The X-Descent Set of a Permutation

Richard P. Stanley
M.I.T. and U. Miami

May 26, 2022

The Descent Set of a Permutation

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descent set of w: $\mathrm{Des}(w) = \{1 \le i \le n-1 \,:\, a_i > a_{i+1}\}$

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Fix n. For $S \subseteq [n-1]$, define

$$F_S = \sum_{\substack{1 \leq i_1 \leq i_2 \leq \cdots \leq i_n \\ i_j < i_{j+1} \text{ if } j \in S}} x_{i_1} x_{i_2} \cdots x_{i_n},$$

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Theorem. $\sum_{w \in \mathfrak{S}_n} F_{\mathrm{Des}(w)} = p_1^n$

X-descent sets

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Example. (a) $X = \{(i,j) \, : \, n-1 \geq i > j \geq 1\}$: $\mathrm{XDes} = \mathrm{Des}$ (the ordinary descent set)

(b) $X = \{(i,j) \in [n] \times [n] \, : \, i \neq j\}$: $\mathrm{XDes}(w) = [n-1]$

A generating function for the XDescent set

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Example. $X = \{(1,3),(2,1),(3,1),(3,2)\}$

w	$\mathrm{XDes}(w)$
123	$\emptyset$
132	$\{1,2\}$
213	$\{1,2\}$
231	$\{2\}$
312	$\{1\}$
321	$\{1,2\}$

$$U_X = F_\emptyset + F_1 + F_2 + 3F_{1,2} = p_1^3 - p_2 p_1 + p_3 = s_3 + s_{21} + 2s_{111}$$

Two easy theorems

Theorem. (a) U_X *is a p-integral symmetric function.*

(b) *Let* $\overline{X} = \mathcal{E}_n - X$. *Then* $\omega U_X = U_{\overline{X}}$.

Proof.

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(b) Exercise.

Special case

record set $\mathrm{rec}(\boldsymbol{w})$ for $w = a_1 \cdots a_n \in \mathfrak{S}_n$:
$\mathrm{rec}(w) = \{0 \leq i \leq n-1 \,:\, a_i > a_j \text{ for all } j < i\}$. Thus always $0 \in \mathrm{rec}(w)$.

record partition $\mathrm{rp}(\boldsymbol{w})$: if $\mathrm{rec}(w) = \{r_0, \ldots, r_j\}_<$, then $\mathrm{rp}(w)$ is the numbers $r_1 - r_0, r_2 - r_1, \ldots, n - r_j$ arranged in decreasing order.

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Theorem (conjectured by **RS**, proved by **I. Gessel**). *Let X have the property that if $(i,j) \in X$ then $i > j$. Then*

$$U_X = \sum_{\substack{w \in \mathfrak{S}_n \\ \mathrm{XDes}(w) = \emptyset}} p_{\mathrm{rp}(w)}.$$

In particular, U_X is p-positive.

An example

$$X = \{(2,1),(3,2),(4,3)\}$$

w	$\mathrm{rec}(w)$
1234	1111
1342	211
1423	31
2314	211
2341	211
2413	31
312**4**	31
314**2**	22
3412	31
4123	4
4231	4

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$$\Rightarrow U_X = p_1^4 + 3p_2p_1^2 + 4p_3p_1 + p_2^2 + 2p_4$$

Connection with chromatic symmetric functions

$\boldsymbol{P}$: partial ordering of $[n]$

$\boldsymbol{Y_P} = \{(i,j) : i >_P j\}$

$\mathbf{inc}(\boldsymbol{P})$: incomparability graph of P, i.e., vertex set $[n]$, edges ij if $i \parallel j$ in P

$\boldsymbol{X_G}$: chromatic symmetric function of the graph G

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Theorem. $U_{Y_P} = X_{\mathrm{inc}(P)}$

Reverse succession-free permutations

Let $X = \{(2,1),(3,2),\ldots,(n,n-1)\}$.

$\boldsymbol{f_n} = \#\{w \in \mathfrak{S}_n \,:\, \mathrm{XDes}(w) = \emptyset\}$ (**rs-free** permutations)

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Example. $n = 4$: $U_X = 11s_4 + 3s_{31} + s_{211} + s_{1111}$

Sketch of proof

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Left-hand side: $\#\{w \in \mathfrak{S}_n \,:\, \mathrm{XDes}(w) = S\}$

Right-hand side: Use

$$s_{i,1^{n-i}} = \sum_{S \in \binom{[n-1]}{n-i}} F_S.$$

To show: $f_i = \#\{w \in \mathfrak{S}_n \,:\, \mathrm{XDes}(w) = S\}$ if $\#S = n-i$.

