EXTREMAL PROBLEMS

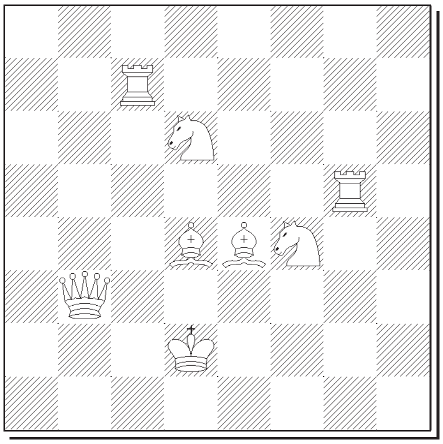
These transparencies were used for the second lecture in a series of two lectures (the first given by Noam Elkies) given at M.I.T. in January, 2000, and are based on a book in preparation. They are available at

www-math.mit.edu/~rstan/trans.html#chess

Construction tasks.

Most number of moves by the eight White officers (with no Black units on the board)

M. Bezzel, 1849



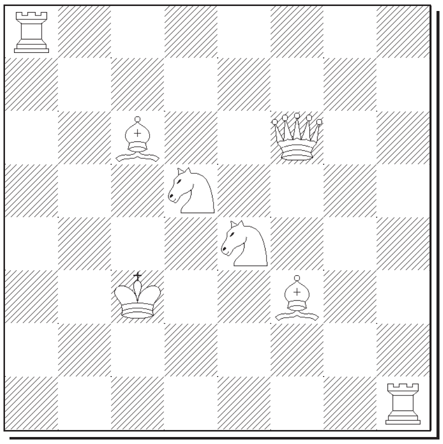
One hundred moves by eight White pieces.

Proved by E. Landau in 1899 to be the maximum possible.

Kling (1849) asked: Can the eight White officers control (guard) every square, occupied or not?

Can only be done with bishops of the same color (unique up to symmetry).

Kling, 1849

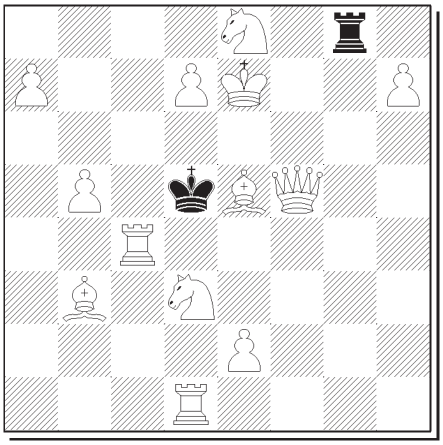


All squares controlled

Later proved impossible with bishops on squares of opposite colors. Then the maximum is 63 squares (144 solutions up to symmetry).

Maximum (known) number of checkmates in one move:

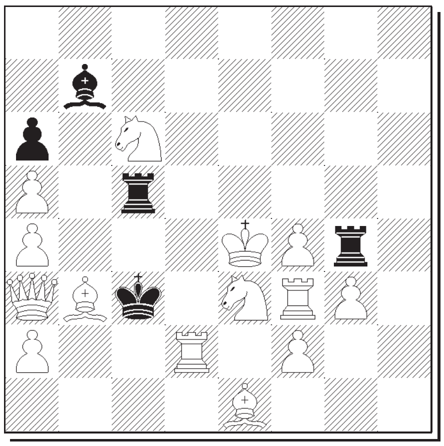
H. Pöllmacher and five others, 1859



47 mates in one move

Maximum number of forced checkmates in one move:

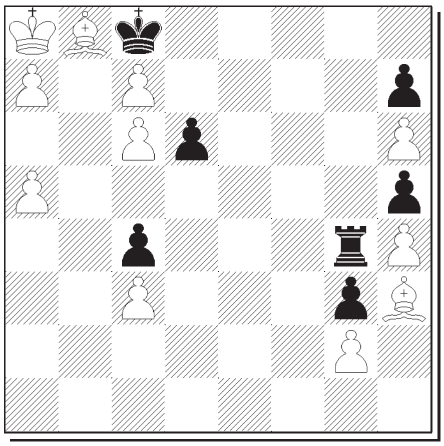
H. H. Cross, 1936



29 forced mates in one move

The ultimate no-brainer:

Noam Elkies, 1994

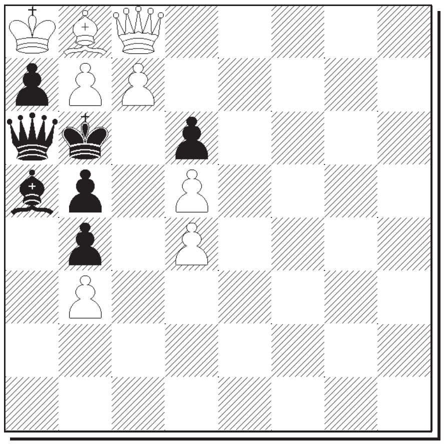


Mate in 7

Can the number 7 be increased without promoted pieces on the board? With promoted pieces a mate in 10 is possible (Elkies).

