SOME OPEN PROBLEMS IN ENUMERATIVE COMBINATORICS

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Fantastically
Original
Approaches
Toward
Algebraic combinatorics

hyperplane arrangement $\mathcal{A}$: finitely many affine hyperplanes in $\mathbb{R}^n$

region of $\mathcal{A}$: connected component (cell) of $\mathbb{R}^n - \bigcup_{H\in\mathcal{A}} H$

$\boldsymbol{\mathcal{R}(\mathcal{A})}$: set of regions of $\mathcal{A}$

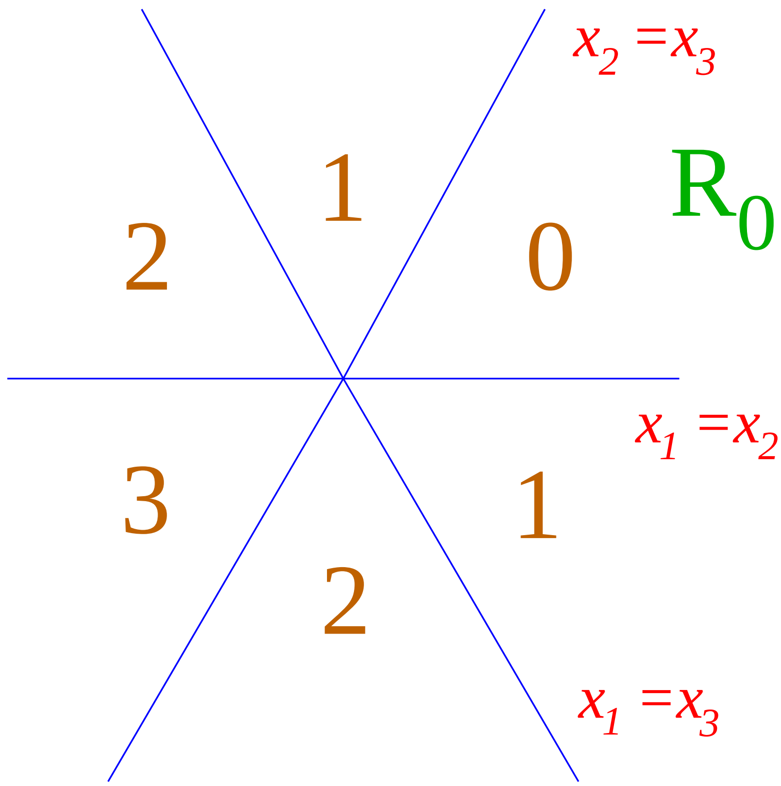
$\boldsymbol{d(R, R')}$, where $R, R' \in \mathcal{R}(\mathcal{A})$:

\# of $H \in \mathcal{A}$ separating R and R'

$\boldsymbol{R_0}$: a fixed base region of $\mathcal{A}$

distance enumerator of $\mathcal{A}$ (w.r.t. R_0):

$$\boldsymbol{D_{\mathcal{A}}(q)} = \sum_{R\in\mathcal{R}(\mathcal{A})} q^{d(R_0,R)}$$



braid arrangement $\mathcal{B}_3$:

