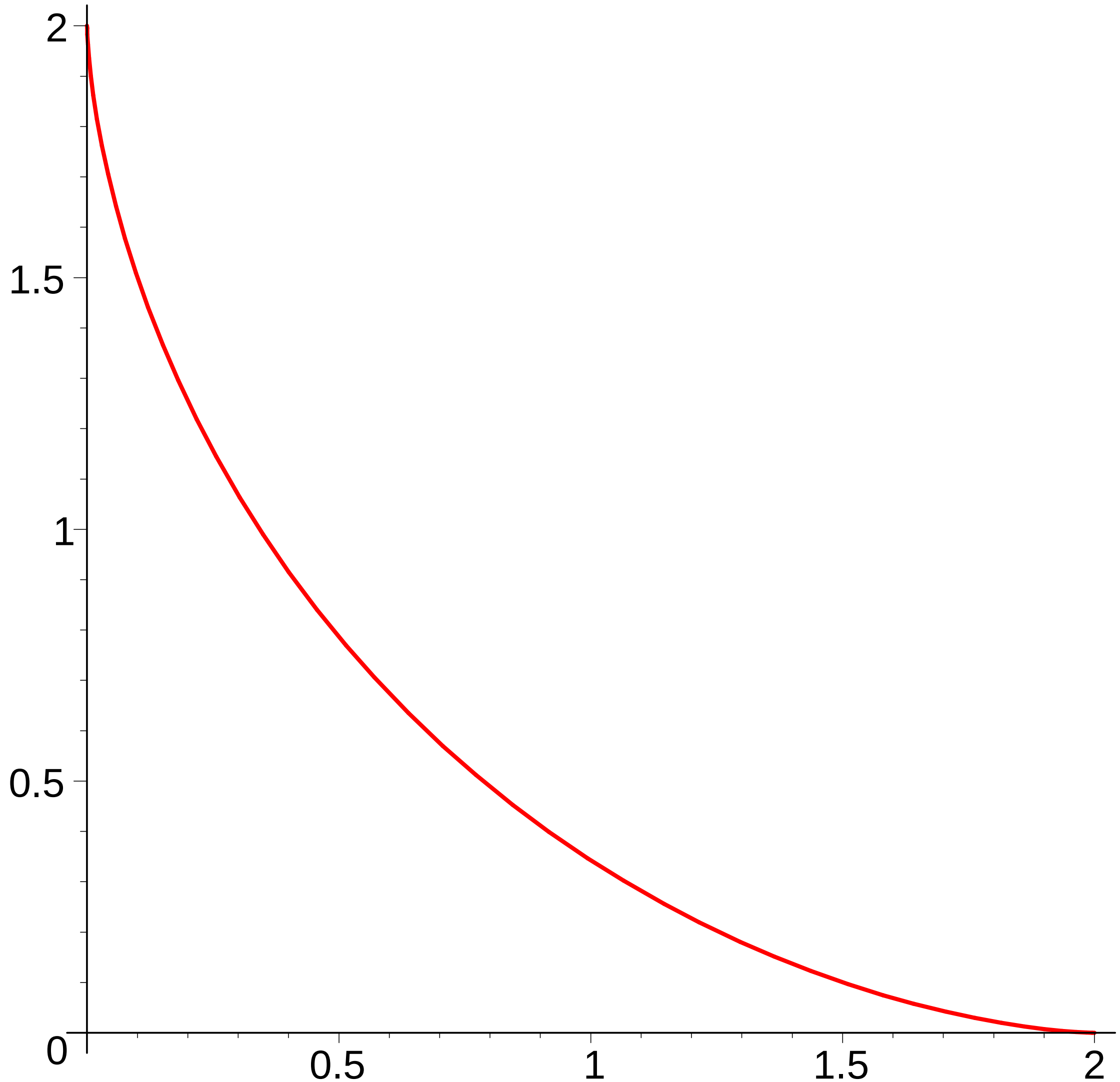
Idea of proof.

$$\begin{aligned} E(n) &= \frac{1}{n!} \sum_{\lambda \vdash n} \lambda_1 \left(f^\lambda\right)^2 \\ &\approx \frac{1}{n!} \max_{\lambda \vdash n} \lambda_1 \left(f^\lambda\right)^2 . \end{aligned}$$

Find “limiting shape” of $\lambda \vdash n$ maximizing λ as $n \to \infty$ using hook-length formula.

A big shape

The limiting curve



Equation of limiting curve

$$\begin{aligned} x &= y + 2\cos\theta \\ y &= \frac{2}{\pi}(\sin\theta - \theta\cos\theta) \end{aligned}$$

$$0 \leq \theta \leq \pi$$

$\mathbf{is}(w) \leq 2$

$$u_k(n) := \#\{w \in \mathfrak{S}_n \,:\, \mathrm{is}_n(w) \leq k\}.$$

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J. M. Hammersley (1972):

$$u_2(n) = C_n = \frac{1}{n+1}\binom{2n}{n},$$

a **Catalan number**.