Ultimate mutual Zugzwang:

H. Hünerkopf, 1972



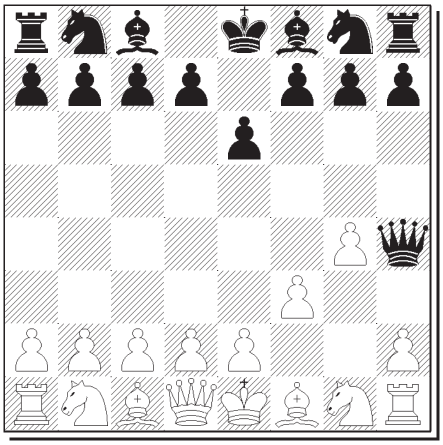
Whoever moves must be mated in one

Can the number 12 of White and Black initial moves be increased?

Open. For what (m, n) does there exist a legal position such that if White moves first then Black *must* checkmate White in exactly m moves, while if Black moves first then White *must* checkmate Black in exactly n moves. (Known only for $m = n = 1$, and for $m = n = 2$ using promoted pieces (Elkies).)

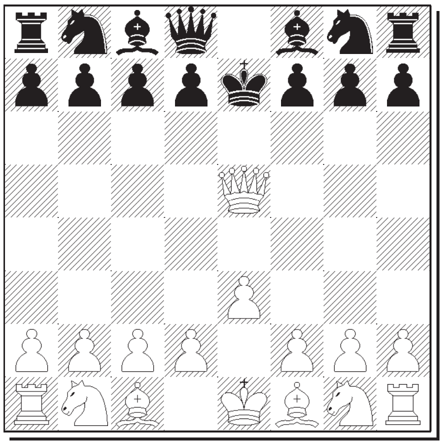
Synthetic games.

Shortest game ending ending in checkmate (Fool's Mate): eight "essentially equivalent" solutions in 2.0 moves (i.e., two White and two Black moves). Two solutions have the final position:



Fool's Mate

Shortest game ending with checkmate by capture: 2.5 moves (two essentially equivalent solutions).



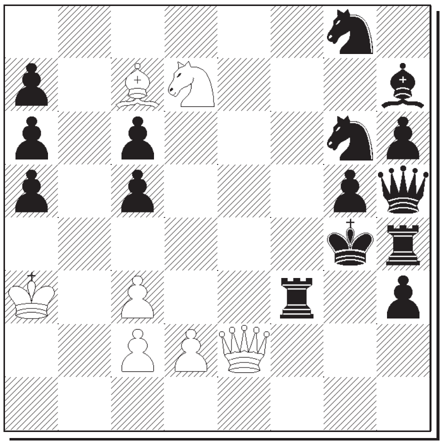
Mate by capture

- Construct a legal game of chess in which Black mates White by discovered check on Black's fourth move.
- Construct a legal game of chess in which Black makes no captures and Black stalemates White on Black's twelfth move.
- Construct a legal game of chess in which Black mates White on Black's fifth move by promoting a pawn to a bishop.
- Construct a legal game of chess in which Black mates White on Black's fifth move by promoting a pawn to a knight.

Length records.

Let n be the largest integer for which there exists a dual-free mate in n, i.e, in a legal position White is to checkmate Black in n move. At least one Black defense forces each White move uniquely. All claims to records derive from:

W. Jørgensen, 1976



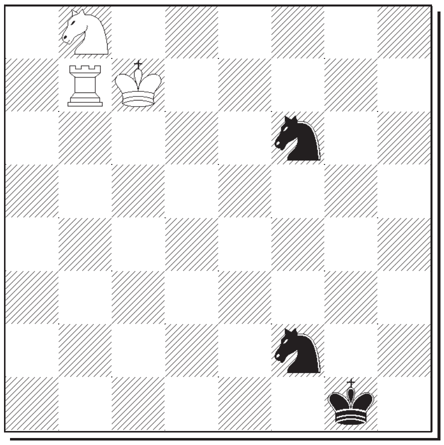
Mate in 200

Longest one with no doubt about soundness: mate in 208 by A. Chéron (1979). A mate in 226 by Jørgensen himself may be sound.

