A generalization of the absolute order and of the lattice of noncrossing partitions

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(with J. Shareshian and M. Wachs)

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The absolute order on $\mathfrak{S}_n$

$\boldsymbol{\kappa(w)}$: number of cycles of $w \in \mathfrak{S}_n$

Example: $\kappa(42513) = \kappa((1,4)(2)(3,5)) = 3$.

Let $u, v \in \mathfrak{S}_n$. Define $\boldsymbol{u < v}$ if there is $i \neq j$ for which $v = (i,j)u$ and $\kappa(v) < \kappa(u)$. This defines the cover relations of the **absolute order** on $\mathfrak{S}_n$, denoted $\mathbf{Abs}(\mathfrak{S}_n)$.

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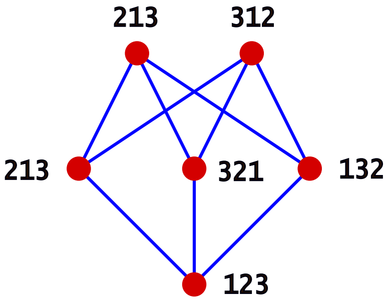

Properties of $\mathrm{Abs}(\mathfrak{S}_n)$

- $\mathrm{Abs}(\mathfrak{S}_n)$ is graded of rank $n-1$. The elements of rank k are those $w \in \mathfrak{S}_n$ with $n-k$ cycles, denoted

$$\mathrm{Abs}(\mathfrak{S}_n)_k = \{w \in \mathfrak{S}_n \,:\, \kappa(w) = n-k\}.$$

  Thus $\#\mathrm{Abs}(\mathfrak{S}_n)_k = \boldsymbol{c(n, n-k)}$, the signless Stirling number of the first kind. In particular, the number of maximal elements of $\mathrm{Abs}(\mathfrak{S}_n)$ is $(n-1)!$. The identity permutation id is the unique minimal element of $\mathrm{Abs}(\mathfrak{S}_n)$.

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- Hence the rank-generating function of $\mathrm{Abs}(\mathfrak{S}_n)$ is

$$\begin{aligned} F(\mathrm{Abs}(\mathfrak{S}_n), q) &:= \sum_{k=0}^{n-1} (\#\mathrm{Abs}(\mathfrak{S}_n)_k) q^k \\ &= (1+q)(1+2q)\cdots(1+(n-1)q). \end{aligned}$$

Further properties

- The Hasse diagram $\boldsymbol{\mathcal{H}(\mathrm{Abs}(\mathfrak{S}_n))}$ of $\mathrm{Abs}(\mathfrak{S}_n)$, considered as a graph, is the **Cayley graph** of $\mathfrak{S}_n$ with respect to the set $\boldsymbol{T_n}$ of transpositions. (Since the generators are involutions, we can replace directed edges $u \to v$ and $v \to u$ by a single edge u—v.)
- Since $\mathcal{H}$ is a Cayley graph, $\mathcal{H}(\mathrm{Abs}(\mathfrak{S}_n))$ is **vertex-transitive**.
- Since T_n is a conjugacy class, it follows from representation theory that the adjacency matrix of $\mathcal{H}(\mathrm{Abs}(\mathfrak{S}_n))$ has integer eigenvalues (that can be described explicitly).

Noncrossing partitions

Recall that a **noncrossing partition** of $[n] = \{1, 2, \ldots, n\}$ is a partition π of the set $[n]$ such that if $i < j$ are in the same block B and $k < l$ are in the same block B', and if $i < k < j < l$, then $B = B'$. Define $\mathbf{NC(}\boldsymbol{n}\mathbf{)}$ to be the set of all noncrossing partitions of n.

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For $\pi, \sigma \in \mathrm{NC}(n)$, define $\boldsymbol{\pi \leq \sigma}$ if π is a refinement of σ. The resulting poset (also denoted $\mathrm{NC}(n)$) is a graded lattice of rank $n - 1$.