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For ≥160 combinatorial interpretations of C_n, see

`www-math.mit.edu/~rstan/ec`

Gessel's theorem

I. Gessel (1990):

$$\sum_{n\geq 0} u_k(n)\frac{x^{2n}}{n!^2} = \det\left[I_{|i-j|}(2x)\right]_{i,j=1}^{k},$$

where

$$I_m(2x) = \sum_{j\geq 0}\frac{x^{m+2j}}{j!(m+j)!},$$

a **hyperbolic Bessel function** of the first kind of order m.

The case $k = 2$

Example. $\displaystyle\sum_{n\geq 0} u_2(n)\frac{x^{2n}}{n!^2}$

$$= I_0(2x)^2 - I_1(2x)^2$$

$$= \sum_{n\geq 0} C_n \frac{x^{2n}}{n!^2}.$$

Painlevé II equation

Baik-Deift-Johansson:

Define $\boldsymbol{u(x)}$ by

$$\frac{d^2}{dx^2}u(x) = 2u(x)^3 + xu(x) \quad (*),$$

with certain initial conditions.

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$(*)$ is the **Painlevé II** equation (roughly, the branch points and essential singularities are independent of the initial conditions).

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1933: died in Paris.

The Tracy-Widom distribution

$$\boldsymbol{F(t)} = \exp\left(-\int_t^\infty (x-t)u(x)^2\,dx\right)$$

where $u(x)$ is the Painlevé II function.

The Baik-Deift-Johansson theorem

Let χ be a random variable with distribution F, and let χ_n be the random variable on $\mathfrak{S}_n$:

$$\chi_n(w) = \frac{\mathrm{is}_n(w) - 2\sqrt{n}}{n^{1/6}}.$$

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Theorem. *As* $n \to \infty$,

$$\chi_n \to \chi \quad \text{in distribution},$$

i.e.,

$$\lim_{n\to\infty} \mathrm{Prob}(\chi_n \le t) = F(t).$$

Expectation redux

Recall $E(n) \sim 2\sqrt{n}$.

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Corollary to BDJ theorem.

$$E(n) = 2\sqrt{n} + \left(\int t \, dF(t) \right) n^{1/6} + o(n^{1/6})$$

$$= 2\sqrt{n} - (1.7711\cdots)n^{1/6} + o(n^{1/6})$$

Proof of BDJ theorem

Gessel's theorem reduces the problem to "just" analysis, viz., the **Riemann-Hilbert problem** in the theory of integrable systems, and the **method of steepest descent** to analyze the asymptotic behavior of integrable systems.

Origin of Tracy-Widom distribution

Where did the Tracy-Widom distribution $F(t)$ come from?

$$F(t) = \exp\left(-\int_t^\infty (x-t)u(x)^2\,dx\right)$$

$$\frac{d^2}{dx^2}u(x) = 2u(x)^3 + xu(x)$$

Gaussian Unitary Ensemble (GUE)

Analogue of normal distribution for $n \times n$ hermitian matrices $M = (M_{ij})$:

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$$Z_n^{-1} e^{-\mathrm{tr}(M^2)} dM,$$

$$dM = \prod_i dM_{ii} \cdot \prod_{i<j} d(\Re M_{ij}) d(\Im M_{ij}),$$

where $\boldsymbol{Z_n}$ is a normalization constant.

Tracy-Widom theorem

Tracy-Widom (1994): let α_1 denote the largest eigenvalue of M. Then

$$\lim_{n\to\infty} \text{Prob}\left(\left(\alpha_1 - \sqrt{2n}\right)\sqrt{2}n^{1/6} \le t\right) = F(t).$$

Random topologies

Is the connection between is(w) and GUE a coincidence?

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Okounkov provides a connection, via the theory of **random topologies on surfaces**. Very briefly, a surface can be described in two ways:

- Gluing polygons along their edges, connected to random matrices via quantum gravity.
- Ramified covering of a sphere, which can be formulated in terms of permutations.

A variation

Alternating sequence of length k:

$$b_1 > b_2 < b_3 > b_4 < \cdots \ b_k$$

$\boldsymbol{E_n}$: number of alternating $w \in \mathfrak{S}_n$ (**Euler number**)

$E_4 = 5$: 2134, 3142, 3241, 4132, 4231

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Désiré André (1840–1917): showed in 1879 that

$$\sum_{n \geq 0} E_n \frac{x^n}{n!} = \sec x + \tan x$$

Alternating subsequences?