See Appendix for solution to above problem.

What about mate in n allowing duals (i.e., at least one Black defense requires n moves by White, but they don't have to be uniquely determined)? Without promoted force, the record for a long time was $n = 257$, by O. T. Blathy (1889). Recently Ken Thompson reported the unique longest mate with RN vs. NN.

Ken Thompson (& computer), 1999 or 2000



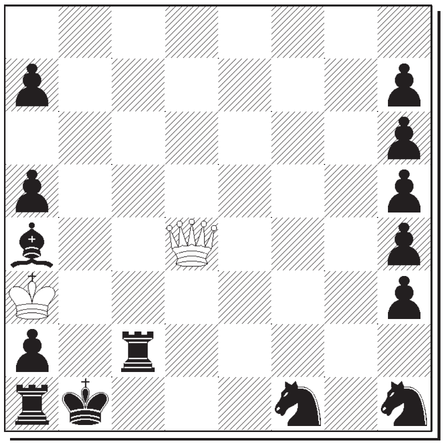
Mate in 262

This can be extended to at least mate in 265 with a prequel.

Conclusion: Human beings will never be able to understand chess completely.

Longest dual-free mate where White has only a queen, based on mate in 127 by O. T. Blathy.

J. Halumbirek (after O. T. Blathy), 1955



Mate in 130

Other dual-free mate records:

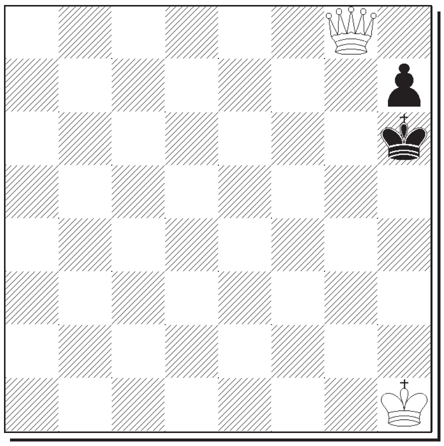
- rook minimal: 69 moves
- bishop minimal: 38 moves
- knight minimal: 42 moves
- pawn minimal: 72 moves

ENUMERATIVE PROBLEMS

Let P be a chess position. Let $f(n)$ be the number of different games (or games ending in checkmate, or "games" where only White moves, etc.) starting from P (with or without specifying who moves first). Transfer-matrix method $\Rightarrow$

Theorem. *Let $F(x) = \sum_{n=0}^{\infty} f(n)x^n$. Then $F(x)$ is a rational function (quotient of two polynomials).*

Example (Elkies). In the following position, let $f(n)$ be the number of ways Black can make n consecutive moves, followed by a checkmate in one move by White. (Black may not move into check.)



Serieshelpmate in n: how many solutions?

Then

$$\begin{aligned}\sum_{n=0}^{\infty} f(n)x^n &= \frac{x^5(2+5x-4x^2-2x^3)}{(1-x^2)(1-2x^2)(1-3x^2+x^4)} \\ &= 2x^5+5x^6+8x^7+28x^8+24x^9+108x^{10}+66x^{11}+\cdots \\ f(2m) &= 3-2^{m+2}+F_{2m+3},\ m>0 \\ f(2m+1) &= 2(F_{2m-1}-1),\end{aligned}$$

where F_n is a Fibonacci number:

$$F_1 = F_2 = 1, \qquad F_{n+1} = F_n + F_{n-1}.$$

Let P be the initial position. Let $f(n)$ be the number of games in n single moves, and $g(n)$ the number ending in checkmate. Then

$$\begin{aligned}
(f(0), f(1), \ldots, f(8)) = (&1, 20, 400, 8902, 197281, 4865609, \\
&119060324, 3195901860, \\
&84998978956, 2439530234167).
\end{aligned}$$

$$(g(4), g(5), g(6)) = (8, 347, 10828),$$

the latter computed in 1897.