$$x_i = x_j, \quad 1 \le i < j \le 3$$

$$D_{\mathcal{B}_3}(q) = 1+2q+2q^2+q^3 = (1+q)(1+q+q^2)$$

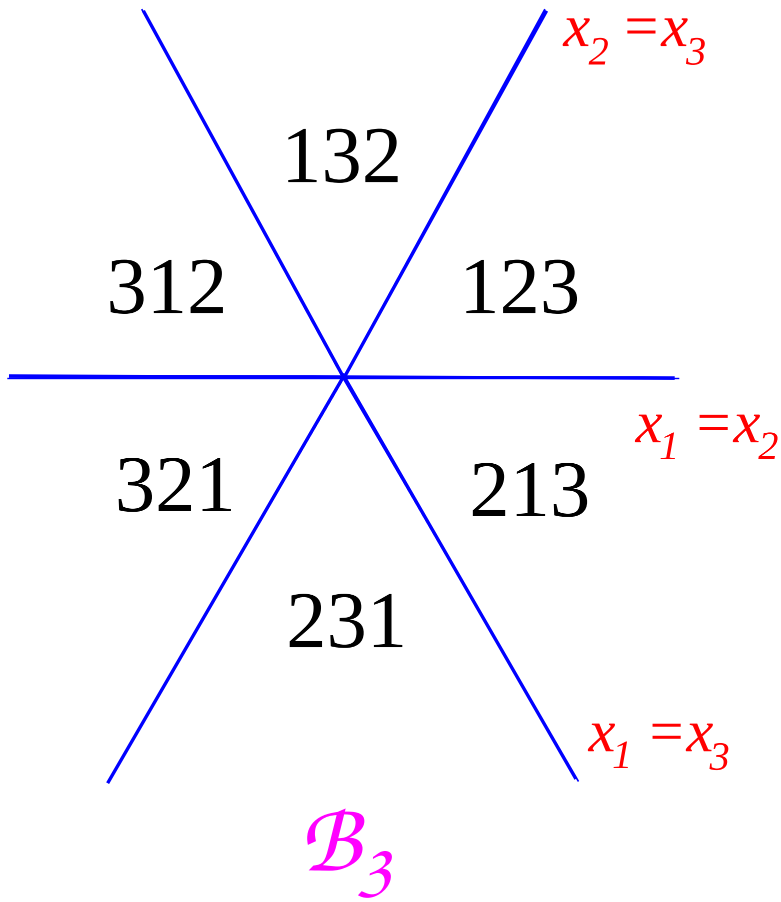
$\mathcal{B}_n$: $x_i = x_j, \quad 1 \le i < j \le n$

regions: $R_w : \ x_{w(1)} > \cdots > x_{w(n)}$, where $w \in \mathfrak{S}_n$

R_0: $x_1 > \cdots > x_n$ (so $w = 12\cdots n$)

$$\begin{aligned} d(R_0, R_w) &= \#\{(i,j) \ : \ i < j, \ w(i) > w(j)\} \\ &= \ell(w) = \mathbf{INV}(w) \end{aligned}$$

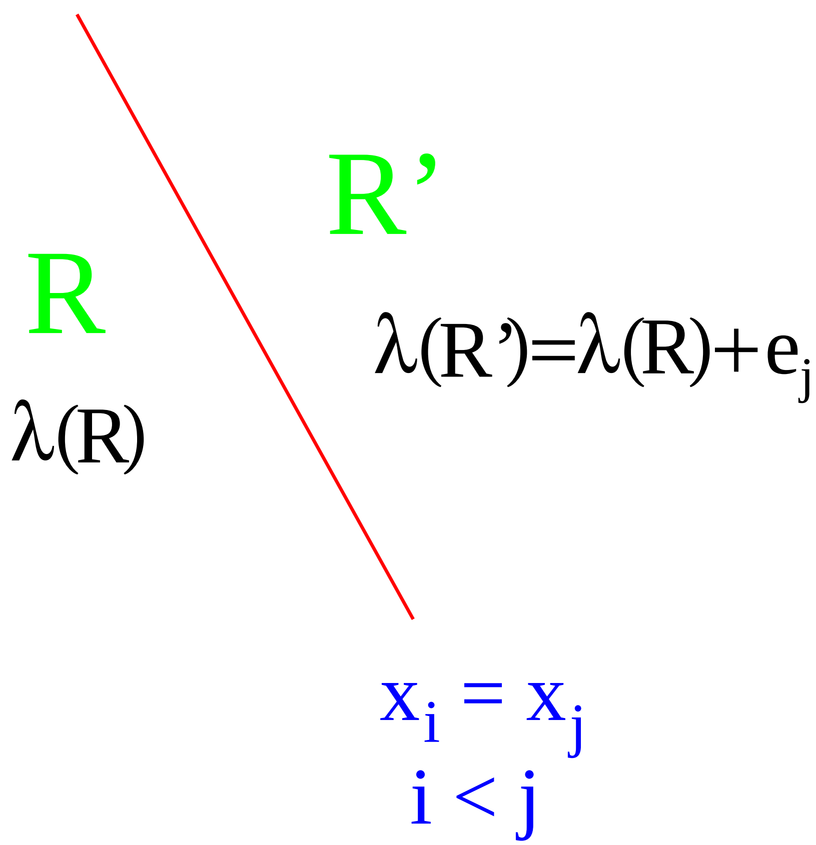
$$\begin{aligned} D_{\mathcal{B}_n}(q) &= \sum_{w \in \mathfrak{S}_n} q^{\ell(w)} \\ &= (1+q)(1+q+q^2) \\ &\qquad \cdots (1+q+\cdots+q^{n-1}), \end{aligned}$$