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Easy fact. For every maximal element (or n-cycle) w of $\mathrm{Abs}(\mathfrak{S}_n)$, the interval $[\mathrm{id}, w]$ is isomorphic to $\mathrm{NC}(n)$. The isomorphism is given e.g. by $84376521 = (1, 8)(2, 4, 7)(3)(5, 6) \mapsto$ 18-247-3-56.

An edge labeling of $\mathrm{NC}(n)$

$[\pi, \tau]$: an edge of the Hasse diagram of $\mathrm{NC}(n)$, i.e., $\pi < \tau$ and no $\sigma \in \mathrm{NC}(n)$ satisfies $\pi < \sigma < \tau$ (τ **covers** π)

τ is obtained from π by merging two blocks B and B'. Let the least element i of $B \cup B'$ belong to B.

$\lambda(\pi, \tau)$: the largest element j of B which is less than every element of B'.

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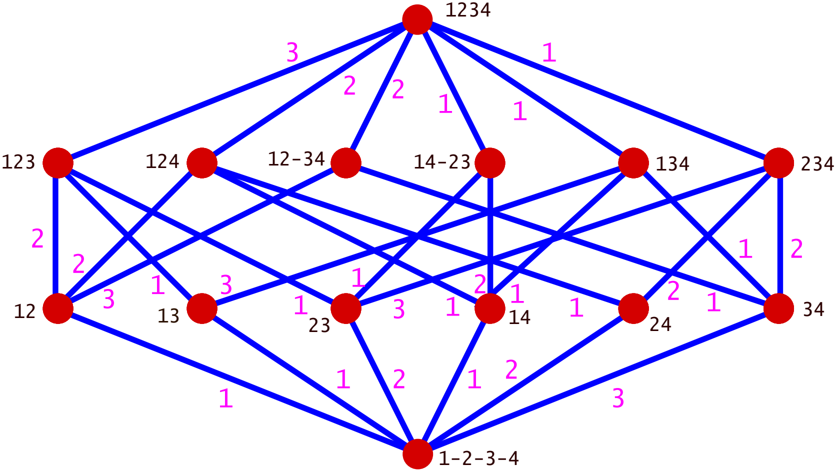
$\hat{0} = \pi_0 < \pi_1 < \cdots < \pi_{n-1} = \hat{1}$: a maximal chain $\boldsymbol{C}$ of $\mathrm{NC}(n)$

$\boldsymbol{\lambda(C)} := (\lambda(\pi_0, \pi_1), \lambda(\pi_1, \pi_2), \ldots, \lambda(\pi_{n-2}, \pi_{n-1}))$

Edge labeling of NC(4)



Parking functions

parking function of length n: a sequence $(a_1, \ldots, a_n) \in \mathbb{P}^n$ whose weakly increasing rearrangement $b_1 \leq b_2 \leq \cdots \leq b_n$ satisfies $b_i \leq i$.

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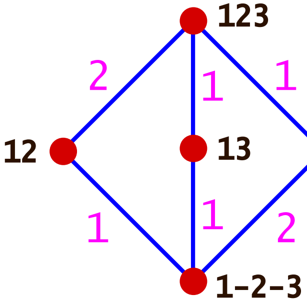
Theorem (RS, 1997). *The multiset of labels $\lambda(C)$ of maximal chains of $\mathrm{NC}(n)$ equals the set $\mathcal{P}(n-1)$ of parking functions of length $n-1$.*

The poset $\mathrm{NC}(3)$



Maximal chain labels: $\{12, 11, 21\} = \mathcal{P}(3)$

Flag f-vectors and flag h-vectors

P: a finite graded poset of rank n with $\hat{0}$ and $\hat{1}$

ρ: $P \to [0, n]$: rank function of P, so $\rho(\hat{0}) = 0$ and $\rho(\hat{1}) = n$

Let $S \subseteq [n-1]$, $\#S = r$. The **flag f-vector** $\alpha_P : 2^{[n-1]} \to \mathbb{Z}$ is defined by

$$\alpha_P(S) = \#\{\hat{0} = t_0 < t_1 < \cdots < t_r < t_{r+1} = \hat{1} :$$
$$S = \{\rho(t_1), \rho(t_2), \ldots, \rho(t_r)\}.$$