Classical ballot problem. Let C_n be the number of ways $2n$ voters can vote sequentially for two candidates A and B, so that each receives n votes and A never trails B during the voting. E.g., $C_3 = 5$:

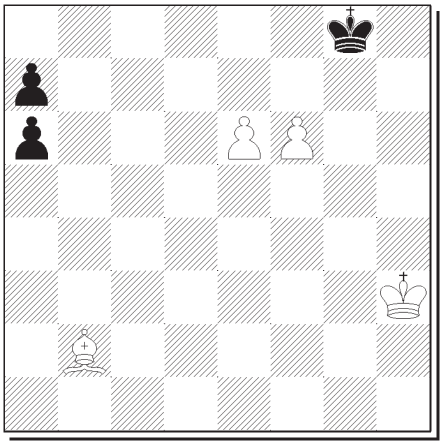
$$AAABBB \quad AABABB \quad AABBAA \quad ABAABB \quad ABABAB.$$

C_n is the *Catalan number*

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

For 66 combinatorial interpretations, see Exercise 6.19 of R. Stanley, *Enumerative Combinatorics*, vol. 2.

K. Väisänen, 1992



Serieshelpmate in 19: how many solutions?

Serieshelpmate in n: Black moves first and makes n consecutive move. White then checkmates Black in one move. Black and White are *cooperating*. Black may not move into check or check White (except possibly on his last move).

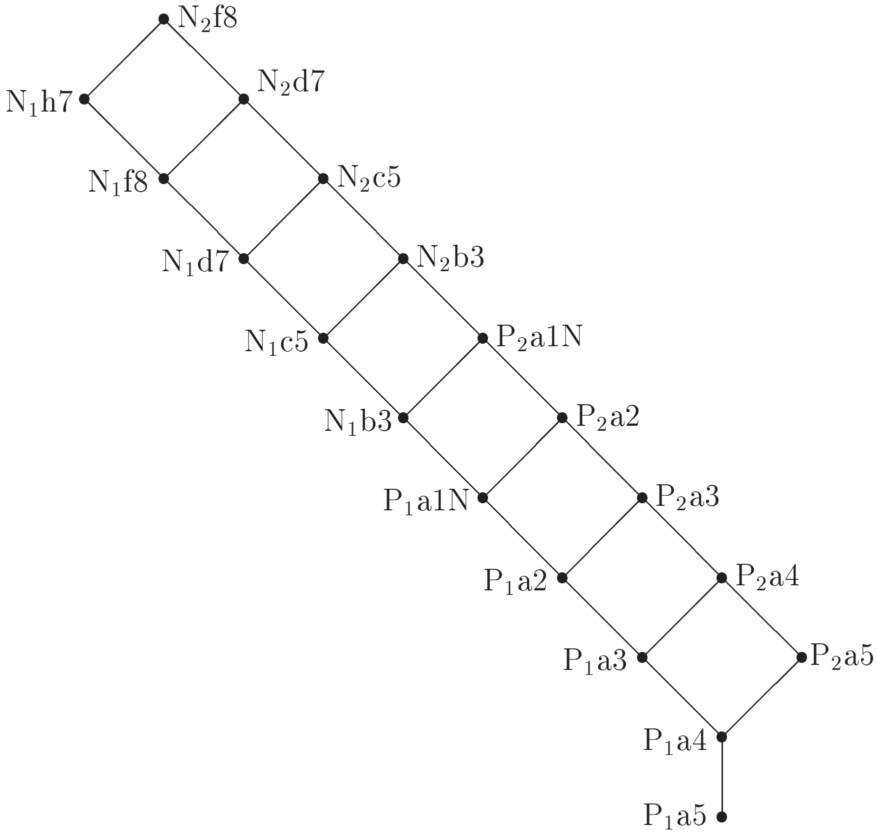
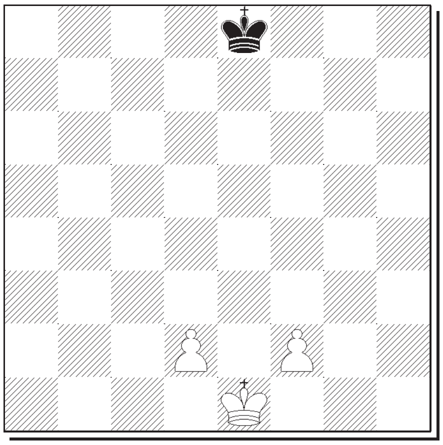


