Basic rules

- Two players: **Blue** and **Red**.
- Perfect information.
- Players move alternately.
- First player unable to move **loses**.
- The game ***must*** terminate.

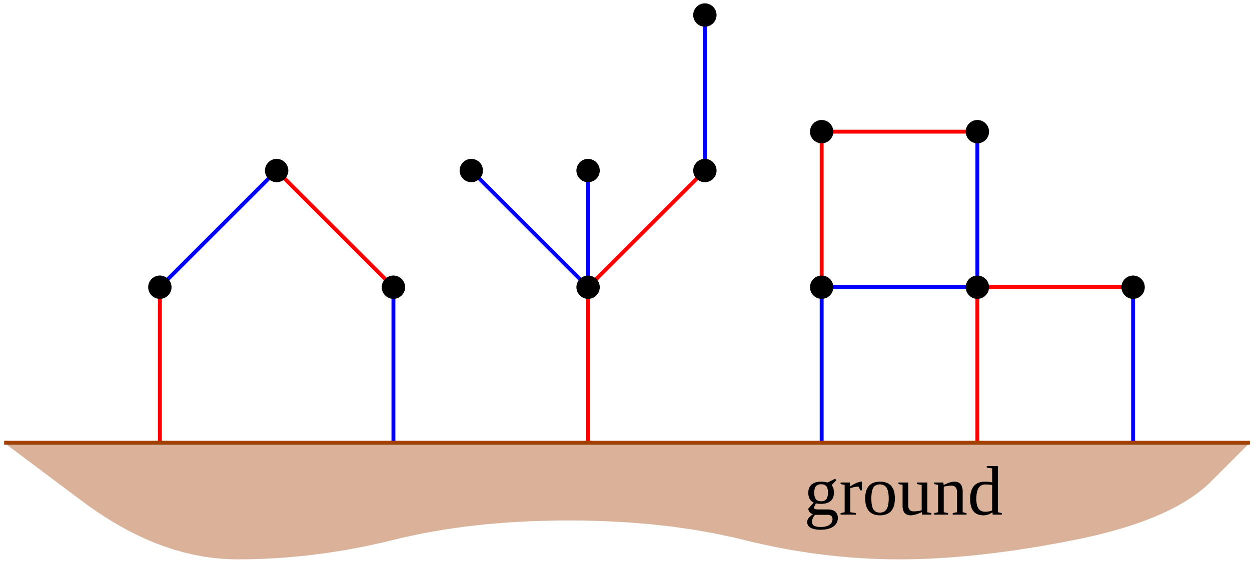
Outcomes (assuming perfect play)

- **Blue** wins (whoever moves first): $G > 0$
- **Red** wins (whoever moves first): $G < 0$
- Mover loses: $G = 0$
- Mover wins: $G \| 0$

Two elegant classes of games

- **number game**: always disadvantageous to move (so never $G \| 0$)
- **impartial game**: same moves always available to each player

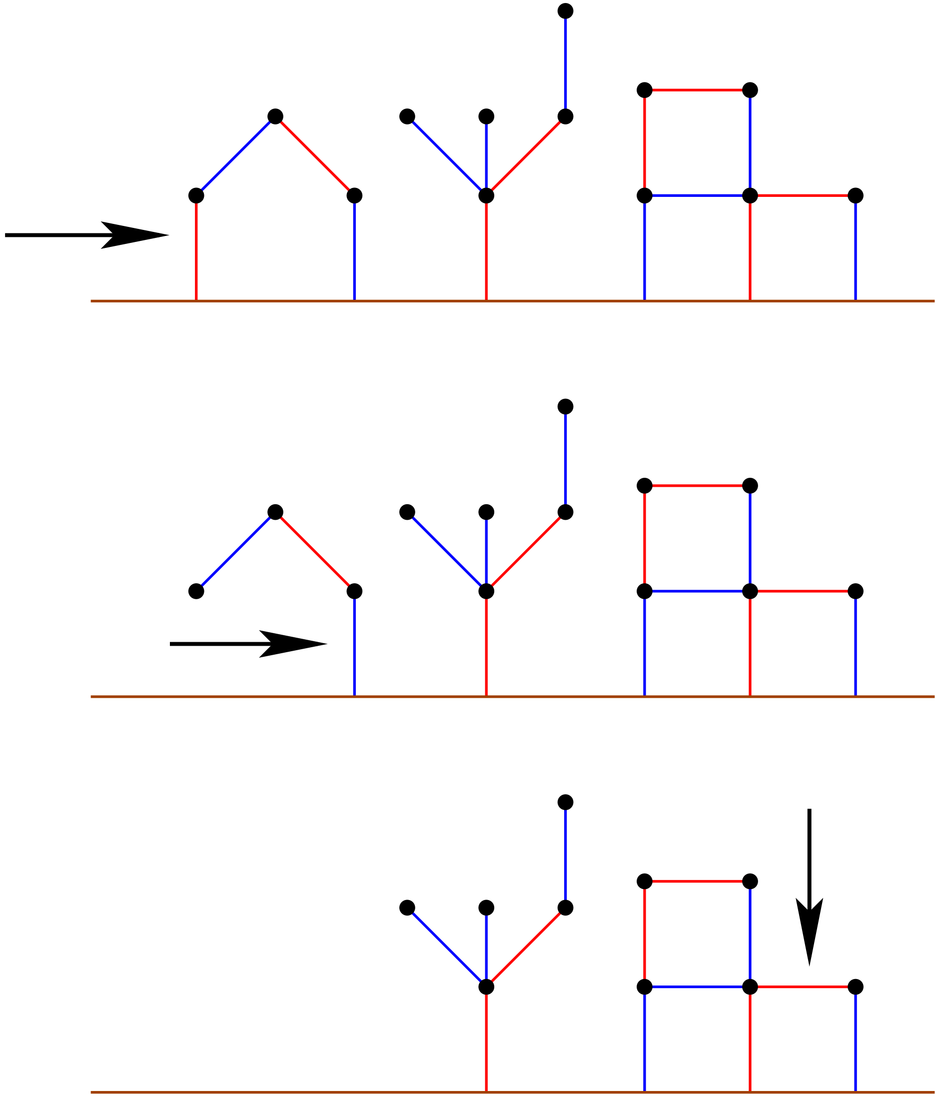
Blue-Red Hackenbush



prototypical number game:

Blue-Red Hackenbush: A player removes one edge of his or her color. Any edges not connected to the ground are also removed. First person unable to move loses.