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The **flag h-vector** $\boldsymbol{\beta_P}: 2^{[n-1]} \to \mathbb{Z}$ is defined by

$$\beta_P(S) = \sum_{T \subseteq S} (-1)^{\#(S-T)} \alpha_P(T).$$

Equivalently,

$$\alpha_P(S) = \sum_{T \subseteq S} \beta_P(T).$$

Weak descent sets

weak descent set $\mathbf{wDes}(\sigma)$ of a sequence $\sigma = (a_1, \ldots, a_n) \in \mathbb{Z}^n$:
$\{i \, : \, 1 \leq i \leq n-1, \;\, a_i \geq a_{i+1}\}$

Definition. Let λ be a labeling of the edges of the Hasse diagram of a graded poset P of rank n with $\hat{0}$ and $\hat{1}$. Then λ is an **R'-labeling** if there is a unique strictly decreasing saturated chain in every interval $[s, t]$ of P.

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Corollary. *Assume that P has an R'-labeling. Then*

$$\begin{aligned} \alpha_P(S) &= \#\{\pi \in \mathcal{P}(n-2) \,:\, \mathrm{wDes}(\pi) \subseteq S\} \\ \beta_P(S) &= \#\{\pi \in \mathcal{P}(n-2) \,:\, \mathrm{wDes}(\pi) = S\} \ (\geq 0) \end{aligned}$$

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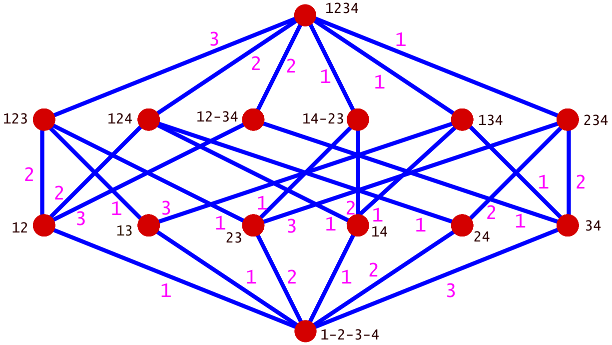

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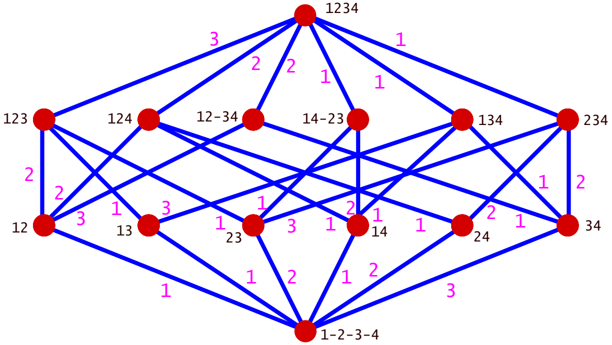

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$\pi \in \mathcal{P}(3)$ with weak descent set $\{1,2\}$:

$$111, \quad 211, \quad 221, \quad 311, \quad 321$$

The Ehrenborg quasisymmetric function

$\boldsymbol{P}$: finite graded poset of rank n with $\hat{0}$ and $\hat{1}$

$$\boldsymbol{E_P} = \sum_{S \subseteq [n-1]} \beta_P(S) L_S,$$

where $\boldsymbol{L_S}$ is the **fundamental quasisymmetric function** indexed by S and n

E_P is the **Ehrenborg quasisymmetric function** of P, a generating function for the flag h-vector values $\beta_P(S)$.

