The X-Descent Set of a Permutation

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The Descent Set of a Permutation

$$w = a_1 a_2 \cdots a_n \in \mathfrak{S}_n$$

descent set of w: $\mathrm{Des}(w) = \{1 \leq i \leq n-1 \,:\, a_i > a_{i+1}\}$

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Want a **generating function** for

$$\beta_n(S) := \#\{w \in \mathfrak{S}_n \,:\, D(w) = S\},$$

i.e.,

$$\sum_{w \in \mathfrak{S}_n} Y_{\mathrm{Des}(w)} = \sum_{S \subseteq [n-1]} \beta_n(S) Y_S,$$

for some (linearly independent) algebraic entities Y_S, $S \subseteq [n-1]$.

Best choice here of Y_S

Fix n. For $S \subseteq [n-1]$, define

$$F_S = \sum_{\substack{1 \le i_1 \le i_2 \le \cdots \le i_n \\ i_j < i_{j+1} \text{ if } j \in S}} x_{i_1} x_{i_2} \cdots x_{i_n},$$

known as **(Gessel's) fundamental quasisymmetric function**.

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known as **(Gessel's) fundamental quasisymmetric function**.

Theorem. $\sum_{w \in \mathfrak{S}_n} F_{\mathrm{Des}(w)} = (x_1 + x_2 + \cdots)^n = p_1^n$

The case $n = 3$

w	$F_{\mathrm{Des}(w)}$
123	$\sum_{1 \leq a \leq b \leq c} x_a x_b x_c$
132	$\sum_{1 \leq a \leq b < c} x_a x_b x_c$
213	$\sum_{1 \leq a < b \leq c} x_a x_b x_c$
231	$\sum_{1 \leq a \leq b < c} x_a x_b x_c$
312	$\sum_{1 \leq a < b \leq c} x_a x_b x_c$
321	$\sum_{1 \leq a < b < c} x_a x_b x_c$
	$(x_1 + x_2 + \cdots)^3$

X-descent sets

$$X \subseteq \mathcal{E}_n := \{(i,j) \,:\, 1 \leq i \leq n,\ 1 \leq j \leq n,\ i \neq j\}$$

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Example. (a) $X = \{(i,j) : n-1 \geq i > j \geq 1\}$: $\mathrm{XDes} = \mathrm{Des}$ (the ordinary descent set)

(b) $X = \{(i,j) \in [n] \times [n] : i \neq j\}$: $\mathrm{XDes}(w) = [n-1]$, where $[n-1] = \{1, 2, \ldots, n-1\}$

Symmetric functions

Symmetric function: $f = f(x_1, x_2, \dots)$, a power series of bounded degree with rational coefficients, invariant under any permutation of the x_i's.

partition of n: $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots)$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$, $\sum \lambda_i = n$, denoted $\lambda \vdash n$

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Example. Power sums: $\boldsymbol{p_k} = \sum_i x_i^k$ (with $p_0 = 1$),

$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots ,$$

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Schur functions s_λ: another $\mathbb{Q}$-basis, not defined here

A generating function for the XDescent set

Define $U_X = \sum_{w \in \mathfrak{S}_n} F_{\mathrm{XDes}(w)}$.

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Example. $n = 3, \; X = \{(1,3),(2,1),(3,1),(3,2)\}$

w	$\mathrm{XDes}(w)$
123	$\emptyset$
132	$\{1,2\}$
213	$\{1,2\}$
231	$\{2\}$
312	$\{1\}$
321	$\{1,2\}$

$$U_X = F_{\emptyset} + F_1 + F_2 + 3F_{1,2} = p_1^3 - p_2 p_1 + p_3 = s_3 + s_{21} + 2s_{111}$$

First easy theorem

Theorem. (a) *U_X is a p-integral symmetric function, i.e., $U_X = \sum_\lambda c_\lambda p_\lambda$, where $c_\lambda \in \mathbb{Z}$.*