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Example. $w = 3247651$, so $S = \{1, 4, 5\}$, $n = 7$, $i = 4$. Factor w:

$$w = 32 \cdot 4 \cdot 765 \cdot 1.$$

Let $1 \to 1$, $32 \to 2$, $4 \to 3$, $765 \to 4$. get

$$w \to 2341 = u. \quad \square$$

A q-analogue for $X = \{(2,1),(3,2),\ldots,(n,n-1)\}$

Let $U_X(q) = \sum_{w \in \mathfrak{S}_n} q^{\mathrm{des}(w^{-1})} F_{\mathrm{XDes}(w)}$, where des denotes the number of (ordinary) descents.

$U_X(q)$ is the generating function for $w \in \mathfrak{S}_n$ by positions of reverse successions and by $\mathrm{des}(w^{-1})$.

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Theorem. $U_X(q) = \sum_{i=1}^{n} q^{n-i} f_i(q) s_{i,1^{n-i}}$

Digraph interpretation

We can also regard X as a **digraph**, with edges $i \to j$ if $(i,j) \in X$. A **Hamiltonian path** in X is a permutation $a_1 a_2 \cdots a_n \in \mathfrak{S}_n$ such that $(a_i, a_{i+1}) \in X$ for $1 \leq i \leq n-1$. Define

$$\mathrm{ham}(\boldsymbol{X}) = \#\ \text{Hamiltonian paths in } X$$

Observation. Let $U_X = \sum_\lambda c_\lambda p_\lambda$. Then

$$\begin{aligned} \mathrm{ham}(X) &= \sum_\lambda \varepsilon_\lambda c_\lambda \\ \mathrm{ham}(\overline{X}) &= \sum_\lambda c_\lambda . \end{aligned}$$

Tomescu's theorem

Theorem (**Tomescu**, 1985). $\operatorname{ham}(X) \equiv \operatorname{ham}(\overline{X}) \pmod 2$

Proof (**D. Grinberg**). Let $U_X = \sum_\lambda c_\lambda p_\lambda$. To prove:

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Obvious since $\varepsilon_\lambda = \pm 1$. $\square$

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tournament: a digraph X with vertex set $[n]$ (say), such that for all $1 \leq i < j \leq n$, exactly one of $(i, j) \in X$ or $(j, i) \in X$.

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Theorem (**D. Grinberg**). *Let X be a tournament. Then*

$$U_X = \sum_w 2^{\mathrm{nsc}(w)} p_{\rho(w)},$$

where w ranges over all permutations in $\mathfrak{S}_n$ of odd order such that every nonsingleton cycle of w is a (directed) cycle of X, and where $\mathbf{nsc}(\boldsymbol{w})$ *denotes the number of nonsingleton cycles of w.*

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Special case of a result for **any** X.

A corollary

Theorem (repeated). *Let X be a tournament. Then*

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Corollary. *If X is a tournament, then*

$$U_X \in \mathbb{Z}[p_1, 2p_3, 2p_5, 2p_7, \ldots].$$

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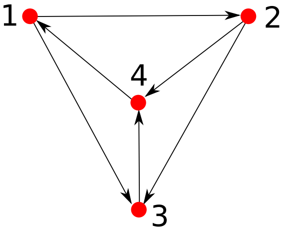
Note. Thus U_X can be written uniquely as a linear combination of Schur's "shifted Schur functions" P_λ, where λ has distinct parts. Can anything worthwhile be said about the coefficients?

An example



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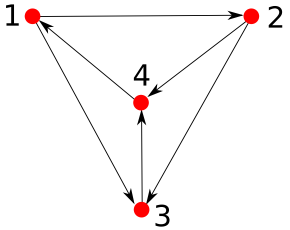

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$$\Rightarrow U_X = p_1^4 + 4p_3p_1 = 5P_4 - 2P_{3,1}$$

An application to Hamiltonian paths

Observation (repeated). Let $U_x = \sum_\lambda \boldsymbol{c_\lambda} p_\lambda$. Then

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Since $c_{1^n} = 1$ for **all** X (immediate from $U_X = \sum_{w \in \mathfrak{S}_n} F_{\operatorname{XDes}(w)}$), we conclude:

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Theorem (**L. Rédei**, 1934) *Every tournament has an odd number of Hamiltionian paths.*

90 years of insight

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created by DALL-E