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Theorem. *Let P be graded of rank n with $\hat{0}$ and $\hat{1}$. Suppose that every interval $[s, t]$ of P is rank-symmetric, i.e., the rank generating function of $[s, t]$ has palindromic coefficients. Then E_P is a symmetric function.*

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Since every interval of $\mathrm{NC}(n)$ is self-dual, we get:

Corollary. *$E_{\mathrm{NC}(n)}$ is a symmetric function.*

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\begin{aligned}
\mathcal{P}(1) &= \mathfrak{S}_{\{1\}} \\
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E_{\mathrm{NC}(2)} &= e_1 \\
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Extends to all n, so:

Theorem. *$E_{\mathrm{NC}(n)}$ is e-positive.*

γ-positivity

A polynomial $P(q) \in \mathbb{R}[q]$ of degree n is **γ-positive** if it can be written

$$P(q) = \sum_{i=0}^{\lfloor n/2 \rfloor} \gamma_i q^i (1+q)^{n-2i}, \quad \gamma_i \geq 0.$$

Note. A γ-positive polynomial is palindromic and unimodal.

γ-positivity and e-positivity

Theorem. *Let P be a finite graded poset of rank n with $\hat{0}$ and $\hat{1}$. If E_P is an e-positive symmetric function, then the rank generating function $F_P(q)$ is γ-positive.*

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and if E_P is a symmetric function, then for $0 \le i \le \lfloor n/2 \rfloor$ we have

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Corollary. $F_{\mathrm{NC}(n)}$ *is γ-positive.* (Moreover,

$$p_i = \frac{1}{n} \binom{n}{i+1} \binom{n}{i},$$

a **Narayana number**, well-known to be γ-positive.)

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- $E_{\mathrm{NC}(n)}$ is an e-positive symmetric function
- e-positivity $\Rightarrow$ $F_{\mathrm{NC}(n)}(q)$ is γ-positive

A generalization

Fix n. Let $\mu = (\mu_1, \mu_2, \ldots, \mu_{n-2})$, $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-2} \geq 0$, $\mu_i \leq n - i - 1$, i.e., μ is a partition contained inside the **staircase** $(n-2, n-3, \ldots, 1)$. Number of such μ's is the **Catalan number** C_{n-1}.

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$$\mathfrak{S}[\mu] := \{w \in \mathfrak{S}_n \ : \ w(i) \leq n - \mu_i\}$$

A generalization

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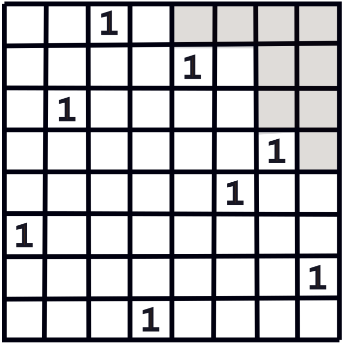
Example. $\mu = \emptyset \Rightarrow \mathfrak{S}[\mu] = \mathfrak{S}_n$

$$\mu = (1) \Rightarrow \mathfrak{S}[\mu] = \{w \in \mathfrak{S}_n \,:\, w(1) \neq n\},$$

so $\#\mathfrak{S}[\mu] = n! - (n-1)! = (n-1)(n-1)!$

Alternative viewpoint

Then $\mathfrak{S}[\mu]$ consists of all $w \in \mathfrak{S}_n$ whose permutation matrix avoids the reflected shape μ in the upper-right corner of an $n \times n$ square grid.

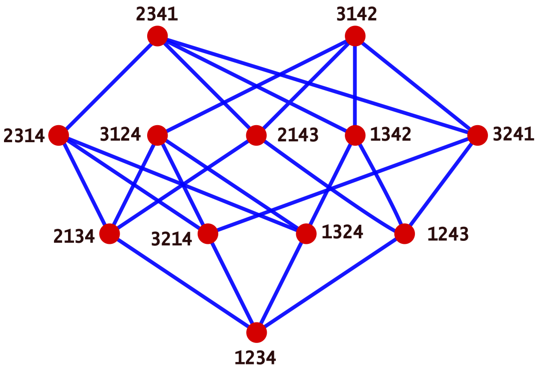


$$n = 8, \quad w = 35276184, \quad \mu = (4, 2, 2, 1)$$

Some subposets of $\mathrm{Abs}(\mathfrak{S}_n)$

$\mathbf{Abs}(\mathfrak{S}_n)[\mu]$: the poset $\mathrm{Abs}(\mathfrak{S}_n)$ restricted to $\mathfrak{S}[\mu]$

$n = 4, \quad \mu = (1, 1)$, so $w(1) \neq 4, \; w(2) \neq 4$.