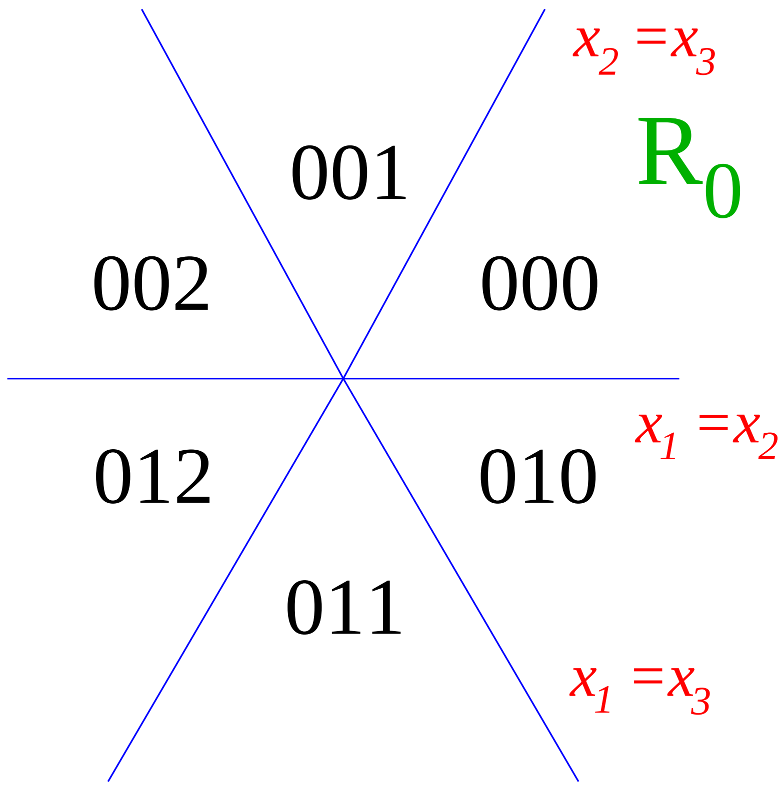

$x_2 = x_3$
132
312
123
$x_1 = x_2$
321
213
231
$x_1 = x_3$
$\mathcal{B}_3$

Alternative labeling rule: Let

$$\lambda(R_0) = 00\cdots 0 \in \mathbb{Z}^n$$

$$\boldsymbol{e_i} = 00\cdots \overset{i}{1} \cdots 0 \in \mathbb{Z}^n$$





$$\lambda(R_w) = (c_1, \ldots, c_n),$$

where

$$c_j = \#\{i \,:\, i < j,\ w = \cdots j \cdots i \cdots\}$$

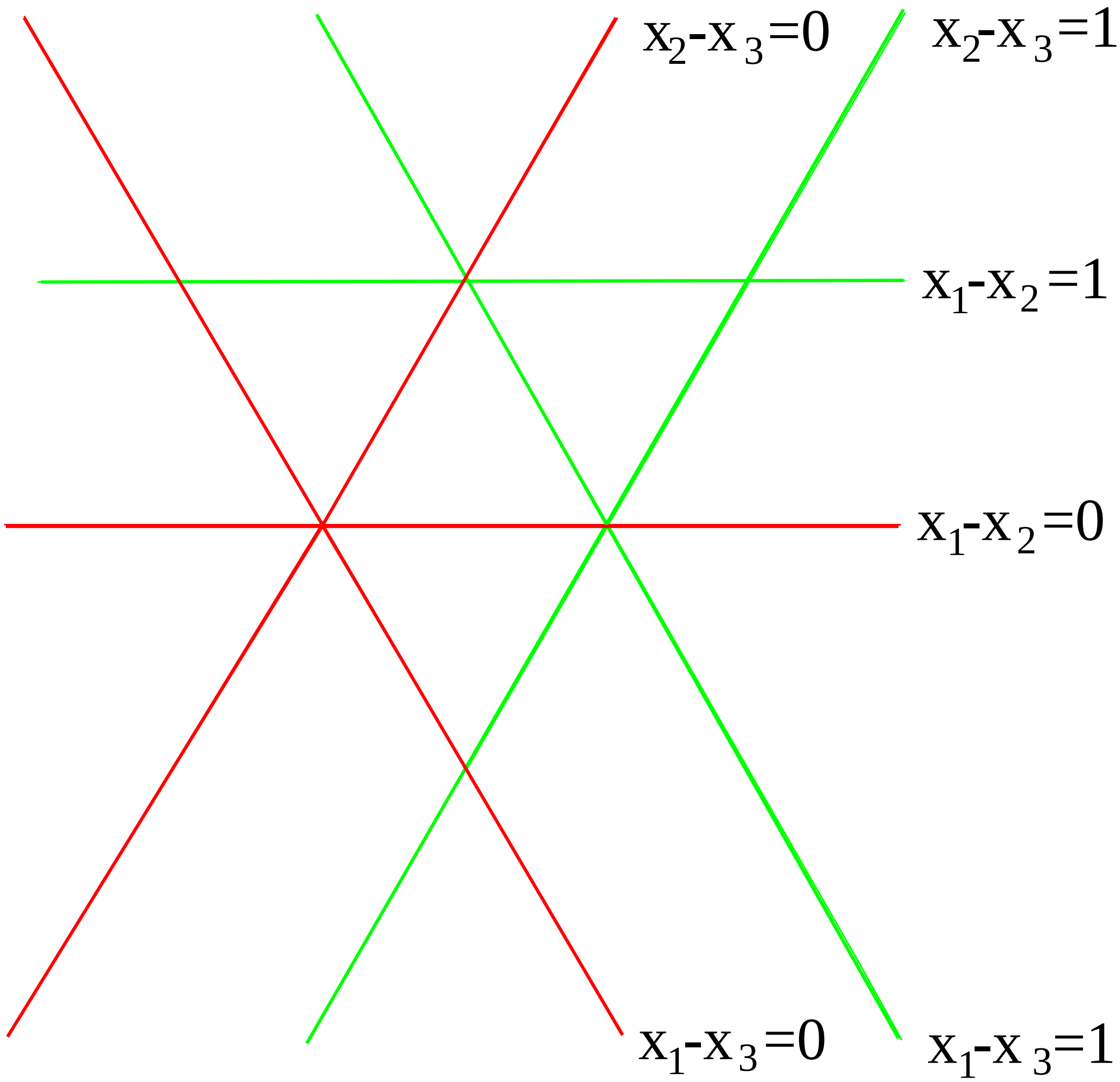