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Proof. Consider the coefficient of a monomial, say $\mathfrak{m} = x_1^2 x_2^3 x_4^2$ (where $n = 7$). Recall

$$\boldsymbol{U_X} = \sum_{w \in \mathfrak{S}_n} F_{\mathrm{XDes}(w)}$$

$$\boldsymbol{F_S} = \sum_{\substack{1 \le i_1 \le i_2 \le \cdots \le i_n \\ i_j < i_{j+1} \text{ if } j \in S}} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

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Let $w = a_1 a_2 \cdots a_7$. Thus $\mathfrak{m}$ appears in $F_{\mathrm{XDes}(w)}$ if and only if $(a_1, a_2), (a_3, a_4), (a_4, a_5), (a_6, a_7) \notin X$.

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Write $w = a_1 a_2 \cdot a_3 a_4 a_5 \cdot a_6 a_7 = u_1 u_2 u_3$ (juxtaposition of words). Then $x_1^3 x_2^2 x_4^2$ appears in $F_{\mathrm{XDes}(w')}$, where $w' = u_2 u_1 u_3$. Generalizing shows that U_X is a symmetric function.

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Also $x_1^2 x_2^3 x_4^2 = \mathfrak{m}$ appears in $F_{\mathrm{XDes}(w'')}$, where $w'' = u_3 u_2 u_1$. Generalizing shows that the coefficient of $x_1^{\alpha_1} x_2^{\alpha_2} \cdots$ in U_X is an integer multiple of $m_1!\, m_2! \cdots$, where $m_i = \#\{j : \alpha_j = i\}$.

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Well-known and easy that this implies U_X is p-integral (given that U_X is a symmetric function). $\square$

Second easy theorem

ω: linear transformation on symmetric functions given by $\omega(p_\lambda) = (-1)^{n-\ell(\lambda)} p_\lambda$ for $\lambda \vdash n$, where $\ell(\lambda) = \#\{i : \lambda_i > 0\}$.

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Theorem. *Let* $\overline{X} = \mathcal{E}_n - X$*. Then* $\omega U_X = U_{\overline{X}}$*.*

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Proof. Exercise.

Special case

record set $\mathrm{rec}(\boldsymbol{w})$ for $w = a_1 \cdots a_n \in \mathfrak{S}_n$:
$\mathrm{rec}(w) = \{0 \leq i \leq n-1 \,:\, a_i > a_j \text{ for all } j < i\}$. Thus always $0 \in \mathrm{rec}(w)$.

record partition $\mathrm{rp}(\boldsymbol{w})$: if $\mathrm{rec}(w) = \{r_0, \ldots, r_j\}_<$, then $\mathrm{rp}(w)$ is the numbers $r_1 - r_0, r_2 - r_1, \ldots, n - r_j$ arranged in decreasing order.

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Theorem (conjectured by **RS**, proved by **I. Gessel**). *Let X have the property that if $(i,j) \in X$ then $i > j$. Then*

$$U_X = \sum_{\substack{w \in \mathfrak{S}_n \\ \mathrm{XDes}(w) = \emptyset}} p_{\mathrm{rp}(w)}.$$

In particular, U_X is p-positive.

An example

$n = 4, \ X = \{(2,1),(3,2),(4,3)\}$

w	rp(w)
1234	1111
1342	211
1423	31
2314	211
2341	211
2413	31
312**4**	31
314**2**	22
3412	31
4123	4
4231	4

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$$\Rightarrow U_X = p_1^4 + 3p_2p_1^2 + 4p_3p_1 + p_2^2 + 2p_4$$

A generalization

Theorem (**D. Grinberg**) *Suppose that* $(i,j) \in X \Rightarrow (j,i) \notin X$. *Then* U_X *is p-positive.*

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Theorem (**D. Grinberg**) *Suppose that* $(i,j) \in X \Rightarrow (j,i) \notin X$*. Then* U_X *is p-positive.*

In fact, Grinberg has a combinatorial interpretation of the coefficients (not given here).