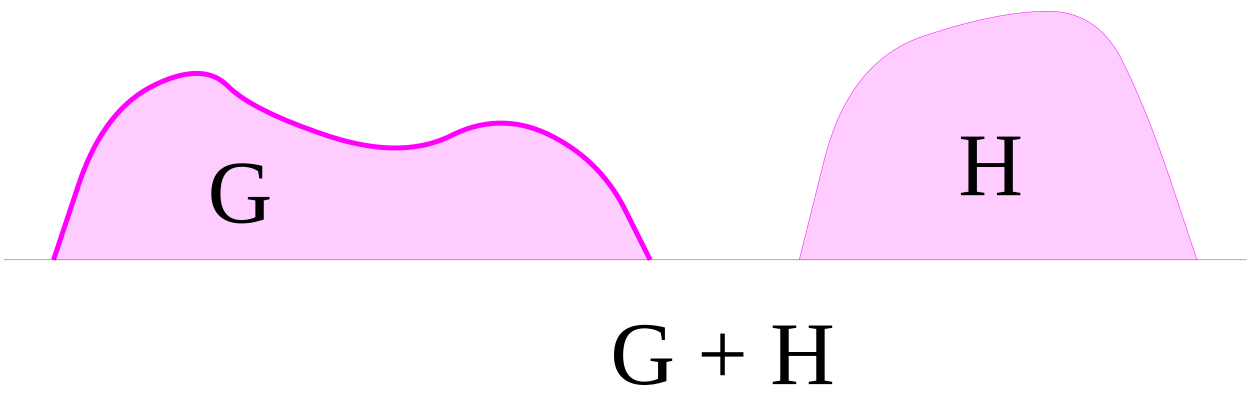
An example



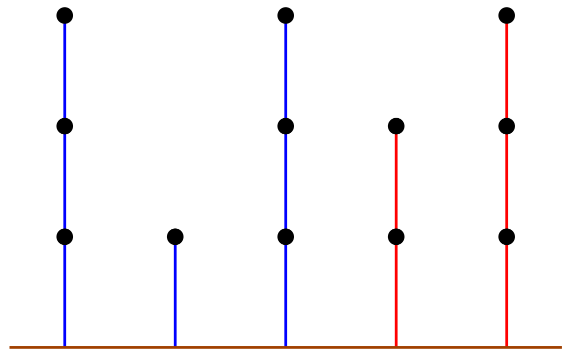
A Hackenbush sum

Let G be a Blue-Red Hackenbush position (or any game). Recall:

- **Blue** wins: $\mathbf{G > 0}$
- **Red** wins: $\mathbf{G < 0}$
- Mover loses: $\mathbf{G = 0}$

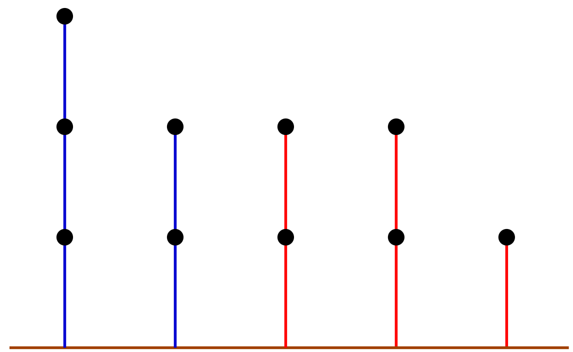


A Hackenbush value



value (to Blue): 3 1 3 −2 −3

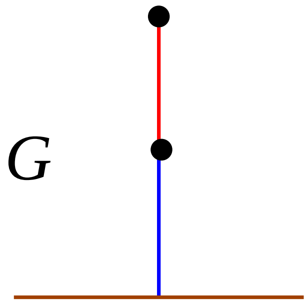
sum: 2 (Blue is two moves ahead), $G > 0$



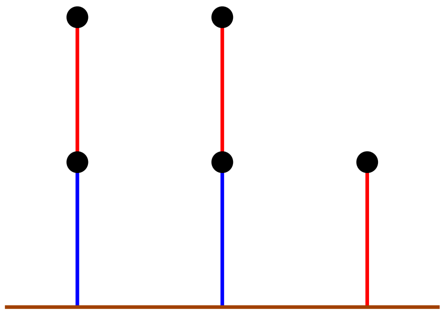
3 2 −2 −2 −1

sum: 0 (mover loses), $G = 0$

1/2



value = ?
clearly >0: Blue wins



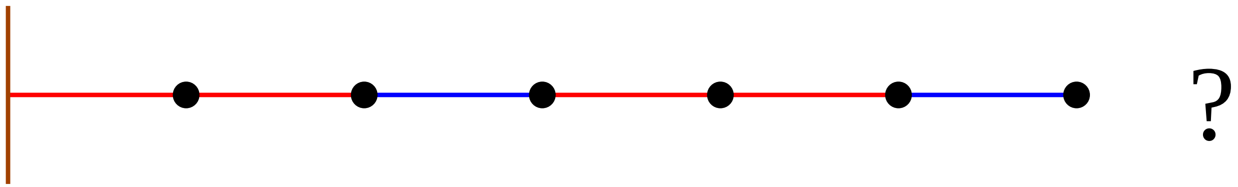
mover loses!

$x + x - 1 = 0$, so $x = 1/2$

Blue is 1/2 move ahead in *G*.