$$d(R_0, R_w) = c_1 + \cdots + c_n$$

$$(c_1, \ldots, c_n) = \textbf{code} \text{ of } w$$

$$\begin{aligned} D_{\mathcal{B}_n}(q) &= \sum_{c_1=0}^{0} \cdots \sum_{c_n=0}^{n-1} q^{c_1+\cdots+c_n} \\ &= (1+q)(1+q+q^2) \\ &\qquad \cdots (1+q+\cdots+q^{n-1}) \end{aligned}$$

Shi arrangement:

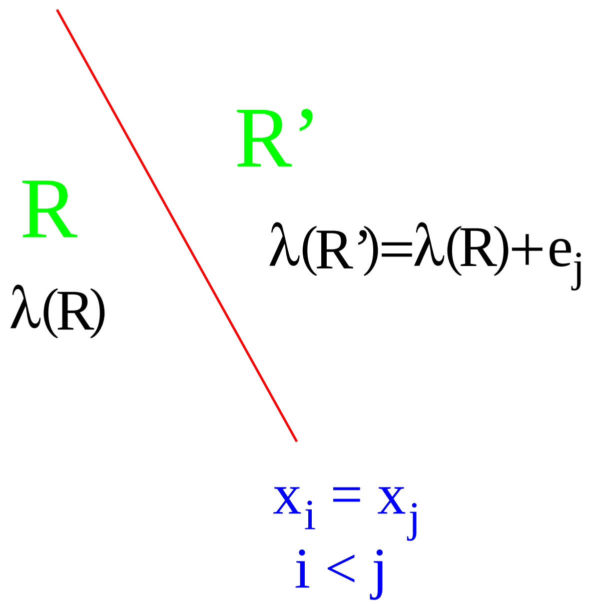
$\mathcal{S}_n$: $x_i - x_j = 0, 1,\ 1 \le i < j \le n$

