THE LAURENT PHENOMENON

$$a_{n-1}a_{n+1} = a_n^2 - (-1)^n, \quad n \geq 1$$

$$a_0 = 1, \quad a_1 = 1$$

A priori $a_n \in \mathbb{Q}$ but actually

$$a_n = F_n \in \mathbb{Z}$$

(Fibonacci number), "explained" by

$$F_n = \alpha a^n + \beta b^n$$

and the addition law for e^x or $\sin x$.

M. Somos, c. 1982: Is there something similar involving addition law for elliptic functions?

First came **Somos-6**. **Somos-4** through **Somos-7**:

$$a_n a_{n-4} = a_{n-1} a_{n-3} + a_{n-2}^2, \quad n \geq 4$$
$$a_i = 1 \text{ for } 0 \leq i \leq 3$$

$$a_n a_{n-5} = a_{n-1} a_{n-4} + a_{n-2} a_{n-3}, \quad n \geq 5$$
$$a_i = 1 \text{ for } 0 \leq i \leq 4$$

$$a_n a_{n-6} = a_{n-1} a_{n-5} + a_{n-2} a_{n-4} + a_{n-3}^2, \quad n \geq 6$$
$$a_i = 1 \text{ for } 0 \leq i \leq 5$$

$$a_n a_{n-7} = a_{n-1} a_{n-6} + a_{n-2} a_{n-5} + a_{n-3} a_{n-4}, \quad n \geq 7$$
$$a_i = 1 \text{ for } 0 \leq i \leq 6.$$

Somos-4 through Somos-7 were conjectured to be integral (now proved), but for Somos-8,

$$a_{17} = 420514/\mathbf{7}.$$

Many similar conjectures, e.g., if

$$1 \le p \le q \le r, \quad k = p + q + r,$$

and

$$a_n a_{n-k} = a_{n-p} a_{n-k+p} + a_{n-q} a_{n-k+q} + a_{n-r} a_{n-k+r},$$

$$a_i = 1, \quad 0 \le i \le k-1,$$

then $a_n \in \mathbb{Z}$ (R. Robinson).

Parameters.

E.g., **generic Somos-4**:

$$a_n a_{n-4} = x a_{n-1} a_{n-3} + y a_{n-2}^2$$

$$a_0 = a,\ a_1 = b,\ a_2 = c,\ a_3 = d.$$

Then

$$a_n \in \mathbb{Z}[a^{\pm 1}, b^{\pm 1}, c^{\pm 1}, d^{\pm 1}, x^{\pm 1}, y^{\pm 1}],$$

an example of the **Laurent phenomenon**.

Note. Coefficients are ≥ 0 (D. Speyer). Also for Somos-5, but open for Somos-6 and Somos-7.

Cluster algebras (Fomin-Zelevinsky).

- Commutative algebras generated by unions of certain subsets called **clusters** (subject to axioms).
- If $C = \{x_1, \ldots, x_n\}$ and C' are clusters and $y \in C'$ then $y = F(x_1, \ldots, x_n)$ for some rational function F.
- In fact, F is a **Laurent polynomial** in $x_1, \ldots, x_n$.
- Developed to create an algebraic framework for dual-canonical bases and total positivity in algebraic groups.
- Techniques could be modified to apply to combinatorial situations such as Somos sequences.

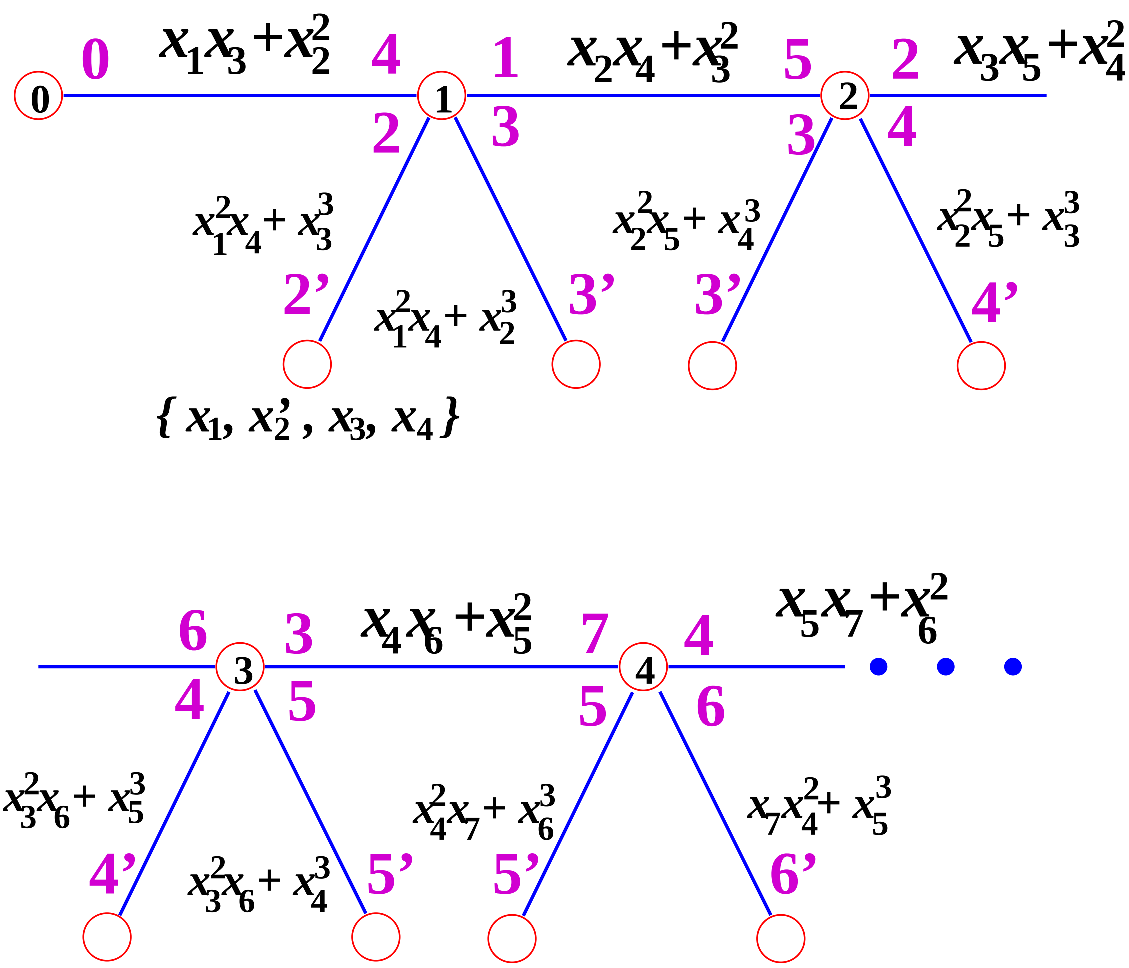
Example. $\boldsymbol{A} = \mathbb{C}[\mathrm{SL}_3/N]$, where $\boldsymbol{N}$ is the subgroup of unipotent upper-triangular matrices. Let

