Generating Functions I Have Known

Richard P. Stanley

June 7, 2024

Most influential

$$\frac{1}{2 - e^x}$$

$$\prod_{n \geq 1} (1 - x^n)^{-n}$$

Three similar generating functions

Theorem (**Cayley, Whitworth**). *Let $f(n)$ be the number of ordered set partitions of $[n] = \{1, 2, \ldots, n\}$ (*Fubini number*), i.e., the number of sequences $(B_1, B_2, \ldots, B_k)$ of sets $B_i \neq \emptyset$, $B_i \cap B_j = \emptyset$ if $i \neq j$, and $\bigcup B_i = [n]$. Then*

$$\sum_{n \geq 0} f(n) \frac{x^n}{n!} = \frac{1}{2 - e^x}.$$

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Theorem (**Kalmár**, 1931). *Let $g(n)$ be the number of ordered factorizations of n, i.e., the number of ways to write $n = a_1 a_2 \cdots a_n$, $a_i > 1$. Then $\sum_{n \geq 1} g(n) n^{-s} = \dfrac{1}{2 - \zeta(s)}$.*

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Theorem. *Let $c(n)$ be the number of compositions of n, i.e., the number of ways to write $n = b_1 + \cdots + b_k$, $b_i \geq 1$ (so $c(n) = 2^{n-1}$ for $n \geq 1$). Then $\sum_{n \geq 1} c(n) x^n = \dfrac{1}{2 - \frac{1}{1-x}}$.*

Chains in posets

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Motivation for much further work on generating functions and posets.

MathOverflow 29490: Gowers

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How many surjections are there from a set of size n?

Asked 13 years, 10 months ago Modified 5 years ago Viewed 18k times

39

It is well-known that the number of surjections from a set of size n to a set of size m is quite a bit harder to calculate than the number of functions or the number of injections. (Of course, for surjections I assume that n is at least m and for injections that it is at most m.) It is also well-known that one can get a formula for the number of surjections using inclusion-exclusion, applied to the sets $X_1, \ldots, X_m$, where for each i the set X_i is defined to be the set of functions that never take the value i. This gives rise to the following expression: $m^n - \binom{m}{1}(m-1)^n + \binom{m}{2}(m-2)^n - \binom{m}{3}(m-3)^n + \ldots$.

Let us call this number $S(n,m)$. I'm wondering if anyone can tell me about the asymptotics of $S(n,m)$. A particular question I have is this: for (approximately) what value of m is $S(n,m)$ maximized? It is a little exercise to check that there are more surjections to a set of size $n-1$ than there are to a set of size n. (To do it, one calculates $S(n,n-1)$ by exploiting the fact that every surjection must hit exactly one number twice and all the others once.) So the maximum is not attained at $m=1$ or $m=n$.

I'm assuming this is known, but a search on the web just seems to lead me to the exact formula. A reference would be great. A proof, or proof sketch, would be even better.

Update. I should have said that my real reason for being interested in the value of m for which S(n,m) is maximized (to use the notation of this post) or m!S(n,m) is maximized (to use the more conventional notation where S(n,m) stands for a Stirling number of the second kind) is that what I care about is the rough size of the sum. The sum is big enough that I think I'm probably not too concerned about a factor of n, so I was prepared to estimate the sum as lying between the maximum and n times the maximum.

co.combinatorics polymath5 Edit tags

Share Cite Edit Follow Close Flag edited Jun 25, 2010 at 13:05 asked Jun 25, 2010 at 10:34 gowers

A reply

55

It seems to be the case that the polynomial $P_n(x) = \sum_{m=1}^n m!S(n,m)x^m$ has only real zeros. (I know it is true that $\sum_{m=1}^n S(n,m)x^m$ has only real zeros.) If this is true, then the value of m maximizing $m!S(n,m)$ is within 1 of $P_n'(1)/P_n(1)$ by a theorem of J. N. Darroch, *Ann. Math. Stat.* **35** (1964), 1317-1321. See also J. Pitman, *J. Combinatorial Theory, Ser. A* **77** (1997), 279-303. By standard combinatorics

$$\sum_{n\geq 0} P_n(x)\frac{t^n}{n!} = \frac{1}{1-x(e^t-1)}.$$

Hence

$$\sum_{n\geq 0} P_n(1)\frac{t^n}{n!} = \frac{1}{2-e^t}$$

$$\sum_{n\geq 0} P_n'(1)\frac{t^n}{n!} = \frac{e^t-1}{(2-e^t)^2}.$$

Since these functions are meromorphic with smallest singularity at $t = \log 2$, it is routine to work out the asymptotics, though I have not bothered to do this.

Update. It is indeed true that $P_n(x)$ has real zeros. This is because $(x-1)^n P_n(1/(x-1)) = A_n(x)/x$, where $A_n(x)$ is an Eulerian polynomial. It is known that $A_n(x)$ has only real zeros, and the operation $P_n(x) \to (x-1)^n P_n(1/(x-1))$ leaves invariant the property of having real zeros.