Simple properties of $\mathrm{Abs}(\mathfrak{S}_n)[\mu]$

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Vertex-transitivity

$\mathrm{Hasse}_n(\mu)$: Hasse diagram of $\mathrm{Abs}(\mathfrak{S}_n)[\mu]$, regarded as an (undirected) graph

Recall that $\mathrm{Hasse}_n(\emptyset)$ (the Hasse diagram of $\mathrm{Abs}(\mathfrak{S}_n)$) is the Cayley graph of $\mathfrak{S}_n$ with respect to the set of all transpositions. Hence it is vertex-transitive and by the representation theory of $\mathfrak{S}_n$ its adjacency matrix has "nice" integer eigenvalues.

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- Vertex-transitive for all 42 μ when $n = 6$ with 9 exceptions.
- (**S. Bang** and **RS**) For any n and $1 \leq k \leq n - 2$, $\mu = (k)$ yields vertex-transitivity.

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Recall that $\mathrm{Hasse}_n(\emptyset)$ has integer eigenvalues with nice descriptions because it's a Cayley graph of $\mathfrak{S}_n$.

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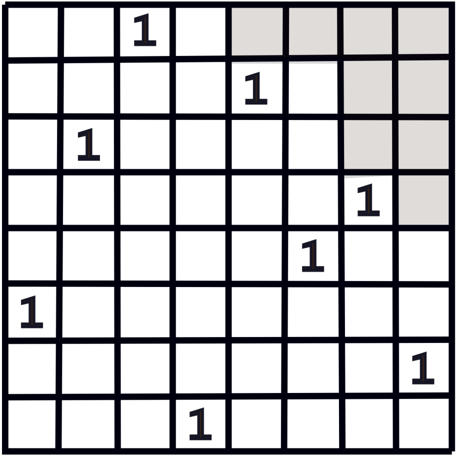
Hanlon develops his work in terms of **rook theory**. Note that $\mathfrak{S}[\mu]$ can be regarded as the set of placements of n rooks on the Ferrers board $[n] \times [n] - \mu$.

Eight rooks on $[8] \times [8] - (4,2,2,1) = (8,8,8,8,7,6,6,4)$



A μ-analogue of $\mathrm{NC}(n)$

Recall that for any maximal element (n-cycle) w of $\mathrm{Abs}(\mathfrak{S}_n)$, we have $[\mathrm{id}, w] \cong \mathrm{NC}(n)$. What about maximal elements of $\mathrm{Abs}(\mathfrak{S}_n)[\mu]$?

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Known: both are graded of rank $n - 1$.

Unknown: number of elements, rank-generating function, flag f-vector, whether they are isomorphic.

Edge labeling

Because $\mu \subseteq (n-2, n-3, \ldots, 1)$, we always have that the cycle $(1, 2, \ldots, n) \in \mathrm{Abs}(\mathfrak{S}_n)[\mu]$. Define

$$\mathrm{NC}(n)[\mu] = [\mathrm{id}, (1, 2, \ldots, n)],$$

the "canonical" μ-generalization of $\mathrm{NC}(n)$. (We don't know what happens for other maximal elements of $\mathrm{Abs}(\mathfrak{S}_n)[\mu]$.)

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Every edge of the Hasse diagram of $\mathrm{NC}(n)[\mu]$ is also an edge of the Hasse diagram of $\mathrm{NC}(n)$. Hence define the edge label $\boldsymbol{\lambda(\pi, \tau)}$ in $\mathrm{NC}(n)[\mu]$ to be the same as for $\mathrm{NC}(n)$.