$$x_1,\ x_2,\ x_3,\ x_{12},\ x_{13},\ x_{23}$$

be Plücker coordinates on SL_3/N. Let $\{x_2\}$ and $\{x_{13}\}$ be the clusters. Then A is the algebra over $\mathbb{C}[x_1, x_3, x_{12}, x_{13}]$ generated by x_2 and x_{13} subject to the exchange relation

$$x_2 x_{13} = x_1 x_{23} + x_3 x_{12}.$$

Example (Somos-4).



spine: infinite path at top

two legs at each spine vertex (except the first)

spine vertices $\boldsymbol{v_0, v_1, \ldots}$

corresponding **cluster**:

$$\boldsymbol{C_i} = \{x_i, x_{i+1}, x_{i+2}, x_{i+3}\}$$

spine edge e has numerical labels $\boldsymbol{a_e}$, $\boldsymbol{b_e}$ and polynomial label $\boldsymbol{P_e}$

leg edge e has numerical label $\boldsymbol{a_e}$, polynomial label $\boldsymbol{P_e}$ and a label $\boldsymbol{b_e} = a'_e$ at bottom

If e connects spine vertex v and leg vertex w, then

$$\boldsymbol{C_w} = (C_v \cup \{x_{a'_e}\}) - \{x_{a_e}\}$$

For any edge $e = vw$ with labels a_e and b_e,

$$C_w = (C_v \cup \{x_{b_e}\}) - \{x_{a_e}\}.$$

If e has labels a, b, P, then regard

$$x_a x_b = P.$$

E.g., leftmost edge of $T \Rightarrow$

$$x_0 x_4 = x_1 x_3 + x_2^2.$$

Thus all x_i, x_i' are rational functions of $C_0 = \{x_0, x_1, x_2, x_3\}$.

What makes these rational functions Laurent polynomials?

- Every internal vertex v_i, $i \geq 1$, has the same degree, namely four, and the four edge labels "next to" v_i are $i, i+1, i+2, i+3$, the indices of the cluster variables associated to v_i.
- The polynomial P_e does not depend on x_{a_e} and x_{b_e}, and is not divisible by any variable x_i or x'_i.

- Write $\bar{P}_e$ for P_e with each variable x_j and x'_j replaced with $x_{\bar{j}}$, where $\bar{j}$ is the least positive residue of j modulo 4. If e and f are consecutive edges of T then the polynomials $\bar{P}_e$ and $\boldsymbol{\bar{P}_{f,0}} := \bar{P}_f|_{x_{\bar{a}_e}=0}$ are relatively prime elements of $\mathbb{Z}[x_1, x_2, x_3, x_4]$.

  Example. The leftmost two top edges of T yield that $x_1x_4 + x_2^2$ and

  $$(x_2x_4 + x_3^2)|_{x_4=0} = x_3^2$$

  are coprime.

- If e, f, g are three consecutive edges of T such that $\bar{a}_e = \bar{a}_g$, then

$$L \cdot \bar{P}_{f,0}^{b} \cdot \bar{P}_e = \bar{P}_g \mid_{x_{\bar{a}_f} \leftarrow \frac{\bar{P}_{f,0}}{x_{\bar{a}_f}}} \qquad (1)$$

where L is a Laurent monomial and $x_{\bar{a}_f} \leftarrow \frac{\bar{P}_{f,0}}{x_{\bar{a}_f}}$ denotes the substitution of $\frac{\bar{P}_{f,0}}{x_{\bar{a}_f}}$ for $x_{\bar{a}_f}$.

Example. Let e be the leftmost leg edge and f, g the second and third spine edges. Thus $\bar{a}_e = \bar{a}_g = 2$ and $\bar{a}_f = 1$. Equation (1) becomes

$$L \cdot (x_2 x_4 + x_3^2)^b_{x_2=0} \cdot (x_1 x_3 + x_2^2) =$$

$$(x_1 x_2^2 + x_3^3) \mid_{x_1 \leftarrow \frac{x_3^2}{x_1}},$$

which holds for $b = 1$ and $L = 1/x_1$, as desired.