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edited Jun 26, 2010 at 16:55

answered Jun 26, 2010 at 0:15

Richard Stanley

Response of Gowers

Thanks for this. It makes me realize that I need a more developed "write down the generating function" reflex (together with some knowledge about how to deal with it once it's written down).

gowers Jun 26, 2010 at 8:57

Response of Terry Tao

1 Given that Tim ultimately only wants to sum m! S(n,m) rather than find its maximum, it is really only P_n(1) which one needs to compute. In principle this is an exercise in the saddle point method, though one which does require a nontrivial amount of effort. – Terry Tao Jun 26, 2010 at 19:03

9 You don't need the saddle point method to find the asymptotic rate of growth of the coefficients of $1/(2-e^t)$. The smallest singularity is at $t = \log 2$. It is a simple pole with residue $-1/2$. Hence

$$P_n(1) \sim \frac{n!}{2(\log 2)^{n+1}}.$$

Using all the singularities $\log 2 + 2\pi i k, k \in \mathbb{Z}$, one obtains an asymptotic series for $P_n(1)$. It can be shown that this series actually converges to $P_n(1)$. – Richard Stanley Jun 26, 2010 at 19:51

11 I quit being lazy and worked out the asymptotics for $P_n'(1)$. The Laurent expansion of $(e^t-1)/(2-e^t)^2$ about $t = \log 2$ begins

$$\frac{e^t-1}{(2-e^t)^2} = \frac{1}{4(t-\log 2)^2} + \frac{1}{4(t-\log 2)} + \cdots$$

$$= \frac{1}{4(\log 2)^2\left(1-\frac{t}{\log 2}\right)^2} - \frac{1}{4(\log 2)\left(1-\frac{t}{\log 2}\right)} + \cdots,$$

whence

$$P_n'(1) = n!\left(\frac{n+1}{4(\log 2)^{n+2}} - \frac{1}{4(\log 2)^{n+1}} + \cdots\right).$$

Thus $P_n'(1)/P_n(1) \sim n/2(\log 2)$. – Richard Stanley Jun 26, 2010 at 21:00

10 Ah, I didn't realise that it was so simple to read off asymptotics of a Taylor series from nearby singularities (though, in retrospect, I implicitly knew this in several contexts). Thanks, I learned something today! – Terry Tao Jun 28, 2010 at 20:26

Food for thought

- Let $f(n)$ be the number of partitions of n such that no part appears exactly once. Then

$$\sum_{n\geq 0} f(n)x^n = \prod_{k\geq 1} \frac{1-x^{6k}}{(1-x^{2k})(1-x^{3k})}.$$

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- Let $f(n)$ be the number of monic polynomials of degree n over $\mathbb{F}_q$ such that no irreducible factor has multiplicity one. Then

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- Let S be the set of all *powerful* positive integers n, i.e., no prime p divides n with multiplicity one. Then

$$\sum_{n\in S} \frac{1}{n^s} = \frac{\zeta(2s)\zeta(3s)}{\zeta(6s)}$$

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$\mathrm{pp}(n)$: number of plane partitions of n

$$\mathrm{pp}(3) = 6: \quad \begin{matrix}3\end{matrix} \quad \begin{matrix}21\end{matrix} \quad \begin{matrix}2\\1\end{matrix} \quad \begin{matrix}111\end{matrix} \quad \begin{matrix}11\\1\end{matrix} \quad \begin{matrix}1\\1\\1\end{matrix}$$

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Gateway to symmetric functions, RSK, P-partitions, combinatorial reciprocity, ...

My own three favorites

$$\frac{1}{2}\left[\left(1+2\sum_{n\geq 1} n^n \frac{x^n}{n!}\right)^{1/2} \times \left(1-\sum_{n\geq 1}(n-1)^{n-1}\frac{x^n}{n!}\right)-1\right]\exp\left(\sum_{n\geq 1} n^{n-2}\frac{x^n}{n!}\right)$$

$$\Psi\left(\frac{1+x}{1-x}\right)^{(E^2+1)/4}, \text{ where } \Psi E^n = E_n$$

$$\frac{z'}{z} = y^2, z(0) = 1, \text{ where } y(2-e^{xy}) = 1$$

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Ehrhart theory, representations of $\mathfrak{S}_n$, hyperplane arrangments

Ordered degree sequences

$\boldsymbol{G}$: simple (no loops, multiple edges) graph on vertex set $[n]$

$\deg(\boldsymbol{i})$: degree (number of adjacent vertices) of vertex i

$\boldsymbol{d(G)} := (\deg(1), \ldots, \deg(n))$, the **ordered degree sequence** of G

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Note. Traditionally one looked at $d(G)_{\text{sorted}}$, the **degree sequence** of G. **Unknown:** number of distinct degree sequences of graphs on $[n]$.

Enumerating ordered degree sequences

$f(n)$: number of distinct ordered degree sequences of graphs on $[n]$

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$f(4) = 54 < 64$, e.g., the three complete matchings have ordered degree sequence $(1, 1, 1, 1)$.