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Corollary. *Every maximal chain label of $\mathrm{NC}(n)[\mu]$ is a parking function, and all maximal chain labels are distinct.*

Maximal chain labels

Crucial lemma. *If* $(a_1, ..., a_i, a_{i+1}, \ldots, a_{n-1})$ *is the label of a maximal chain of* $\mathrm{NC}(n)[\mu]$, *then so is* $(a_1, ..., a_{i+1}, a_i, \ldots, a_{n-1})$.

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Corollary. *Fix* μ. *There is a set* M_μ *of (weakly) increasing parking functions of length* $n-1$ *such that the set of maximal chain labels of* $\mathrm{NC}(n)[\mu]$ *is given by*

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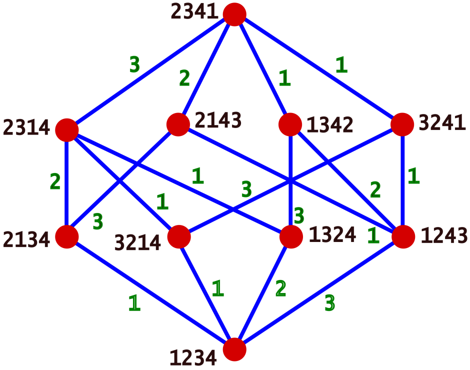
Corollary. *The edge labelling* λ *of* $\mathrm{NC}(n)[\mu]$ *is an* R'*-labeling.*

The example $n = 4$ and $\mu = (1,1)$ $(w(1), w(2) \neq 4)$



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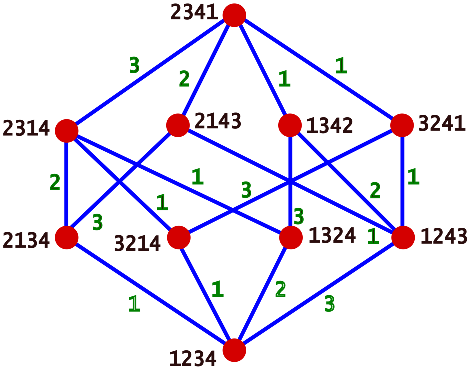

maximal chain labels:

$$\{123, 132, 213, 231, 312, 321, 113, 131, 311\} = \mathfrak{S}_{1,2,3} \cup \mathfrak{S}_{1,1,3}$$

$$\Rightarrow E_{\mathrm{NC}(4)[1,1]} \;=\; e_1^3 + e_2 e_1 .$$

γ-positivity

e-positivity of $E_{\mathrm{NC}(n)[\mu]}$ implies:

Corollary. *For all n and μ, the rank-generating function $F_{\mathrm{NC}(n)[\mu]}(q)$ of $\mathrm{NC}(n)[\mu]$ is γ-positive.*

The graph $\Gamma_{n,\mu}$

Identify μ with a reflected Young diagram in the upper-right corner of $[n] \times [n]$.

$\Gamma_{n,\mu}$: the graph on the vertex set $[n]$ with edges ij, where $i < j$ and $(i,j) \notin \mu$.

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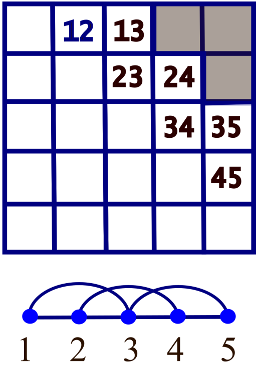

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The graphs $\Gamma_{n,\mu}$ are exactly the "canonically labelled" **unit interval graphs**. These are the graphs for which **Tatsuyuki Hikita** proved have e-positive chromatic symmetric functions.

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No connection known between chromatic symmetric functions and this talk.

The e-expansion of $E_{\mathrm{NC}(n)[\mu]}$

Theorem. *The coefficient of* e_λ *in* $E_{\mathrm{NC}(n)[\mu]}$ *is the number of noncrossing partitions* ρ *of* $[n-1]$ *of type* λ *such that if* $1 \le a < b \le n-1$ *and* a, b *are in the same block of* ρ*, then* $\{a, b+1\}$ *is an edge of* $\Gamma_{n,\mu}$.