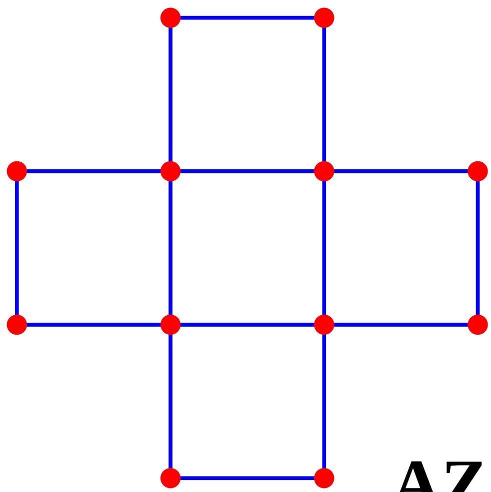
By "periodicity," only finitely many need be checked.

Since $x_i x_{i+4} = x_{i+1} x_{i+3} + x_{i+2}^2$, x_n is just the nth term of Somos-4 with generic initial conditions x_0, x_1, x_2, x_3. Hence **Somos-4 satisfies the Laurent phenomenon**.

Combinatorial proofs. Let e.g. $a_0, a_1, \ldots$ be the Somos-4 sequence. Can we interpret a_n combinatorially and prove combinatorially that

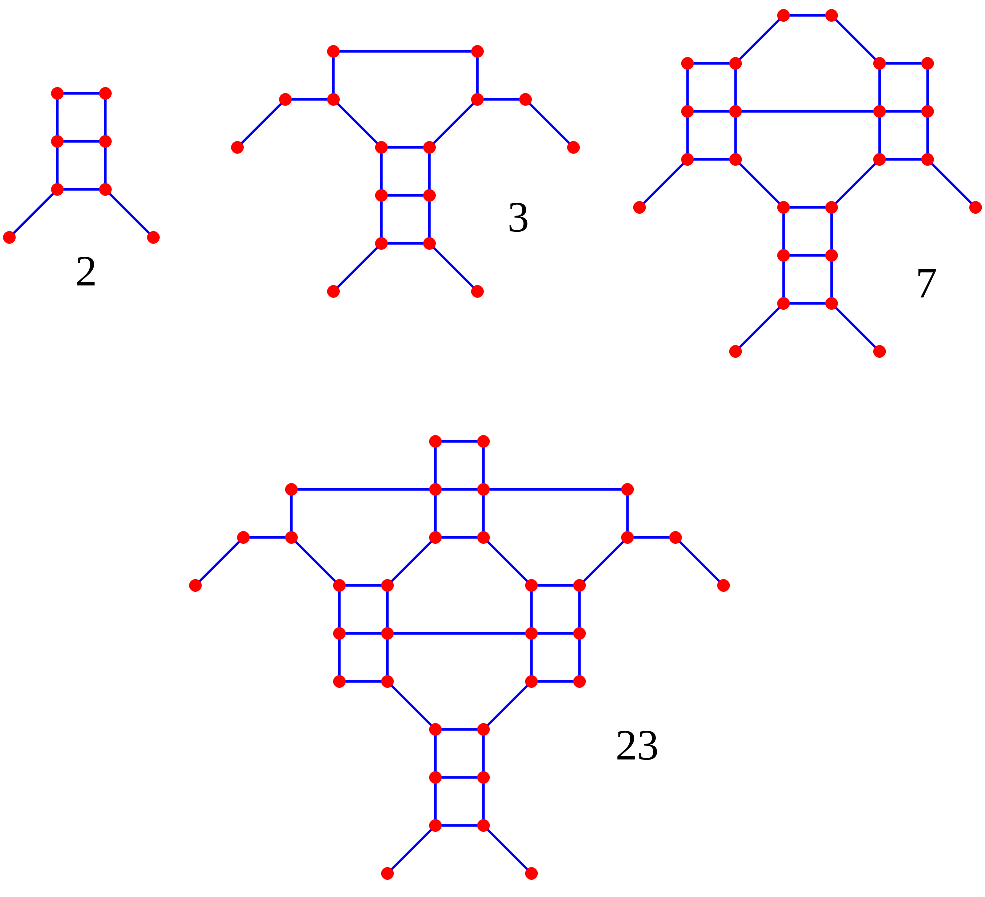
$$a_n a_{n-4} = a_{n-1} a_{n-3} + a_{n-2}^2?$$

Clue. a_n grows quadratically exponentially, as does the number $2^{\binom{n}{2}}$ of complete matchings in the Aztec diamond graph AZ_n.



AZ_3

Project REACH (Propp) and Bousquet-Mélou–Propp–West: a_n is the number of matchings in the **Somos-4 graph** $\boldsymbol{S_n}$.