A mysterious generating function

Theorem (**RS**, 1991).
$$\sum_{n\geq 0} f(n)\frac{x^n}{n!} = \frac{1}{2}\left[\left(1+2\sum_{n\geq 1} n^n\frac{x^n}{n!}\right)^{1/2}\right.$$
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Corollary (**Kotsovec**, 2013).

$$f(n) \sim \frac{\Gamma(3/4)n^{n-\frac{1}{4}}}{2^{3/4}\sqrt{\pi e}}$$

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Lemma. $f(n) = \sum_G \max\{1, 2^{c(G)-1}\}$, *where* G *ranges over all simple graphs on* $[n]$ *such that every connected component has at most one cycle, which is of odd length, and* $c(G)$ *is the number of cycles of* G.

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* A felicitous term due to **Adriano Garsia**.

Connection with zonotopes

Idea of proof. $f(n)$ is the number of integer points in the Minkowski sum Z_n of the line segments between the origin and the vectors $e_i + e_j$, $1 \leq i < j \leq n$, where e_i is the ith unit coordinate vector.

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Is there a combinatorial proof?

Alternating fixed-point free involutions

Theorem (**Ramanujan**). *As* $x \to 0+$,

$$2\sum_{n\geq 0}\left(\frac{1-x}{1+x}\right)^{n(n+1)} \sim 1 + x + x^2 + 2x^3 + 5x^4 + 17x^5 + \cdots := \boldsymbol{F(x)}.$$

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Euler number $\boldsymbol{E_n}$: $\sum_{n\geq 0} E_n \frac{x^n}{n!} = \sec x + \tan x$

Theorem (**B. Berndt**, 1998, rephrased). *Let E be an indeterminate, and let Ψ be the linear operator defined by $\Psi(E^n) = E_n$. Then*

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W. F. Galway, 1999: let $F(x) = \sum f(n)x^n$. Find a combinatorial interpretation of $f(n)$.

Answer to Galway question

Theorem (**RS**, 2007) $f(n)$ *is the number of fixed-point free involutions (i.e., n cycles of length two)* $w = a_1 a_2 \cdots a_{2n}$ *in* $\mathfrak{S}_{2n}$ *that are* alternating, *i.e.,* $a_1 > a_2 < a_3 > a_4 < \cdots > a_{2n}$.

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Proof idea. There is a character χ of $\mathfrak{S}_n$ (due to **H. O. Foulkes**) such that for all $w \in \mathfrak{S}_n$,

$$\chi(w) = 0 \text{ or } \pm E_k.$$

Now use known results on combinatorial properties of characters of $\mathfrak{S}_n$.

Generalize?

E.g.,

$$\lim_{x \to 0+} 2 \sum_{n \geq 0} \left(\frac{1-x}{1+ax} \right)^{n(n+b)} .$$

Interval orders

$\boldsymbol{P} = \{I_1, \ldots, I_n\}$: closed intervals of positive length in $\mathbb{R}$

Define $I_i < I_j$ if I_i lies entirely to the left of I_j (**interval order**)

$$\boldsymbol{\ell}_1, \ldots, \boldsymbol{\ell}_{\boldsymbol{n}} \in \mathbb{R}_+, \ \boldsymbol{\ell} = (\ell_1, \ldots, \ell_n)$$

$\mathcal{I}_{\ell}$: set of **labelled** interval orders with intervals of lengths $\ell_1, \ldots, \ell_n$.

Unit interval orders

If $\ell_1 = \cdots = \ell_n$, number of **unlabelled** interval orders is C_n (Catalan number)

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Number $f(n)$ of **labelled** interval orders given by

$$\sum_{n \geq 0} f(n) \frac{x^n}{n!} = C(1 - e^{-x}),$$

where $C(x) = \sum_{n \geq 0} C_n x^n = \frac{1 - \sqrt{1-4x}}{2x}$.

Generic lengths

Let $\ell_1, \ldots, \ell_n$ be **generic**, e.g., linear independent over $\mathbb{Q}$. Let $c_n = \#\mathcal{I}_\ell$, the number of labelled interval orders of size n with generic interval lengths. Set

$$z = \sum_{n \geq 0} c_n \frac{x^n}{n!} = 1 + x + 2\frac{x^2}{2!} + 19\frac{x^3}{3!} + 195\frac{x^4}{4!} + \cdots.$$

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Theorem (**RS**, 1998). *Define* $y = 1 + x + 5\frac{x^2}{2!} + \cdots$ *by*

$y(2 - e^{xy}) = 1$. *Then* z *is the unique power series satisfying*

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Idea of proof. First show that c_n is the number of regions of a certain hyperplane arrangement. Then use the machinery of hyperplane arrangements. □

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- Call two sequences ℓ, m of lengths *equivalent* if they produce sets of isomorphic labelled interval orders. Describe the equivalence classes.
- Can one characterize the interval orders obtained when $\ell_1 \ll \ell_2 \ll \cdots \ll \ell_n$ or when all the ℓ_i's are almost equal and generic?
- What about **unlabelled** interval orders with generic interval lengths? In particular, does the number of them depend only on n?

Thanks, merci, danke, tēnā koe, спасибо, 谢谢, . . .

Clara Chan
Atsuko Kida
Alejandro Morales
Satomi Okazaki
Tom Roby
Lauren Williams

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all other organizers and participants!