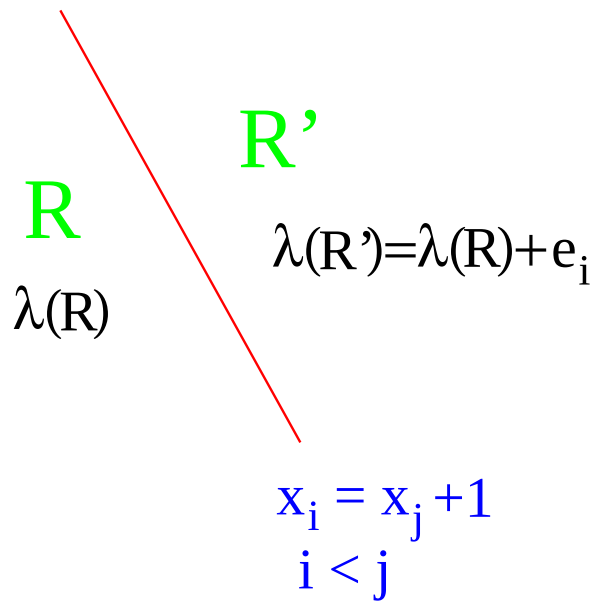


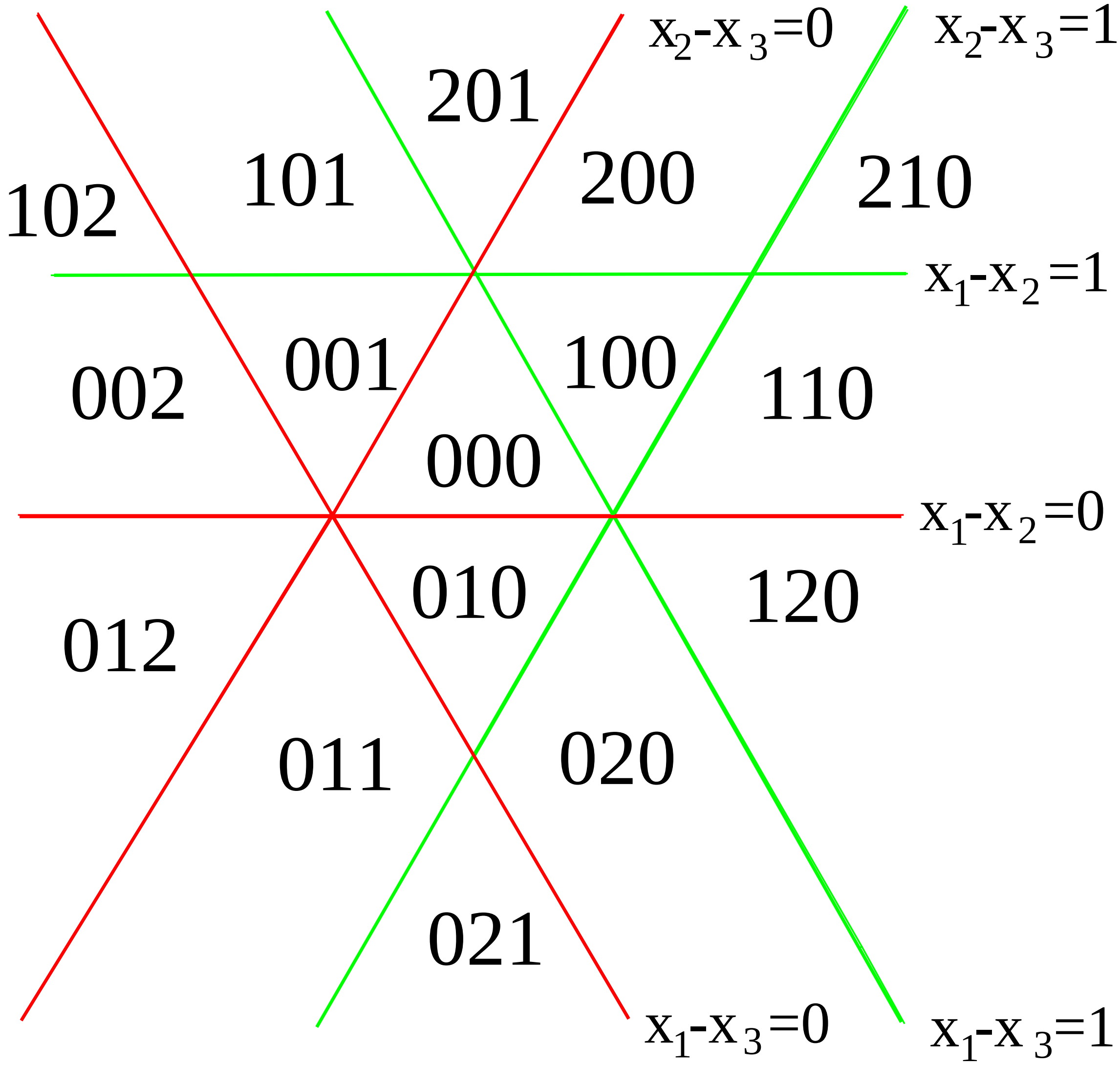
$\boldsymbol{R_0}$: $x_1 > x_2 > \cdots > x_n > x_1 - 1$