Connection with chromatic symmetric functions

$\boldsymbol{P}$: partial ordering of $[n]$

$\boldsymbol{Y_P} = \{(i,j) \,:\, i >_P j\}$

$\mathbf{inc}(\boldsymbol{P})$: incomparability graph of P, i.e., vertex set $[n]$, edges ij if $i \parallel j$ in P

$\boldsymbol{X_G}$: chromatic symmetric function of the graph G (generalizes the chromatic polynomial)

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Theorem. $U_{Y_P} = \omega X_{\mathrm{inc}(P)}$

Succession-free permutations

Let $X = \{(1,2),(2,3),\ldots,(n-1,n)\}$ (**successions**).

$f_n = \#\{w \in \mathfrak{S}_n \,:\, \mathrm{XDes}(w) = \emptyset\}$ (**succession-free** permutations)

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Theorem. $\displaystyle U_X = \sum_{i=1}^{n} f_i\, s_{i,1^{n-i}}$

(generating function for $w \in \mathfrak{S}_n$ by positions of successions, i.e, the **succession set** of w)

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Example. $n = 4$: $U_X = 11s_4 + 3s_{31} + s_{211} + s_{1111}$

Sketch of proof

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Left-hand side: $\#\{w \in \mathfrak{S}_n \,:\, \mathrm{XDes}(w) = S\}$

Right-hand side: Use

$$s_{i,1^{n-i}} = \sum_{S \in \binom{[n-1]}{n-i}} F_S.$$

To show: $f_i = \#\{w \in \mathfrak{S}_n \,:\, \mathrm{XDes}(w) = S\}$ if $\#S = n - i$.

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Will define a bijection (for fixed n and i)

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Example. $w = 5641237$, so $S = \{1, 4, 5\}$, $n = 7$, $i = 4$. Factor w:

$$w = 56 \cdot 4 \cdot 123 \cdot 7.$$

Let $123 \to 1$, $4 \to 2$, $56 \to 3$, $7 \to 4$: get

$$w \to 3214 = u. \quad \square$$

A q-analogue for $X = \{(1,2),(2,3),\dots,(n-1,n)\}$

Let $U_X(q) = \sum_{w \in \mathfrak{S}_n} q^{\mathrm{asc}(w^{-1})} F_{\mathrm{XDes}(w)}$, where asc denotes the number of (ordinary) ascents.

Thus $U_X(q)$ is the generating function for $w \in \mathfrak{S}_n$ by succession set and by $\mathrm{asc}(w^{-1})$. Define

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Theorem. $U_X(q) = \sum_{i=1}^{n} q^{n-i} f_i(q) s_{i,1^{n-i}}$

Digraph interpretation

We can also regard X as a **digraph**, with edges $i \to j$ if $(i, j) \in X$.

A **Hamiltonian path** in X is a permutation $a_1 a_2 \cdots a_n \in \mathfrak{S}_n$ such that $(a_i, a_{i+1}) \in X$ for $1 \le i \le n-1$. Define

$$\textbf{ham}(\boldsymbol{X}) = \#\ \text{Hamiltonian paths in } X$$

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NOTE.

- $w \in \mathfrak{S}_n$ is a Hamiltonian path in X if and only $\mathrm{XDes}(w) = [n-1]$.
- w is a Hamiltonian path in $\overline{X}$ if and only if $\mathrm{XDes}(w) = \emptyset$.

Connection with U_X

Theorem. *Let* $U_X = \sum_\lambda c_\lambda p_\lambda$. *Then* $\mathrm{ham}(\overline{X}) = \sum_\lambda c_\lambda$.

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Proof. Recall $U_X = \sum_{w \in \mathfrak{S}_n} F_{\mathrm{XDes}(w)}$. Since $w \in \mathfrak{S}_n$ is a Hamiltonian path in $\overline{X}$ if and only if $\mathrm{XDes}(w) = \emptyset$,

$$\mathrm{ham}(\overline{X}) = \#\{w \in \mathfrak{S}_n : \mathrm{XDes}(w) = \emptyset\}.$$

Note

$$[x_1^n]F_S = \begin{cases} 1, & S = \emptyset \\ 0, & \text{otherwise.} \end{cases}$$

Also for $\lambda \vdash n$, $[x_1^n]p_\lambda = 1$.