TORIC SCHUR FUNCTIONS

$\mathbf{Gr}_{\boldsymbol{kn}}$: **Grassmann variety** of k-subspaces of $\mathbb{C}^n$

$$\dim_{\mathbb{C}} \mathrm{Gr}_{kn} = k(n-k)$$

$\boldsymbol{H^*(\mathbf{Gr}_{kn})} = H^*(\mathrm{Gr}_{kn};\mathbb{Z})$: cohomology ring (fundamental object for **Schubert calculus**)

basis for $H^*(\mathrm{Gr}_{kn})$: **Schubert classes** $\boldsymbol{\sigma_\lambda}$, where $\lambda = (\lambda_1,\dots,\lambda_k)$ and

$$\lambda \subseteq \boldsymbol{k \times (n-k)},$$

i.e.,

$$n-k \geq \lambda_1 \geq \cdots \geq \lambda_k \geq 0.$$

Let $\boldsymbol{P_{kn}}$ be the set of all such partitions λ, so

$$\#P_{kn} = \mathrm{rank}\, H^*(\mathrm{Gr}_{kn}) = \binom{n}{k}.$$

$\boldsymbol{\Omega_\lambda} \subset \mathrm{Gr}_{kn}$: **Schubert variety**, defined by bounds on $\dim X \cap V_i$, for $X \in \mathrm{Gr}_{kn}$, where

$$\{0\} = V_0 \subset V_1 \subset \cdots \subset V_n = \mathbb{C}^n$$

is a fixed flag.

Multiplication in $H^*(\mathrm{Gr}_{kn})$:

$$\sigma_\mu \sigma_\nu = \sum_{\lambda \in P_{kn}} c_{\mu\nu}^{\lambda} \sigma_\lambda,$$

where $c_{\mu\nu}^{\lambda}$ is a **Littlewood-Richardson coefficient**.

$$\Rightarrow c_{\mu\nu}^{\lambda} = \#\left(\tilde{\Omega}_\mu \cap \tilde{\Omega}_\nu \cap \tilde{\Omega}_{\lambda^\vee}\right),$$

where $\tilde{\boldsymbol{\Omega}}_{\boldsymbol{\nu}}$ is a generic translate of Ω_ν and $\boldsymbol{\lambda}^\vee$ is the **complementary partition**

$$\lambda^\vee = (n-k-\lambda_k, \ldots, n-k-\lambda_1).$$

$\mathbf{QH^*(Gr_{kn})}$: **quantum deformation** of $H^*(\mathrm{Gr}_{kn})$

$\boldsymbol{\Lambda_k}$: ring of symmetric polynomials over $\mathbb{Z}$ in $x_1, \ldots, x_k$.

$$\Lambda_k = \mathbb{Z}[e_1, \ldots, e_k],$$

where $\boldsymbol{e_i}$ is the ith **elementary symmetric function** in $x_1, \ldots, x_k$.

$\boldsymbol{h_i}$: sum of all monomials of degree i (**complete symmetric function**)

$$H^*(\mathrm{Gr}_{kn}) \cong \Lambda_k/(h_{n-k+1}, \ldots, h_n)$$

$$\mathrm{QH}^*(\mathrm{Gr}_{kn}) \cong \Lambda_k \otimes \mathbb{Z}[q]/(h_{n-k+1}, \ldots, h_{n-1}, h_n + (-1)^k q)$$

classical case: $q = 0$

$$H^*(\mathrm{Gr}_{kn}) \cong \Lambda_k/(h_{n-k+1}, \ldots, h_n)$$

Basis $\boldsymbol{B_{kn}}$ for $\Lambda_k/(h_{n-k+1}, \ldots, h_n)$:

Let λ be a partition.

semistandard Young tableau (SSYT) of shape λ:

$\leqslant$

$\wedge$

1	**1**	**3**	**4**
2	**4**	**4**	**6**
4	**6**	**9**	
6			

$$\lambda/\mu = (4, 4, 3, 1)$$

$$x^T = x_1^2 x_2 x_3 x_4^4 x_6^3 x_9$$

Schur function $\boldsymbol{s_\lambda}$ of shape λ:

$$s_\lambda = \sum_T x^T,$$

summed over all SSYT T of shape λ.

$$B_{kn} = \{s_\lambda \,:\, \lambda \subseteq k \times (n-k)\},$$

$$\begin{aligned} H^*(\mathrm{Gr}_{kn}) &\overset{\cong}{\to} \Lambda_k/(h_{n-k+1}, \ldots, h_n) \\ \sigma_\lambda &\mapsto s_\lambda \end{aligned}$$

$$s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda$$

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}$$

$$\begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$$

$$\begin{aligned} s_{21}(a,b,c) &= a^2b + ab^2 + a^2c + ac^2 \\ &\quad + b^2c + bc^2 + 2abc \\ s_{21} &= \sum_{i \neq j} x_i^2 x_j + 2 \sum_{i<j<k} x_i x_j x_k \\ s_{21}^2 &= s_{42} + s_{33} + s_{411} + 2s_{321} + s_{222} \\ &\quad + s_{3111} + s_{2211} \\ &\to s_{42} + s_{33} \text{ in } H^*(\mathrm{Gr}_{26}). \end{aligned}$$

basis for $\mathrm{QH}^*(\mathrm{Gr}_{kn})$ remains

$$\{\sigma_\lambda : \lambda \subseteq k \times (n-k)\}$$

quantum multiplication:

$$\sigma_\mu * \sigma_\nu = \sum_{d \geq 0} \sum_{\substack{\lambda \vdash |\mu|+|\nu|-dn \\ \lambda \in P_{kn}}} q^d \boldsymbol{C_{\mu\nu}^{\lambda,d}} \sigma_\lambda,$$

where $C_{\mu\nu}^{\lambda,d} \in \mathbb{Z}$.