Labeling rule:

$$\lambda(R_0) = 00\cdots 0 \in \mathbb{Z}^n$$







$$D_{\mathcal{S}_3}(q) = 1 + 3q + 6q^2 + 6q^3$$

Theorem. *Let*

$$\alpha = (a_1, \ldots, a_n) \in \mathbb{N}^n.$$

TFAE:

- $\lambda(R) = \alpha$ for some (unique) $R \in \mathcal{R}(\mathcal{S}_n)$.
- If $b_1 \leq \cdots \leq b_n$ is the increasing rearrangement of $a_1, \ldots, a_n$, then $b_i \leq i - 1$.
- α is a permutation of the coordinates of some label of a region of $\mathcal{B}_n$.
- $(a_1 + 1, \ldots, a_n + 1)$ is a **parking function**.

Corollary (of the theory of parking functions). *Let*

$$I_n(q) = \sum_F q^{\mathrm{INV}(F)},$$

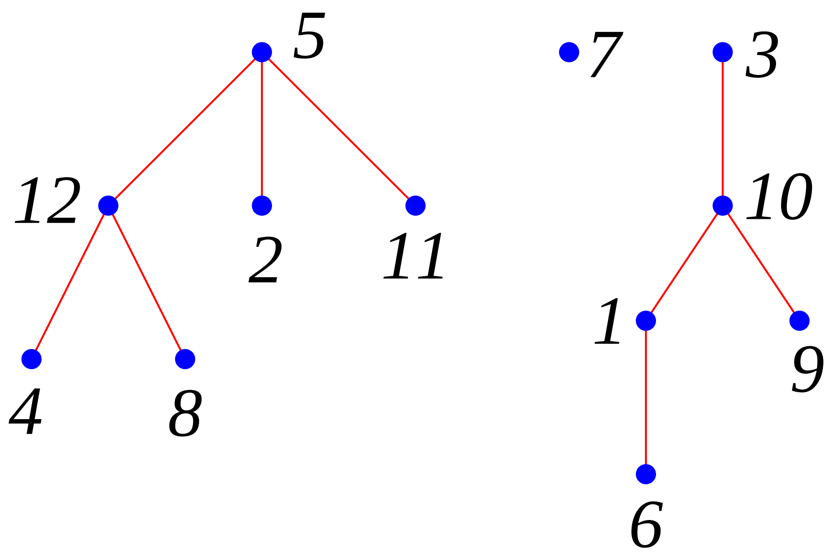
where F ranges over all rooted forests on the vertices $1, \ldots, n$ and

$$\mathrm{INV}(F) = \#\{(i,j) : i > j, i \text{ lies on the path from } j \text{ to the root}\}$$

Then

$$D_{\mathcal{S}_n}(q) = q^{\binom{n}{2}} I_n(1/q)$$

(so $\#\mathcal{R}(\mathcal{S}_n) = (n+1)^{n-1}$).