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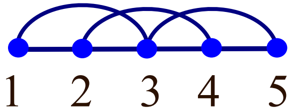

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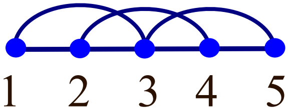

valid noncrossing partitions: 1-2-3-4, 12-3-4, 23-1-4, 34-1-2

$$\Rightarrow E_{\mathrm{NC}(5)[2,1]} = e_1^4 + 3e_2e_1^2$$

Rank generating function of $E_{\mathrm{NC}(n)[\mu]}$

Corollary. *Let* γ_i *denote the number of noncrossing matchings of* $[n-1]$ *of with* i *edges (and hence* $n-2i$ *singleton vertices) such that if* $1 \leq a < b \leq n-1$ *and* a, b *are in the same block of* ρ*, then* $\{a, b+1\}$ *is an edge of* $\Gamma_{n,\mu}$*. Then*

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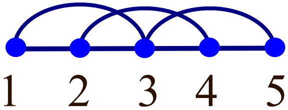

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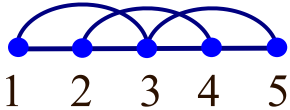

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$$\begin{aligned} \Rightarrow F_{\mathrm{NC}(5)[2,1]} &= (1+q)^4 + 3q(1+q)^2 \\ &= 1 + 7q + 12q^2 + 7q^3 + q^4. \end{aligned}$$

Real zeros

Aside. Does $F_{\mathrm{NC}(n)[\mu]}$ have only real zeros?

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True for $n \leq 6$.

Noncrossing bond lattices

G: simple graph on the vertex set $[n]$

connected partition of G: a partition ρ of $[n]$ such that the restriction of G to every block of ρ is connected

bond lattice L_G of G: the set of connected partitions of G, ordered by refinement

Note. L_G is a geometric lattice with many remarkable properties.

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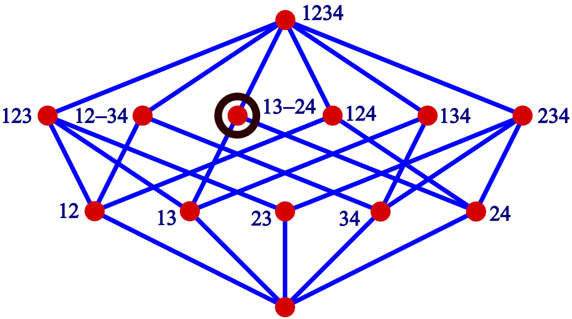
Definition. The **noncrossing bond lattice** $\mathrm{NB}(G)$ of G is the subposet of L_G whose elements are noncrossing partitions.

Connection with $\mathrm{NC}(n)[\mu]$

(ordinary) bond lattice $L_{\Gamma_{4,(1)}} \cong K_4 - e$:



$$n = 4, \quad \mu = (1), \quad L_{\Gamma_{4,(1)}} - 13\text{-}24 = \mathrm{NC}(4)[(1)]$$

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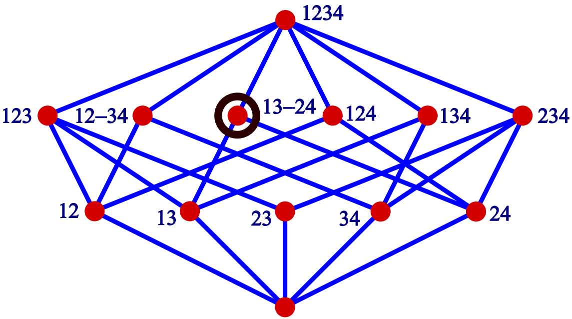

$$n = 4, \quad \mu = (1), \quad L_{\Gamma_{4,(1)}} - 13\text{-}24 = \mathrm{NC}(4)[(1)]$$

Theorem. *The noncrossing bond lattice of $\Gamma_{n,\mu}$ is isomorphic to* $\mathrm{NC}(n)[\mu]$.

The final slide