$C_{\mu\nu}^{\lambda,d}$: number of rational curves of degree d in Gr_{kn} meeting $\tilde{\Omega}_\mu \cap \tilde{\Omega}_\nu \cap \tilde{\Omega}_{\lambda^\vee}$, a **3-point Gromov-Witten invariant**

Naively, a **rational curve of degree r in Gr_{kn}** is a set

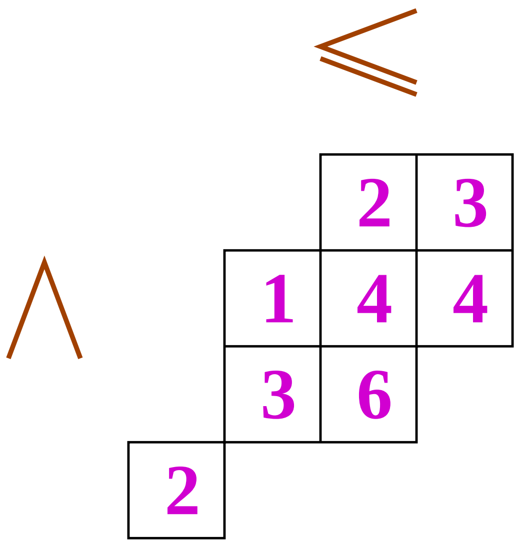
$$C = \left\{ (f_1(s,t), f_2(s,t), \ldots, f_{\binom{n}{k}}(s,t)) \in P^{\binom{n}{k}-1}(\mathbb{C}) \ : \ s,t \in \mathbb{C} \right\},$$

where $f_1(x,y), \ldots, f_{\binom{n}{k}}(x,y)$ are homogeneous polynomials of degree d such that $C \subset \mathrm{Gr}_{kn}$.

Rational curve of degree $d = 0$ is a point.

Let λ/μ be a **skew partition**, i.e., $\mu \subseteq \lambda$.

semistandard Young tableau (SSYT) of shape λ/μ:



$$\lambda/\mu = (4,4,3,1)/(2,1,1)$$

$$x^T = x_1 x_2^2 x_3^2 x_4^2 x_6$$

skew Schur function $\boldsymbol{s_{\lambda/\mu}}$ of shape λ/μ:

$$s_{\lambda/\mu} = \sum_T x^T,$$

summed over all SSYT T of shape λ/μ.

$$s_\lambda = s_{\lambda/\emptyset}$$

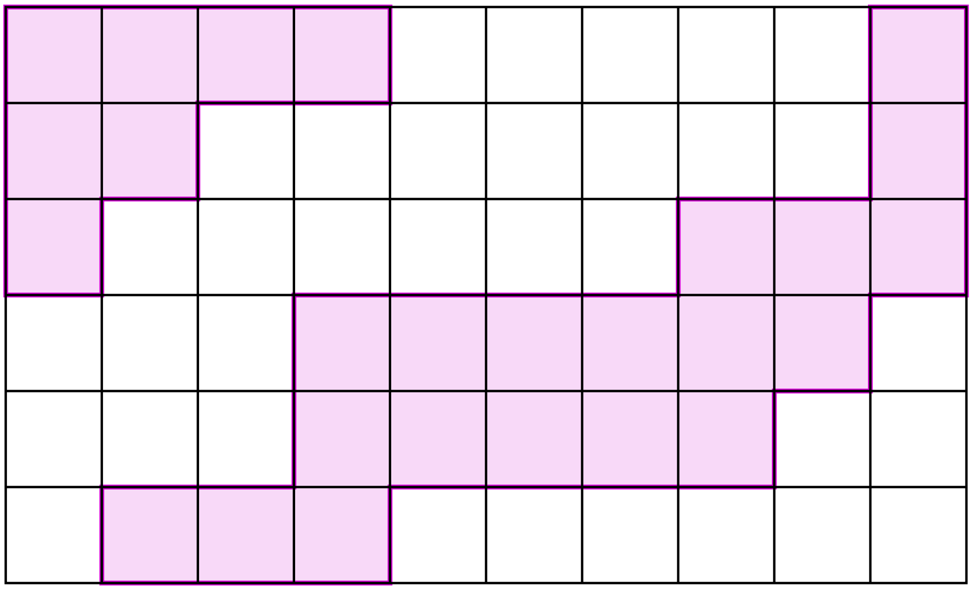
$$s_{\lambda/\mu} = \sum_\nu c^\lambda_{\mu\nu} s_\nu, \qquad (2)$$

where $c^\lambda_{\mu\nu}$ is a Littlewood-Richardson coefficient, i.e.,

$$s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda.$$

Want to generalize (2) to $C^{\lambda,d}_{\mu\nu}$.

toric shape τ in a 6×10 rectangle:

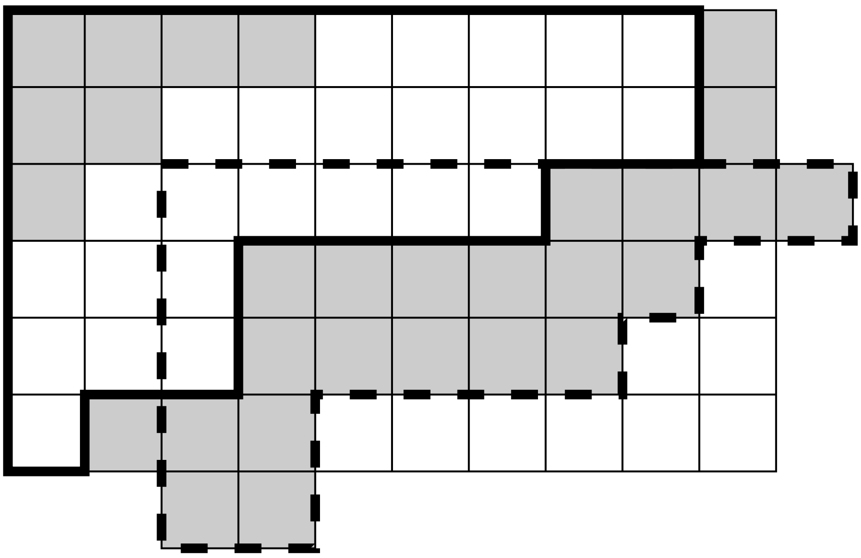


semistandard toric tableau (SSTT):