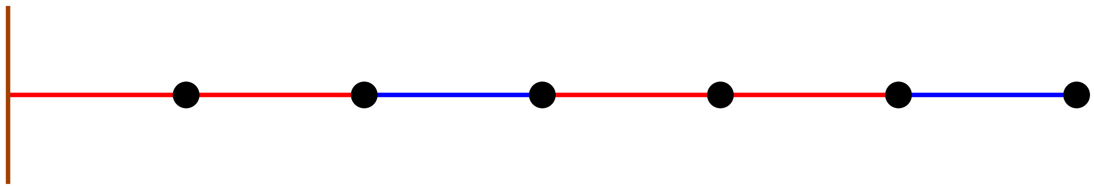
Another position

What about



Another position

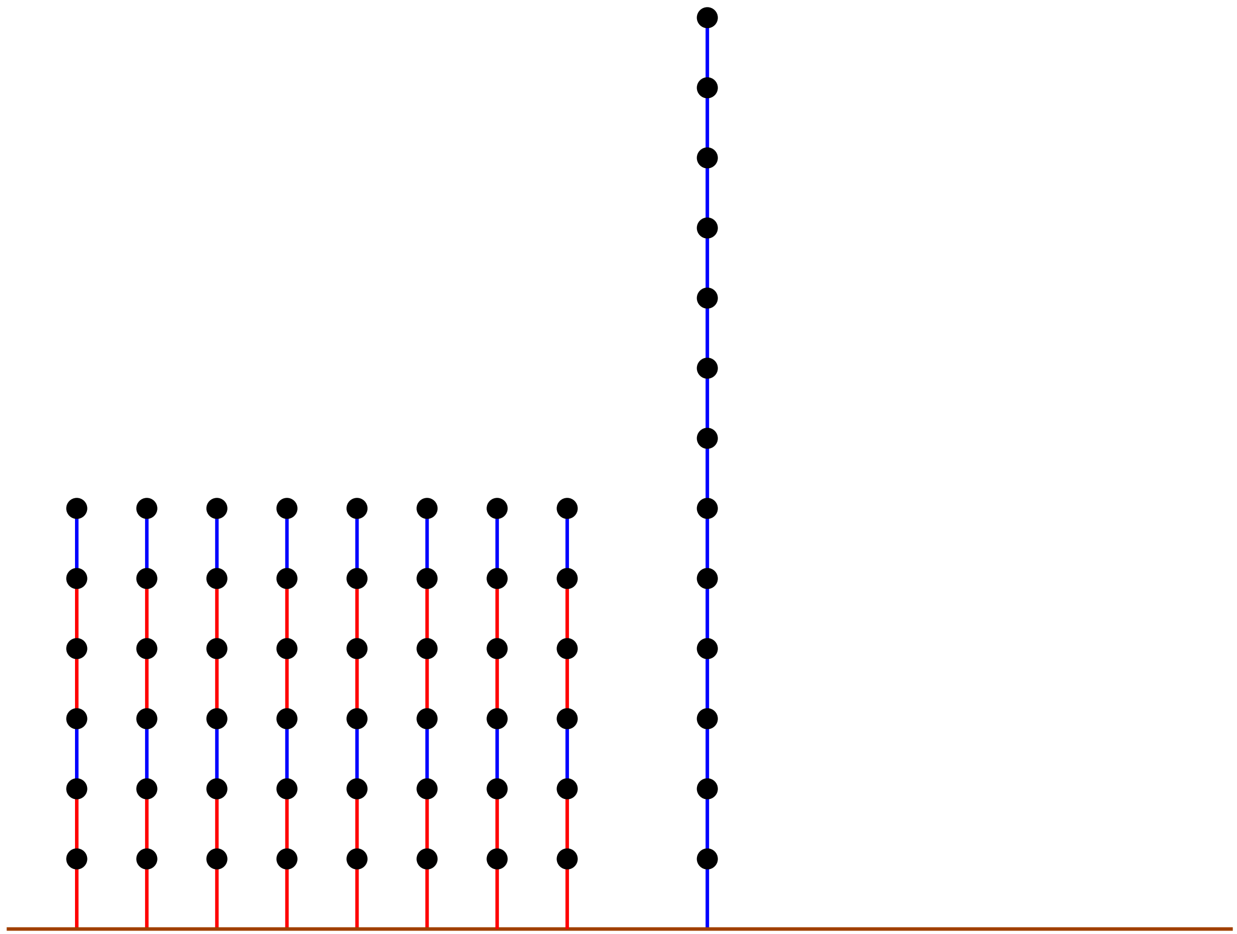
What about



?

Clearly $G < 0$.

$-13/8$



8x + 13 = 0 (mover loses!)

x = -13/8

b and r

How to compute the value $v(G)$ of any Blue-Red Hackenbush position G?

Let b be the **largest** value of any position to which **Blue** can move. Let r be the **smallest** value of any position to which **Red** can move. (We will always have $b < r$.)

The simplicity rule

The Simplicity Rule. **(a)** *If there is an integer* n *satisfying* $b < n < r$*, then* $v(G)$ *is the closest such integer to 0.*

(b) *Otherwise* $v(G)$ *is the (unique) rational number* x *satisfying* $b < x < r$ *whose denominator is the smallest possible power of 2.*

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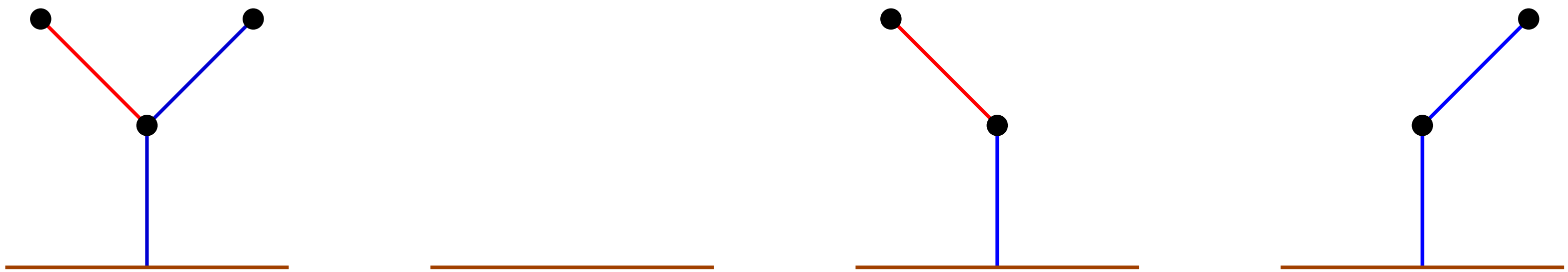
Moreover, $v(G + H) = v(G) + v(H)$.

Some examples

Examples.

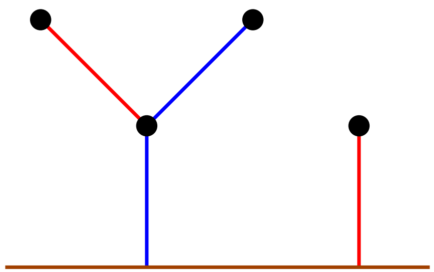
b	r	x
$2\frac{3}{4}$	$6\frac{1}{2}$	3
-5	$2\frac{5}{8}$	0
0	1	$\frac{1}{2}$
$\frac{1}{4}$	$\frac{5}{16}$	$\frac{9}{32}$
$\frac{1}{4}$	$\frac{7}{16}$	$\frac{3}{8}$
$-2\frac{7}{8}$	$-2\frac{3}{32}$	$-2\frac{1}{2}$