Figure 1: Solution poset

Number of solutions $= C_9 = \frac{1}{10}\binom{18}{9} = 4862.$

RETROGRADE ANALYSIS

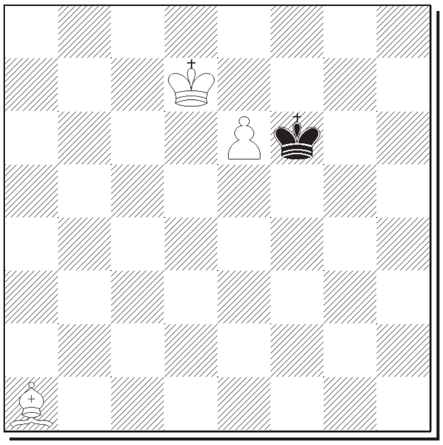
R. Smullyan, 1979



Monochrome chess. A White bishop stands at e3 or e4. Which?

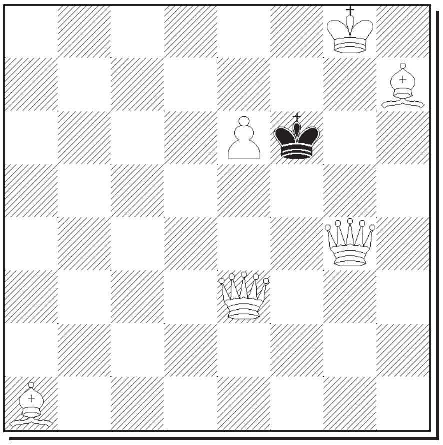
Monochrome chess: every move is to a square of the same color (so e.g. knights never move).

N. Hoëg, 1916



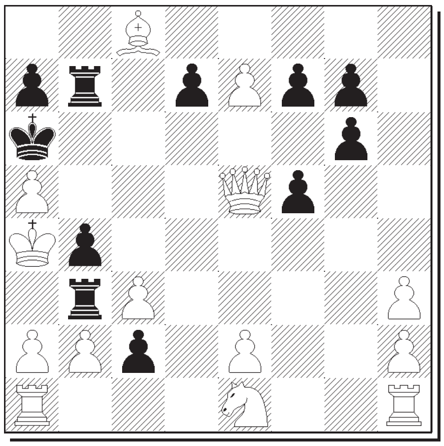
What was the last move?

N. Petrović, 1954



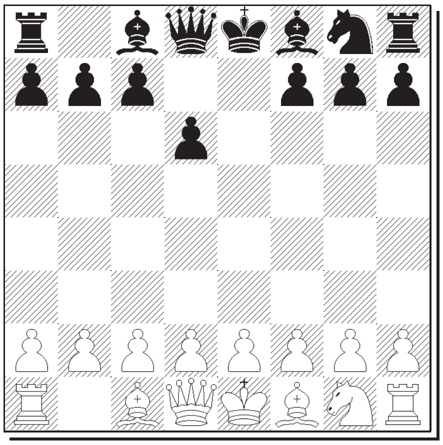
The last six single moves?

J. Furman, 1973



White's Rooks have exchanged places

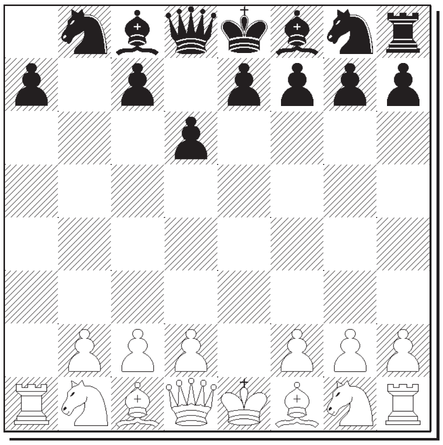
G. Schweig, 1938



Position after the 4th move of Black. How did the game go?
I.e., proof game in 4.0 moves

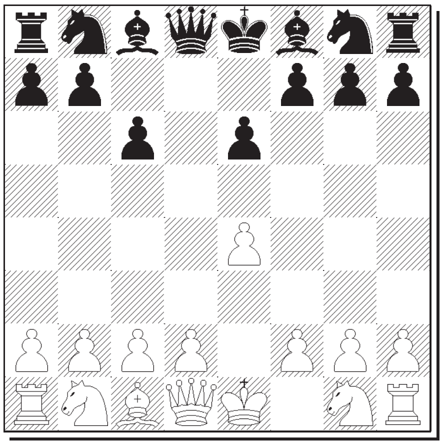
Phoenix theme:

R. Müller, 1985



Proof game in 6.5 moves

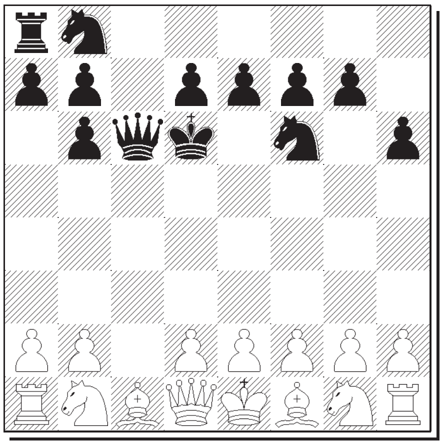
Tibor Orban, 1976



Proof game in exactly 4.0 moves

Solution in 3.5 moves is easy!

G. Donati, 1998



Proof game in 13.5 moves

A nice puzzle!

APPENDIX

Solution to mate in 200 by Jørgensen: We sketch the solution to this complex problem following C. J. Morse. Ample study is necessary for full appreciation. Certain maneuvers are used repeatedly and are indicated after their first appearance by ellipses (...). The solution begins 1. Qe6+ Rf5 2. Bh2. Suppose that White's king were now at d1 and that the five Black queenside pawns had no moves. If Black plays Kf3, then Qe2 is mate. The only alternative for Black is to move one of his knights. If he moves the knight on g6, then Ne5 is mate. If Black instead moves the knight on g8, then White mates in two by Nf6+ Kf3, Qe2 mate. Hence White's goal is to manuever his king to d1 while capturing or immobilizing Black's queenside pawns. However, White may not move his king to the fourth rank, since then Black can move his king off the fourth rank with discovered check. The solution continues 2... Kf3 (other moves shorten the solution) 3. Qe3+ (not 3. Qxf5+ Kg2!, and Black can free his queen. It is essential to keep Black's pieces bottled up.) Kg2 (of course not Kg4 4. Qg3 mate) 4. Qg1+ Kf3 5. Qf1+ Kg4 (not Ke4 6. Qd3 mate) 6. Qe2+ Rf3 7. Qe6+ Rf5 8. Ka2 (not 8. Ka4 Kf3+! nor 8. c4 Kf3 leading to 12. Qe2+ Rf3+!) Kf3 ... 14. Kc1 Kf3 ... 20. Kd1 a4 (not Kf3 21. Ke2 mate, nor c4 which allows 21. Ke1 a4 22. Kd1) 21. Kc1 (not 21. Ke1 a3!, and Black will be able to queen his pawn and check White before White can play Ke4 followed by either Qxf5 mate or by mating with the knight if Black moves a knight) Kf3 ... 27. Kd1 a5 (c4 is playable here and at similar positions up to move 92, but not a3 which allows 65. Ke1) 28. Kc1 Kf3 ... 34. Kd1 a6 (here a3 is an alternative as well as c4) 35. Kc1 Kf3 ... 41. Kd1 a3 42. Kc1 Kf3 ... 48. Kb1 (not 48. Kd1 a2!) Kf3 ... 54. Ka2 Kf3 ... 60. Kxa3 Kf3 ... 66. Kb2 Kf3 ... 72. Kc1 Kf3 ... 78. Kd1 a4 ... 85. Kd1 a5 ... 92. Kd1 c4 93. Kc1 Kf3 ... 96. Qf1+ Kg4 (not Ke4 97. e3+ Kd5 98. Qxf5+ leading to short mate) ... 99. Kd1 c5 ... 106. Kd1 a3 107. Kc1 Kf3 ... 113. Kb1 Kf3 ... 116. Qf1+ Ke4 (this can be played without shortening the solution when the White king is not guarding d2) 117. Qxc4+ (not 117. d3+ Ke3 118. Bg1+ Kd2!) Kf3 118. Qf1+ Kg4 ... 120. Qe6+ Rf5 121 Ka2 Kf3 ... 127. Kxa3 Kf3 ... 145. Kd1 a4 ... 152. Kd1 c4 ... 159. Kd1 (not 159. Kb2, leading to 168. Qf1+ Ke4 and 173. Kxa4 Kf3+!) a3 160. Kc1 Kf3 ... 166. Kb1 Kf3 ... 169. Qf1+ Ke4 (this can be delayed until move 187) ... 174. Ka2 Kf3 ... 180. Kxa3 Kf3 ... 198. Kd1 (finally reaching the desired position!) Ng8 moves 199. Nf6+ Kf3 200. Qe2 mate!