2	2	4	6						
3	5								
4							1	2	4
			1	2	2	2	2	5	
			3	3	4	4	4		
	1	2	4						

the toric shape

$$\tau = \boldsymbol{\lambda}/\boldsymbol{d}/\boldsymbol{\mu}$$
$$= (9,7,6,2,2,0)/2/(9,9,7,3,3,1) :$$



toric Schur function:

$$\boldsymbol{s_{\lambda/d/\mu}} = \sum_T x^T,$$

summed over all SSTT of shape $\lambda/d/\mu$

Theorem. *Let $\lambda/d/\mu$ be a toric shape contained in a $k \times (n-k)$ torus. Then*

$$s_{\lambda/d/\mu}(x_1, \ldots, x_k) = \sum_{\nu \in P_{kn}} C_{\mu\nu}^{\lambda,d} s_\nu(x_1, \ldots, x_k).$$

Compare the case $d = 0$: If

$$\lambda/\mu \subseteq k \times (n-k),$$

then

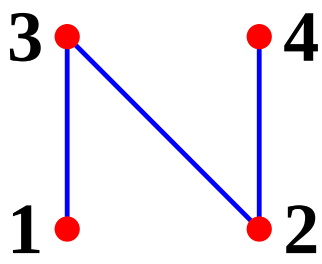
$$s_{\lambda/\mu}(x_1, , \ldots, x_k) = \sum_{\nu \in P_{kn}} c_{\mu\nu}^{\lambda} s_\nu(x_1, \ldots, x_k).$$

SIGN IMBALANCE

$\boldsymbol{P}$: partial ordering of $1, 2, \ldots, n$

$\boldsymbol{\mathcal{L}_P}$: set of **linear extensions** of P, regarded as permutations

$$a_1 \cdots a_n \in \mathfrak{S}_n$$



$$\mathcal{L}_P = \{1243, 1234, 2134, 2143, 2413\}$$

F. Ruskey (1989): does $\exists$ linear ordering of $\mathcal{L}_P$ such that any two consecutive terms differ by an (adjacent) transposition?

Let

$$\boldsymbol{\varepsilon_w} = \begin{cases} 1, & \text{if } w \text{ is even} \\ -1, & \text{if } w \text{ is odd.} \end{cases}$$

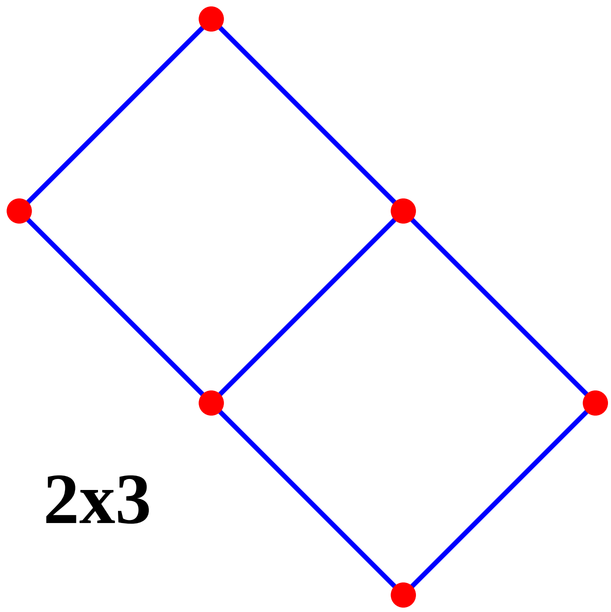
Define the **imbalance** $\boldsymbol{I_P}$ by

$$I_P = \sum_{w \in \mathcal{L}_P} \varepsilon_w.$$

Note. $|I_P|$ depends only on P **up to isomorphism**.

Note. If $\exists$ a Ruskey ordering of $\mathcal{L}_P$, then $I_P = 0, \pm 1$ (P is **sign-balanced**).

$\boldsymbol{r}$: r-element chain



2x3

Ruskey conjectured (1992):

$$I_{\boldsymbol{r}\times\boldsymbol{s}} = 0 \Leftrightarrow r, s > 1,\ r \equiv s \,(\mathrm{mod}\, 2).$$

Easy for r, s even (Ruskey). Proof in general by D. White (2002). In fact:

Theorem. *Let* $r \not\equiv s \,(\mathrm{mod}\, 2)$. *Then*

$$I_{\boldsymbol{r}\times\boldsymbol{s}} = g^{\nu},$$

where $\boldsymbol{g^{\nu}}$ is the number of **standard shifted Young tableaux (SShYT)** of shape

$$\nu = \left(\frac{r+s-1}{2}, \frac{r+s-3}{2}, \ldots, \frac{|r-s|+3}{2}, \frac{|r-s|+1}{2}\right).$$

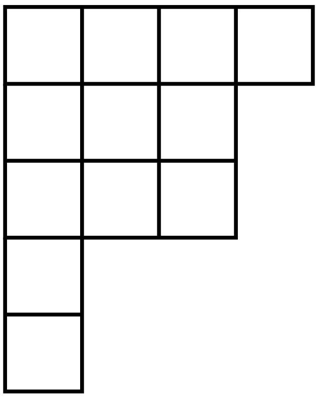
E.g., $P = \mathbf{8} \times \mathbf{3}$, $\nu = (5, 4, 3)$.