A Hackenbush computation



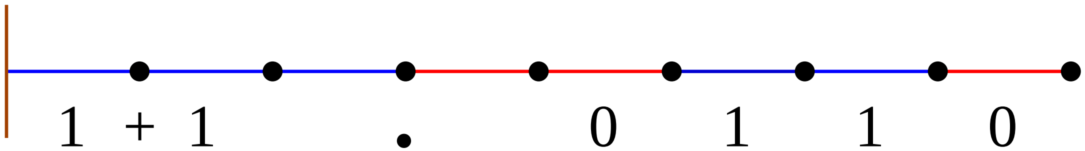
value x 0 1/2 2

b = 1/2, r = 2, x = 1



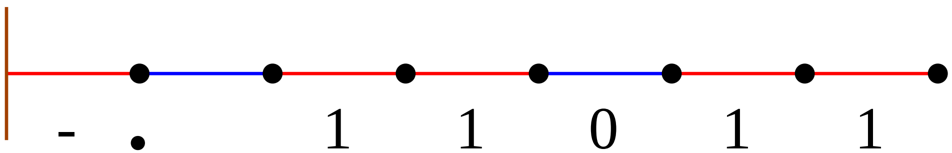
1 - 1 = 0 (mover loses)

Value of Blue-Red strings



1 + 1 . 0 1 1 0 1

= 2 + 1/4 + 1/8 + 1/32 = 2 13/32



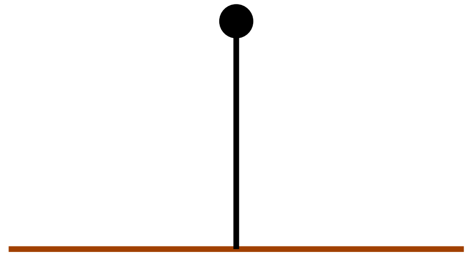
- . 1 1 0 1 1 1

= - (1/2 + 1/4 + 1/16 + 1/32 + 1/64) = - 55/64

Impartial Hackenbush

Now suppose there are also **black** edges, which either player can remove. A game with all **black** edges is called an **impartial** (Hackenbush) game. At any stage of such a game, the two players always have the same available moves.

$*1$



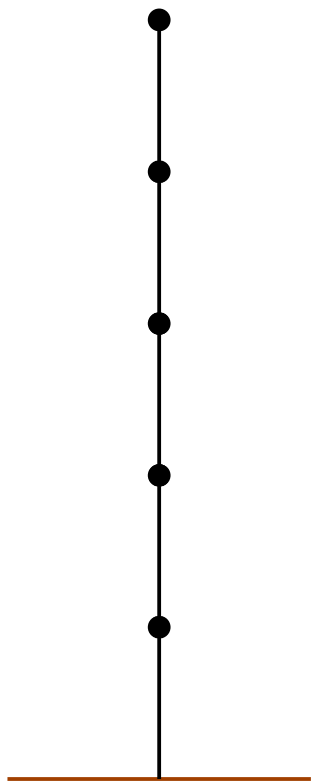
$*1$

Mover wins! Not a number game.

Neither $= 0$, < 0, or > 0.

Two outcomes of any impartial game: mover wins or mover loses.

Denote by $*n$ (star n) the impartial game with one chain of length n.



*5

A nice fact

We can still assign a number with useful properties to an impartial game, based on the following fact.

Fact. Given any (finite) impartial game G, there is a unique integer $n \geq 0$ such that mover loses in the sum of G and $*n$, i.e.,

$$G + *n = 0.$$

The Sprague-Grundy number

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The Sprague-Grundy number

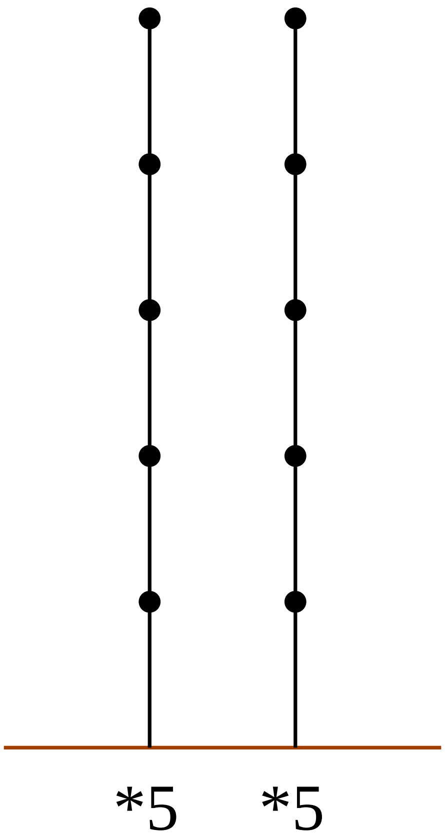
Fact. Given any (finite) impartial game G, there is a unique integer $n \geq 0$ such that mover loses in the sum of G and $*n$, i.e.,

$$G + *n = 0.$$

Denote this integer by $\boldsymbol{N(G)}$, the **Sprague-Grundy** number of G.

NOTE: Mover loses (i.e., $G = 0$) if and only if $N(G) = 0$.

A simple example



Mover loses (second player copies first player).

$N(*5) = 5$

$*5 + *5 = 0$

In general, $N(*n) = n$.

Nim addition

Nim addition. Define $m \oplus n$ by writing m and n in binary, adding **without carrying** (mod 2 addition in each column), and reading result in binary.

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Example. $13 \oplus 11 \oplus 7 \oplus 4 = 5$

$$
\begin{array}{rcl}
 & & \underline{8\,4\,2\,1} \\
13 & = & 1\,1\,0\,1 \\
11 & = & 1\,0\,1\,1 \\
7 & = & 1\,1\,1 \\
4 & = & 1\,0\,0 \\
\hline
5 & = & 0\,1\,0\,1
\end{array}
$$

The Nim-Sum Theorem

Nim-sum Theorem. Let G and H be impartial games. Then

$$N(G+H) = N(G) \oplus N(H).$$

Nim

Nim: sum of $*n$'s.