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Take coefficient of x_1^n on both sides of

$$U_X = \sum_{w \in \mathfrak{S}_n} F_{\mathrm{XDes}(w)} = \sum_\lambda c_\lambda p_\lambda. \qquad \square$$

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Proof. Recall $\omega p_\lambda = (-1)^{n-\ell(\lambda)} p_\lambda$ and $\omega U_X = U_{\overline{X}}$. Now apply ω to $U_X = \sum_\lambda c_\lambda p_\lambda$ and use previous theorem:

$$\mathrm{ham}(\overline{X}) = \sum_\lambda c_\lambda. \qquad \square$$

Berge's theorem

Theorem (**C. Berge**). $\operatorname{ham}(X) \equiv \operatorname{ham}(\overline{X}) \pmod 2$

Proof (**D. Grinberg**). Let $U_X = \sum_\lambda c_\lambda p_\lambda$. To prove:

$$\sum (-1)^{n-\ell(\lambda)} c_\lambda \equiv \sum c_\lambda \pmod 2.$$

Berge's theorem

Theorem (**C. Berge**). $\mathrm{ham}(X) \equiv \mathrm{ham}(\overline{X}) \pmod 2$

Proof (**D. Grinberg**). Let $U_X = \sum_\lambda c_\lambda p_\lambda$. To prove:

$$\sum (-1)^{n-\ell(\lambda)} c_\lambda \equiv \sum c_\lambda \pmod 2.$$

Obvious since $(-1)^{n-\ell(\lambda)} = \pm 1$. $\square$

Tournaments

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Theorem (**D. Grinberg**). *Let X be a tournament. Then*

$$U_X = \sum_w 2^{\mathrm{nsc}(w)} p_{\rho(w)},$$

where w ranges over all permutations in $\mathfrak{S}_n$ of odd order such that every nonsingleton cycle of w is a (directed) cycle of X, and where $\mathbf{nsc}(\boldsymbol{w})$ *denotes the number of nonsingleton cycles of w.*

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Special case of a result for **any** X.

A corollary

Theorem (repeated). *Let X be a tournament. Then*

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Corollary. *If X is a tournament, then*

$$U_X \in \mathbb{Z}[p_1, 2p_3, 2p_5, 2p_7, \ldots].$$

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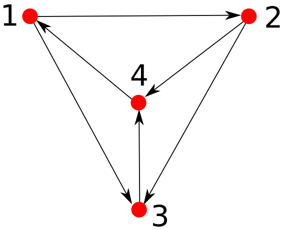
Note. Thus U_X can be written uniquely as a linear combination of Schur's "shifted Schur functions" P_λ, where λ has distinct parts. Can anything worthwhile be said about the coefficients?

An example



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(3,4)
(4,1)

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$(1)(2)(3)(4)$	p_1^4
$(1,2,4)(3)$	$2p_3p_1$
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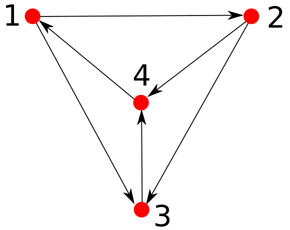

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$$\Rightarrow U_X = p_1^4 + 4p_3p_1 = 5P_4 - 2P_{3,1}$$

An application to Hamiltonian paths

Observation (repeated). Let $U_x = \sum_\lambda c_\lambda p_\lambda$. Then

$$\mathrm{ham}(X) = \sum_\lambda (-1)^{n-\ell(\lambda)} c_\lambda.$$

Theorem (repeated). *Let X be a tournament. Then*

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Rédei's theorem

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Since $c_{1^n} = 1$ for **all** X (immediate from $U_X = \sum_{w \in \mathfrak{S}_n} F_{\operatorname{XDes}(w)}$), we conclude:

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Theorem (**L. Rédei**, 1934) *Every tournament has an odd number of Hamiltionian paths.*

The final slide