1	2	3	5	7
	4	6	9	11
		8	10	12

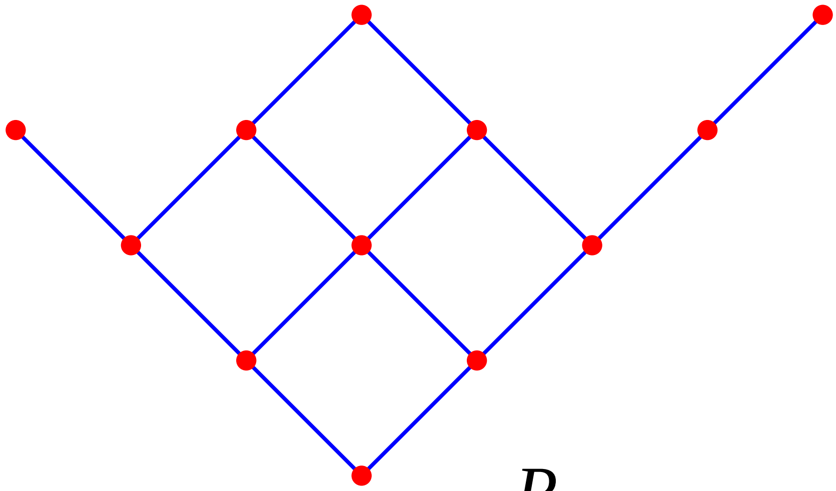
$$I_P = g^{5,4,3} = \frac{12!}{9 \cdot 8 \cdot 7 \cdot 5 \cdot 4^2 \cdot 3^3 \cdot 2^2 \cdot 1}$$

$$= 110$$

Generalize to partitions λ:



$\lambda = (4,3,3,1,1)$



P_λ

Write $\boldsymbol{I_\lambda} = I_{P_\lambda}$.

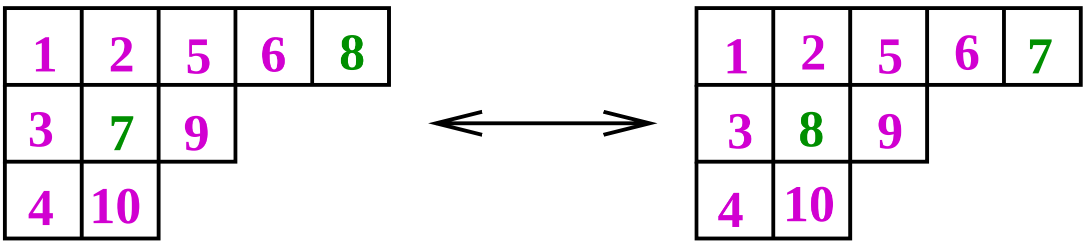
standard Young tableau (SYT)

of shape λ:

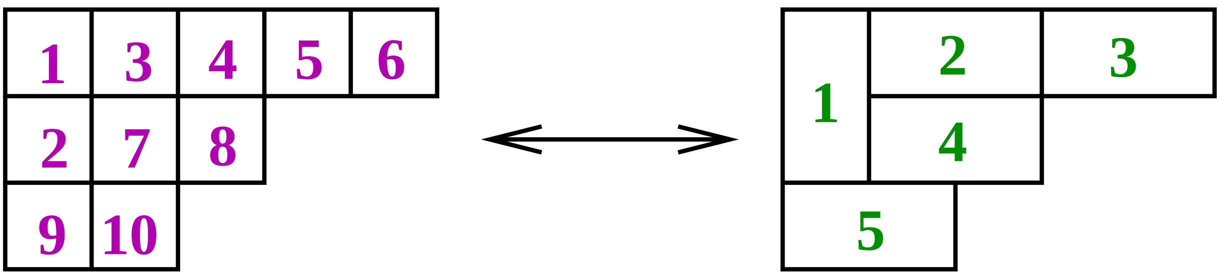
1	3	4	12
2	6	8	
5	9	11	
7			
10			

$$\lambda = (4,3,3,1,1)$$

Involution on SYT's T of shape λ: interchange smallest $2i-1, 2i$ possible; otherwise T is fixed.

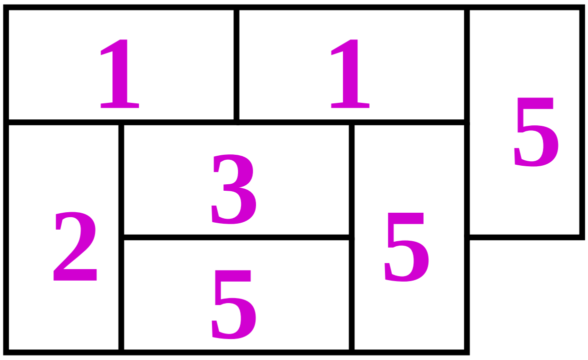


survivors are **standard domino tableaux** of shape λ:



Domino Schur functions (Carré, Leclerc, Lascoux, Thibon, Kirillov, T. Lam, . . .)

semistandard domino tableau (SSDT) D of shape $(5,5,4)$:



$$\boldsymbol{x}^{\boldsymbol{D}} = x_1^2 x_2 x_3 x_5^3$$

$$\textbf{spin}(\boldsymbol{D}) = \frac{1}{2}(v(\lambda)-v(D)) = \frac{1}{2}(5-3) = 1,$$

where

$\boldsymbol{v}(\boldsymbol{D})$ = # vertical dominos in D

$\boldsymbol{v}(\boldsymbol{\lambda})$ = max # of vertical dominos in a domino tiling of shape λ

Let $\lambda \vdash 2m$. Define

$$\boldsymbol{G_\lambda(x;q)} = \sum_D q^{\mathrm{spin}(D)} x^D,$$

summed over all SSDT D of shape λ. (Analogous definition for $\lambda \vdash 2m+1$, with momino in upper-left corner.)