Nim

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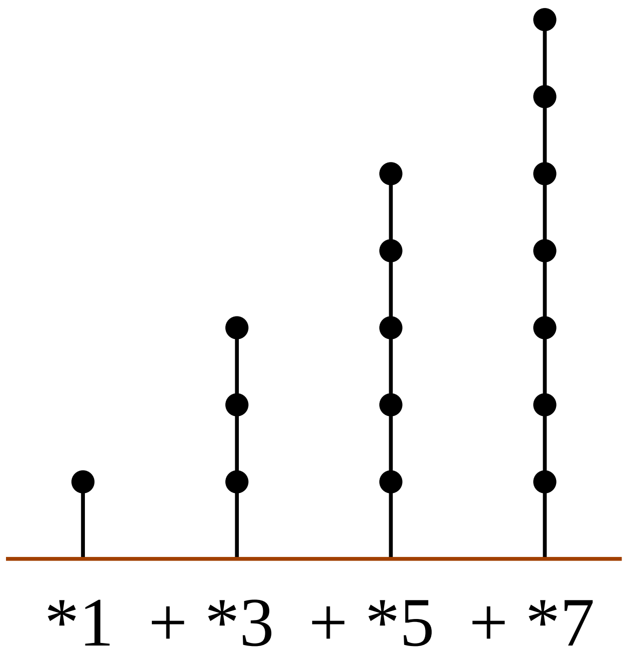
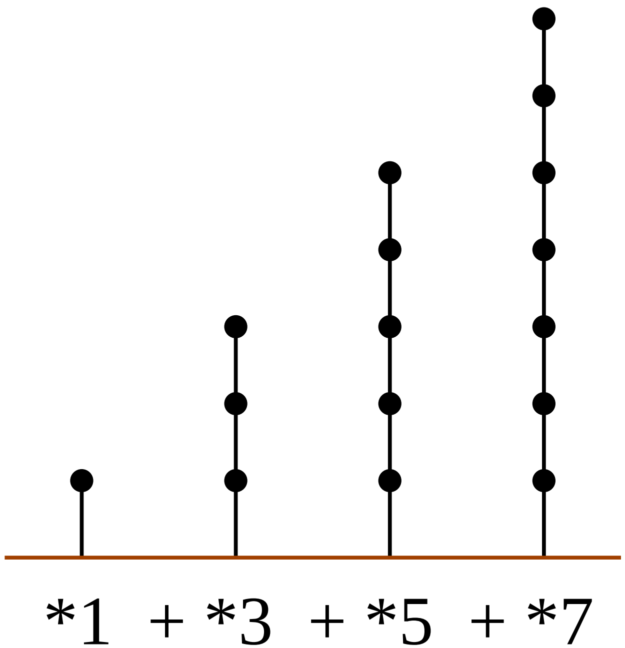
Last Year at Marienbad:



$*1 + *3 + *5 + *7$



			4 2 1
1	=		1
3	=		1 1
5	=		1 0 1
7	=		1 1 1
0	=		0 0 0,

so mover loses!

A Nim example

How to play Nim:

$$G = *23 + *18 + *13 + *7 + *5$$

23 = 10111			
18 = 10010			
13 = 1101	→	0111 = 7	
7 = 111			
5 = 101			

A Nim example

How to play Nim:

$$G = *23 + *18 + *13 + *7 + *5$$

23 = 10111	
18 = 10010	
13 = 1101	→ 0111 = 7
7 = 111	
5 = 101	

Only winning move is to change $*13$ to $*7$.

The minimal excludant

How to compute $N(G)$ in general: If S is a set of nonnegative integers, let $\boldsymbol{mex}(S)$ (the **minimal excludant** of S) be the least nonnegative integer not in S.

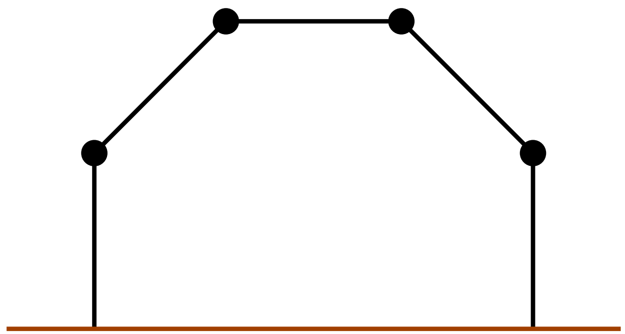
$$\begin{aligned} \text{mex}\{0,1,2,5,6,8\} &= 3 \\ \text{mex}\{4,7,8,12\} &= 0. \end{aligned}$$

The Mex Rule

Mex Rule (analogue of Simplicity Rule). Let S be the set of all Sprague-Grundy numbers of positions that can be reached in one move from the impartial game G. Then

$$N(G) = \mathrm{mex}(S).$$

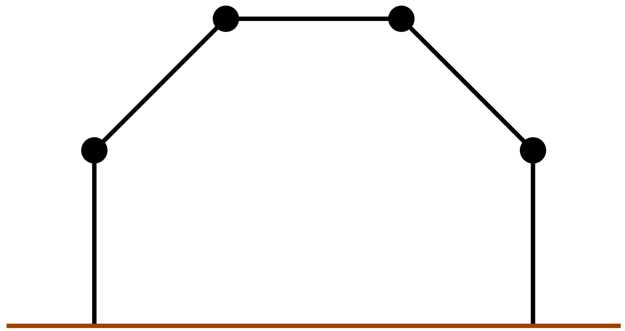
A mex example



Can move to 4, $1 \oplus 3 = 2$, and $2 \oplus 2 = 0$. Thus

$$N(G) = \text{mex}\{0, 2, 4\} = 1.$$

A mex example



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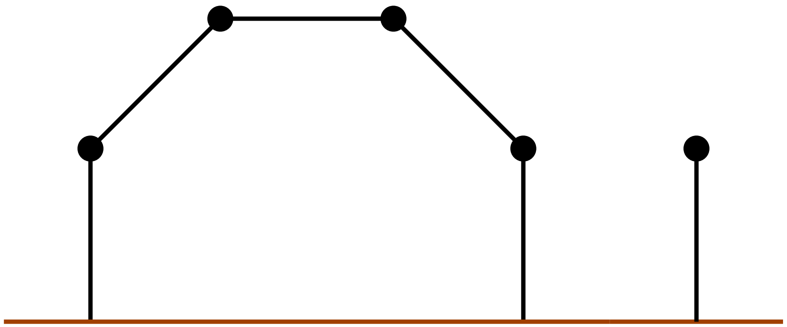
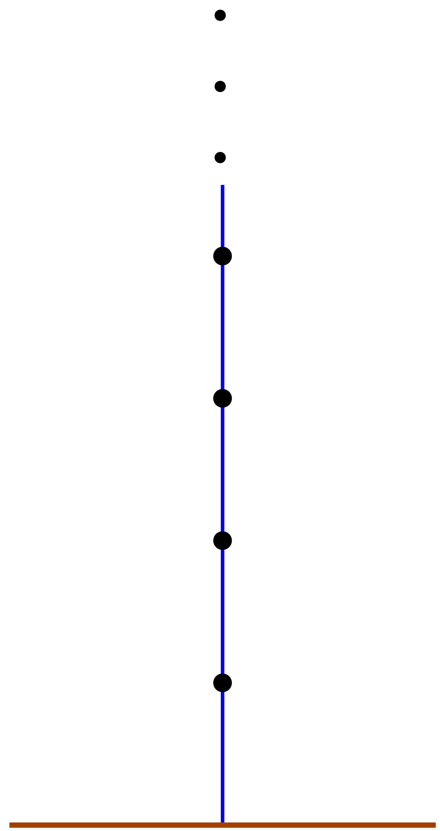