$\mathbf{as}(\boldsymbol{w})$ = length of longest alternating subseq. of w

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$$w = \mathbf{5}6\mathbf{2}1\mathbf{8}\mathbf{3}\mathbf{4}7 \Rightarrow \mathrm{as}(w) = 5$$

The main lemma

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$$\boldsymbol{a_k(n)} = \#\{w \in \mathfrak{S}_n \,:\, \mathrm{as}(w) = k\}$$

$$\begin{aligned} \boldsymbol{b_k(n)} &= a_1(n) + a_2(n) + \cdots + a_k(n) \\ &= \#\{w \in \mathfrak{S}_n \,:\, \mathrm{as}(w) \leq k\}. \end{aligned}$$

Recurrence for $a_k(n)$

$$\Rightarrow a_k(n) = \sum_{j=1}^{n} \binom{n-1}{j-1}$$

$$\sum_{2r+s=k-1} \left(a_{2r}(j-1) + a_{2r+1}(j-1)\right) a_s(n-j)$$

$B(x,t)$ and $A(x,t)$

Define

$$\begin{aligned} B(x,t) &= \sum_{k,n\geq 0} b_k(n) t^k \frac{x^n}{n!} \\ A(x,t) &= \sum_{k,n\geq 0} a_k(n) t^k \frac{x^n}{n!} \end{aligned}$$

The main generating function

Theorem.

$$B(x,t) = \frac{2/\rho}{1 - \frac{1-\rho}{t}e^{\rho x}} - \frac{1}{\rho}$$

$$A(x,t) = (1-t)B(x,t),$$

where $\boldsymbol{\rho} = \sqrt{1-t^2}$.

Formulas for $b_k(n)$

Corollary.

$$\begin{aligned} \Rightarrow \quad b_1(n) &= 1 \\ b_2(n) &= n \\ b_3(n) &= \tfrac{1}{4}(3^n - 2n + 3) \\ b_4(n) &= \tfrac{1}{8}(4^n - (2n-4)2^n) \\ &\vdots \end{aligned}$$

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$$\vdots$$

no such formulas for longest increasing subsequences

Mean (expectation) of $\mathrm{as}(w)$

$$\boldsymbol{D(n)} = \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \mathrm{as}(w),$$

the **expectation** of $\mathrm{as}(w)$ for $w \in \mathfrak{S}_n$

A formula for $D(n)$

$$\sum_{n\geq 1} D(n)x^n = \frac{\partial}{\partial t}A(x,1)$$

$$= \frac{6x-3x^2+x^3}{6(1-x)^2}$$

$$= x + \sum_{n\geq 2}\frac{4n+1}{6}x^n.$$

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$$\Rightarrow D(n) = \frac{4n+1}{6}, \quad n \geq 2$$

Comparison of $E(n)$ and $D(n)$

$$D(n) = \frac{4n+1}{6}, \quad n \geq 2$$

$$E(n) \sim 2\sqrt{n}$$

Variance of $\mathrm{as}(w)$

$$\boldsymbol{V(n)} = \frac{1}{n!}\sum_{w\in\mathfrak{S}_n}\left(\mathrm{as}(w) - \frac{4n+1}{6}\right)^2,\ n \geq 2$$

the **variance** of $\mathrm{as}(n)$ for $w \in \mathfrak{S}_n$

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Corollary.

$$V(n) = \frac{8}{45} n - \frac{13}{180}, \ n \geq 4$$

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Corollary.

$$V(n) = \frac{8}{45}n - \frac{13}{180}, \ n \geq 4$$

similar results for higher moments

A new distribution?

$$\boldsymbol{P(t)} = \lim_{n\to\infty} \mathrm{Prob}_{w\in\mathfrak{S}_n}\left(\frac{\mathrm{as}_n(w) - 2n/3}{\sqrt{n}} \le t\right)$$

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Stanley distribution?

Limiting distribution

Theorem (Pemantle, Widom, (Wilf)).

$$\lim_{n\to\infty} \mathrm{Prob}_{w\in\mathfrak{S}_n}\left(\frac{\mathrm{as}(w)-2n/3}{\sqrt{n}} \le t\right)$$

$$= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{t\sqrt{45}/4} e^{-s^2}\,ds$$

(Gaussian distribution)

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(Gaussian distribution)

k-alternating sequences

Given $k \geq 1$, define a sequence $a_1 a_2 \cdots a_n$ of integers to be **k-alternating** if

$$a_i > a_{i+1} \Leftrightarrow i \equiv 1 \pmod{k}.$$

k-alternating sequences

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Example. 61482572 is 3-alternating

$a_k(w)$ and $E_k(n)$

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$$\begin{aligned} \boldsymbol{E_k(n)} &= \text{expectation of } a_k(w) \\ &= \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} a_k(w) \end{aligned}$$

A problem

$E_k(n)$ interpolates between $E(n) \sim 2\sqrt{n}$ and $D(n) \sim 2n/3$. Is there a sharp cutoff between $c\sqrt{n}$ and cn behavior, or do we get intermediate values like cn^α, $\frac{1}{2} < \alpha < 1$, say for $k = \sqrt{n}$?

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Similar questions for the limiting distribution: do we interpolate between Tracy-Widom and Gaussian?

A variant

Same questions if we replace k-alternating with:

$$a_i > a_{i+1} \Leftrightarrow \lfloor i/k \rfloor \text{ is even.}$$

E.g., $k = 3$:

$$a_1 > a_2 > a_3 < a_4 < a_5 > a_6 > a_7 < \cdots$$


THE END