Related to Hall-Littlewood symmetric functions, quantum affine algebras, unipotent varieties, real Schubert varieties, . . .

Proposition. *Let $\lambda \vdash n$. Then*

$$[x_1 \cdots x_n] G_\lambda(x;-1) = \pm I_\lambda.$$

Connection with real Schubert varieties (Eremenko-Gabrielov). Let $\lambda \subseteq \boldsymbol{k} \times (\boldsymbol{n}-\boldsymbol{k})$. Let $\boldsymbol{\Omega_\lambda(\mathbb{C})}$ be the corresponding Schubert variety for $\mathrm{Gr}_{kn}(\mathbb{C})$.

The **Wronski map**

$$\boldsymbol{W}(f_1, \ldots, f_{n-k}) = \begin{vmatrix} f_1 & \cdots & f_{n-k} \\ f_1' & \cdots & f_{n-k}' \\ \cdots & \cdots & \cdots \\ f_1^{(n-k-1)} & \cdots & f_p^{(n-k-1)} \end{vmatrix},$$

where $\deg f_i < n$, may be regarded as a map

$$\boldsymbol{\phi} : \mathrm{Gr}_{kn}(\mathbb{C}) \to \mathbb{C}P^{k(n-k)}.$$

Restrict to Ω_λ.

Schubert (1886): $\deg \phi|_{\Omega_\lambda} = \boldsymbol{f^\lambda}$, the number of SYT of shape λ (elegant **hook-length formula**).

What about $\Omega_\lambda(\mathbb{R})$? Milnor (1965) defined $\deg \phi$ for maps $\phi : X \to Y$ of **real** spaces satisfying certain orientability conditions.

Let

$$\boldsymbol{\phi}_{\mathbb{R}} : \mathrm{Gr}_{kn}(\mathbb{R}) \to \mathbb{R}P^{k(n-k)}.$$

Restrict to Ω_λ. When orientability conditions are satisfied (e.g., $\lambda = \boldsymbol{k \times (n-k)}$),

$$\deg \phi_{\mathbb{R}}|_{\Omega_\lambda} = \boldsymbol{I^\lambda}.$$

Sample application (conjectured by RS, proved by T. Lam and J. Sjöstrand, independently):

Theorem. $\sum_{\lambda\vdash n} I_\lambda = 2^{\lfloor n/2\rfloor}$

(special case of weighted version)

$$\begin{array}{cccc} 1\,2\,3 & \begin{array}{l}1\,2\\3\end{array} & \begin{array}{l}1\,3\\2\end{array} & \begin{array}{l}1\\2\\3\end{array} \end{array}$$

$$\begin{array}{cccc} 123 & 123 & 132 & 123 \\ 1 & 1 & -1 & 1 \end{array}$$

$$1+1-1+1=2=2^{\lfloor 3/2\rfloor}$$

Connection with shifted tableaux. Recall that if $r \not\equiv s \pmod 2$ then

$$I_{\boldsymbol{r}\times\boldsymbol{s}} = g^{\nu}$$

for a certain SShYT ν. What about other λ?

Conjecture (Eremenko-Gabrielov). For fixed $\ell(\lambda)$ and parity of each λ_i, there is a "nice" formula for I_λ in terms of g^{ν}'s.

Example. Let $g^{r,s,t} = 0$ unless

$$r > s > t \geq 0,$$

etc. Then

$$\begin{aligned}
I_{2a,2b,2c} &= g^{a,b,c} - g^{a+1,b,c-1} \\
I_{2a,2b+1,2c+1} &= g^{a+1,b,c} + g^{a,b+1,c} \\
I_{2a,2b+1,2c} &= 0 \text{ (easy)} \\
I_{2a,2b,2c,2d} &= g^{a,b,c,d} - g^{a+1,b,c-1,d} \\
&\quad - g^{a,b+1,c,d-1} \\
&\quad - g^{a+1,b+1,c-1,d-1} \\
&\quad - 2g^{a+1,b,c,d-1}.
\end{aligned}$$

(Can be proved by induction.)

Explicit formulas not known.

Refinement of previous conjecture. If f is a symmetric function, let

$$\begin{aligned} \boldsymbol{f(x/x)} &= f(p_{2i-1} \to 2p_{2i-1}, p_{2i} \to 0) \\ &= f(X - X). \end{aligned}$$

$\boldsymbol{Q_\lambda(x)}$: Schur's shifted Q-function

$$[x_1 \cdots x_n] Q_\lambda(x) = 2^n g^\lambda$$

Example.

$$I_{2a,2b,2c} = g^{a,b,c} - g^{a+1,b,c-1}$$

$$\begin{aligned} \pm G_{2a,2b,2c}(x/x; -1) &= Q_{a,b,c}(x) \\ &\quad - Q_{a+1,b,c-1}(x) \end{aligned}$$