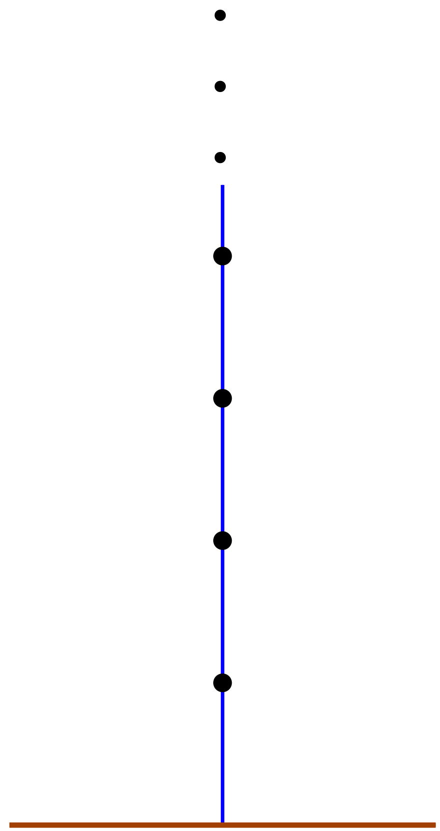
mover loses!

The infinite



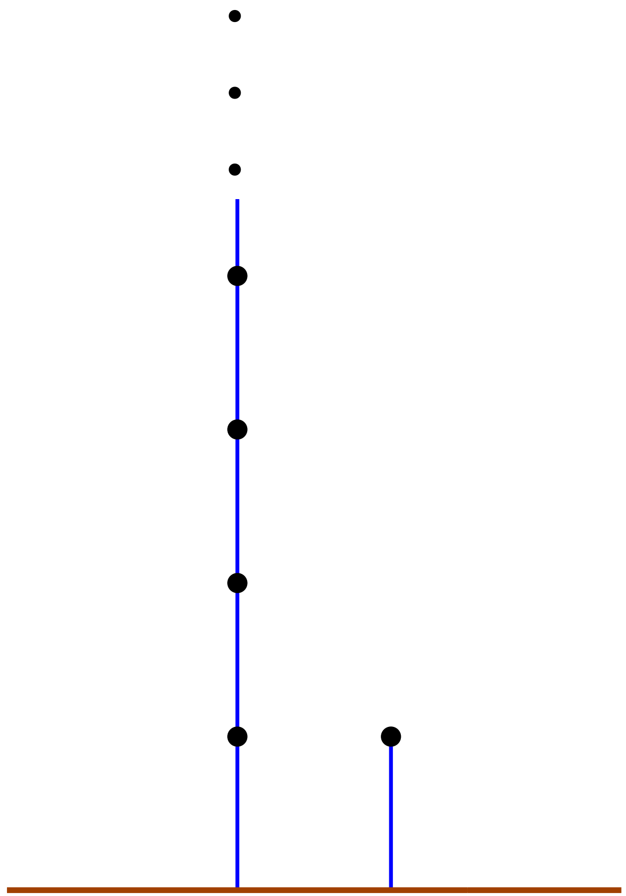
Clearly $v(G) > n$ for all n, i.e., G is **infinite**. Say $v(G) = \boldsymbol{\omega}$.

The infinite



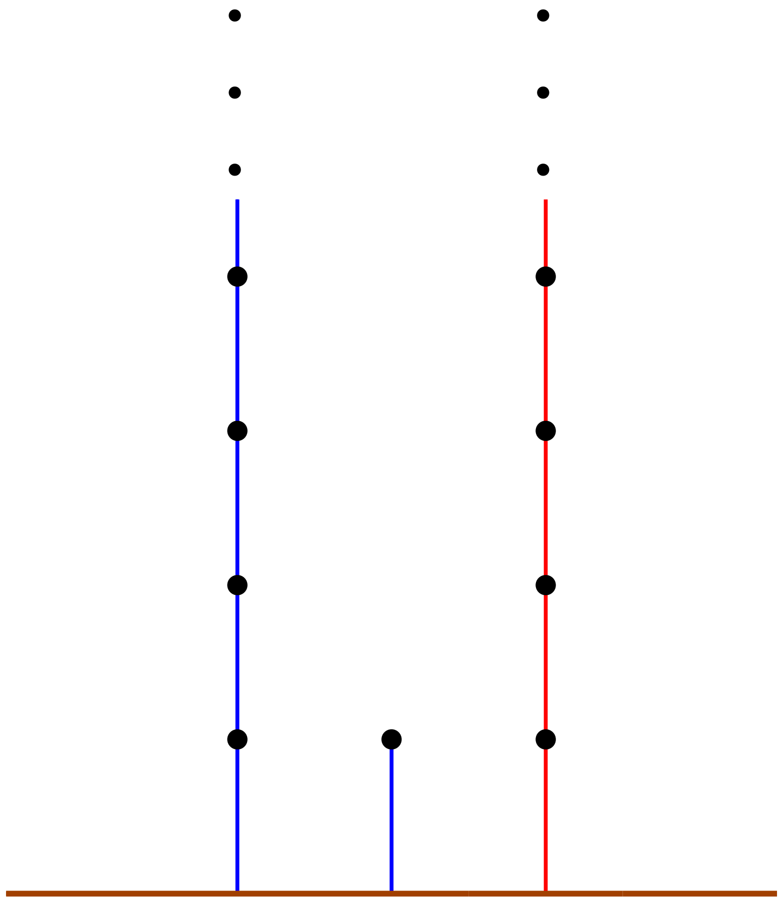
Clearly $v(G) > n$ for all n, i.e., G is **infinite**. Say $v(G) = \omega$. (Still ends in finitely many moves.)

Even more infinite



Clearly $G - \omega > 0$, so G is more infinite than ω.
Call $v(G) = \boldsymbol{\omega + 1}$.

$\omega + 1$ *versus* $-\omega$

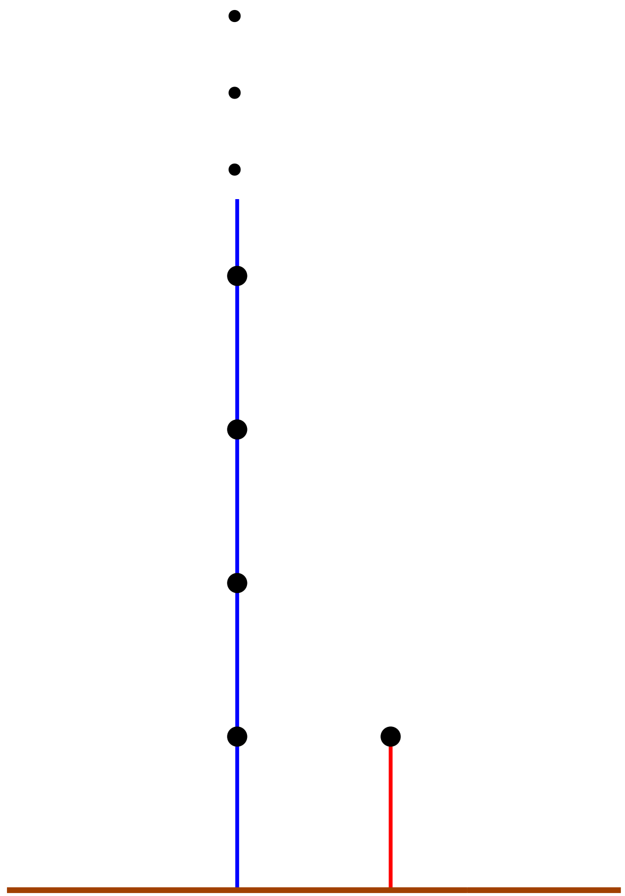


Blue takes the isolated edge to win.

Ordinal games

Similarly, every **ordinal number** is the value of a game.

$\omega - 1$



We can do more! $v(G) = \omega - 1$.

A big field

Can extend ordinal numbers to an **abelian group** $\mathcal{N}$.

A big field

Can extend ordinal numbers to an **abelian group** $\mathcal{N}$.

Conway defined the product of $G \cdot H$ of any two games. This turns $\mathcal{N}$ into a field. We can extend this to a real closed field, each element of which is a number game . . . .

The field $\mathcal{I}$

The product $G \cdot H$ turns the set $\{*0, *1, *2, \dots\}$ into a field $\mathcal{I}$! Since $*n + *n = 0$, char$(\mathcal{I}) = 2$.

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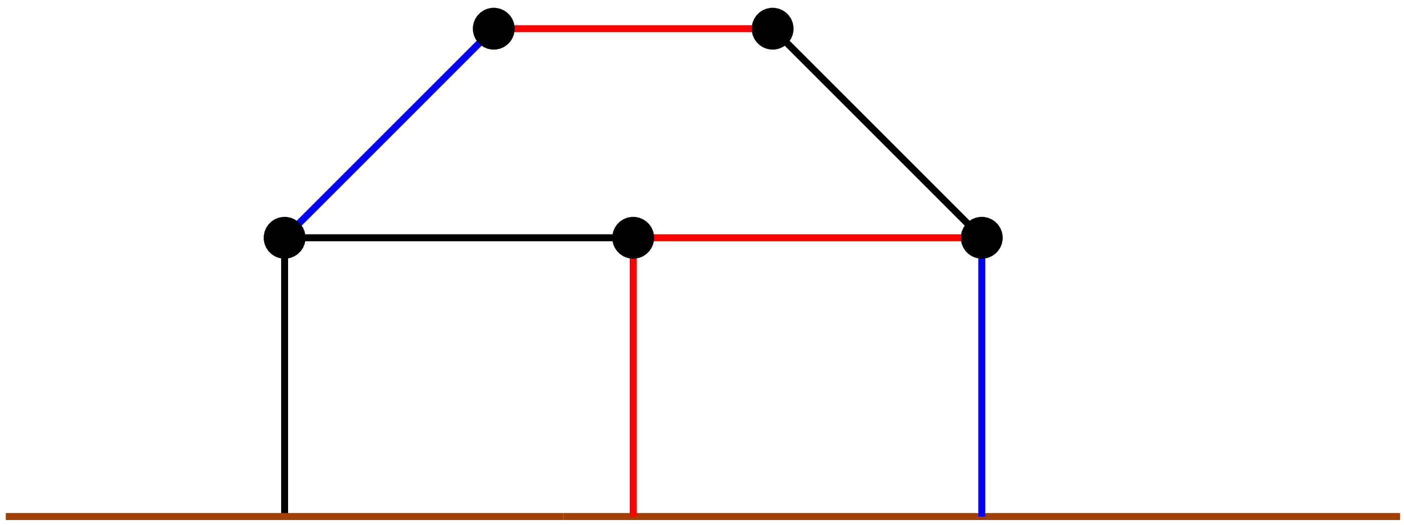
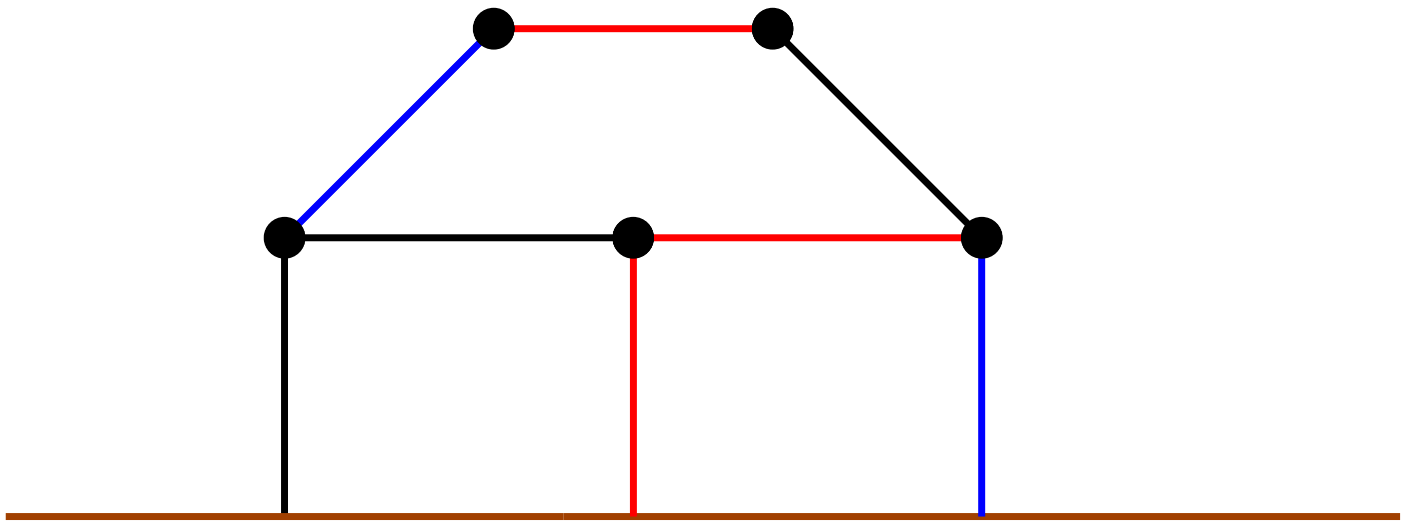
In fact, $\mathcal{I}$ is the quadratic closure of $\mathbb{F}_2$.

Mixed games



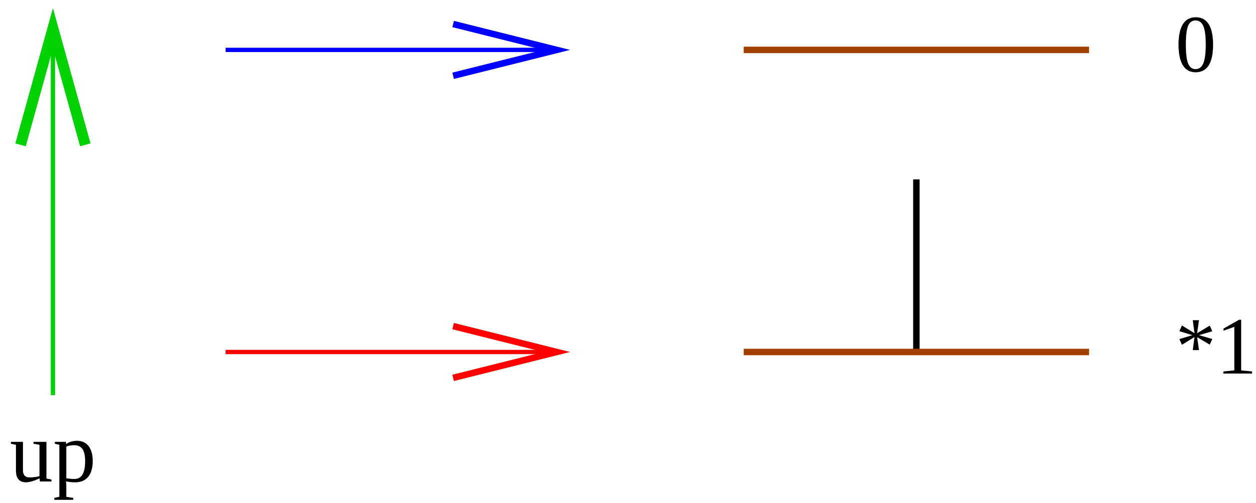
Mixed games



Much more complicated!

Up

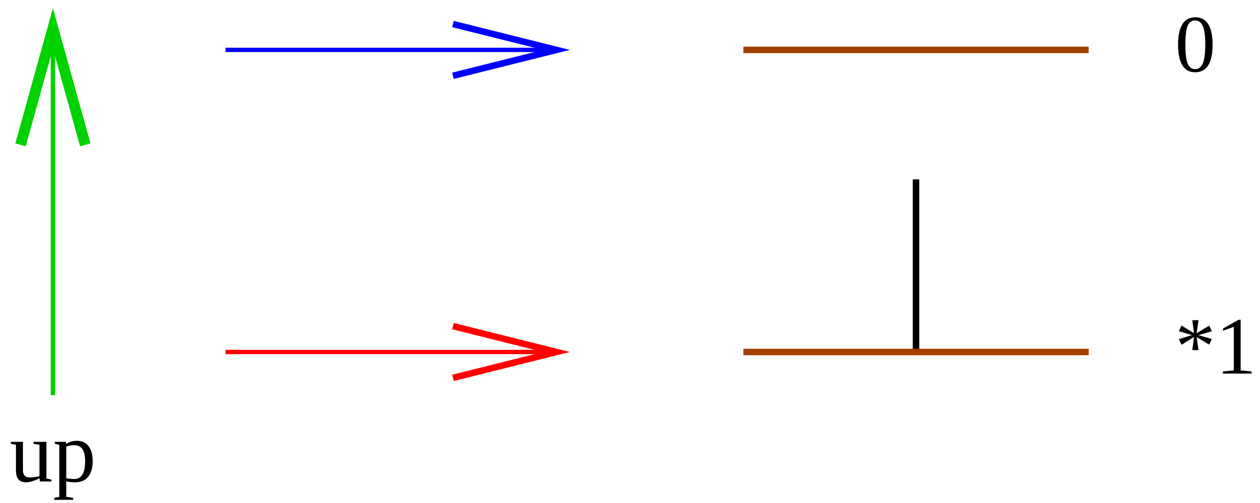
Not a Hackenbush game.



Outcome?

Up

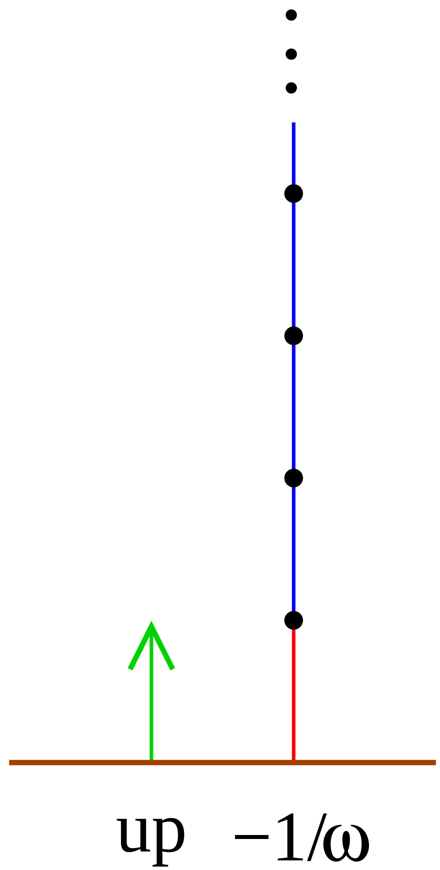
Not a Hackenbush game.



Outcome?

Blue wins, so $\uparrow > 0$.

$\uparrow$ is very tiny



Red wins, so $0 < \uparrow < 1/\omega$. In fact, for **any** number game $G > 0$, we have $0 < \uparrow < G$.

Caveat

Caveat. $\uparrow$ is not a number game. **Red** can move to $*1$, where it is **not** disadvantageous to move.


Darn!
That's
the
end...
H

References

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2. E. R. Berlekamp, J. H. Conway, and R. K. Guy, *Winning Ways*, Academic Press, New York, 1982. The Bible of the theory of mathematical games, with many general principles and interesting examples.

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6. R. J. Nowakowski, ed., *Games of No Chance*, MSRI Publications **29**, Cambridge University Press, New York/Cambridge, 1996. Many articles on mathematical games, including some applications to “real” games such as chess and Go. There are two sequels.

How can I get these slides?

Slides available at:

www-math.mit.edu/∼rstan/transparencies/games.pdf