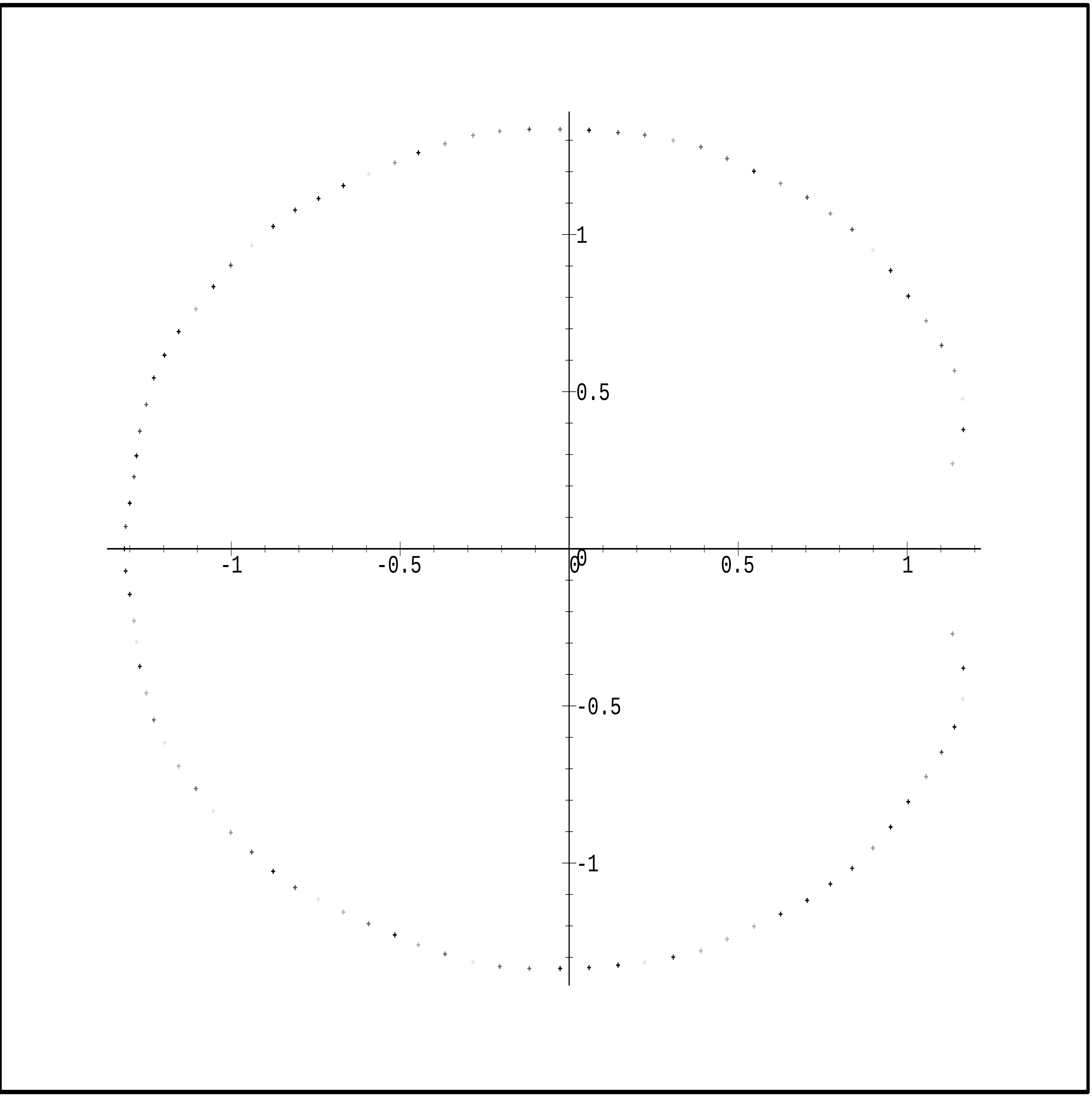
Inversions: $(5,4),\ (5,2),\ (12,4),\ (12,8)$

$(3,1),\ (10,1),\ (10,6),\ (10,9)$

$\text{inv}(F) = 8$

$$\sum_{n\geq 0} I_n(q)(q-1)^n \frac{x^n}{n!} = \frac{\sum_{n\geq 0} q^{\binom{n+1}{2}} \frac{x^n}{n!}}{\sum_{n\geq 0} q^{\binom{n}{2}} \frac{x^n}{n!}}$$

- asymptotics of coefficients of $I_n(q)$?
- unimodality or log-concavity of coefficients? (log-concave for $n \leq 50$)
- zeros of $I_n(q)$


-1
-0.5
0
0.5
1
0.5
-0.5
-1

Let $R, R' \in \mathcal{R}(\mathcal{S}_n)$. Define

$$\boldsymbol{d_1(R, R')} = \#\{H \in \mathcal{S}_n : H \text{ separates } R \text{ and } R', \text{ and } H \text{ has the form } x_i = x_j\}$$

$$\boldsymbol{d_2(R, R')} = \#\{H \in \mathcal{S}_n : H \text{ separates } R \text{ and } R', \text{ and } H \text{ has the form } x_i = x_j + 1\}.$$

$a \backslash b$	0	1
0	1	1
1	1	

$a \backslash b$	0	1	2	3
0	1	1	2	1
1	2	2	2	
2	2	2		
3	1			

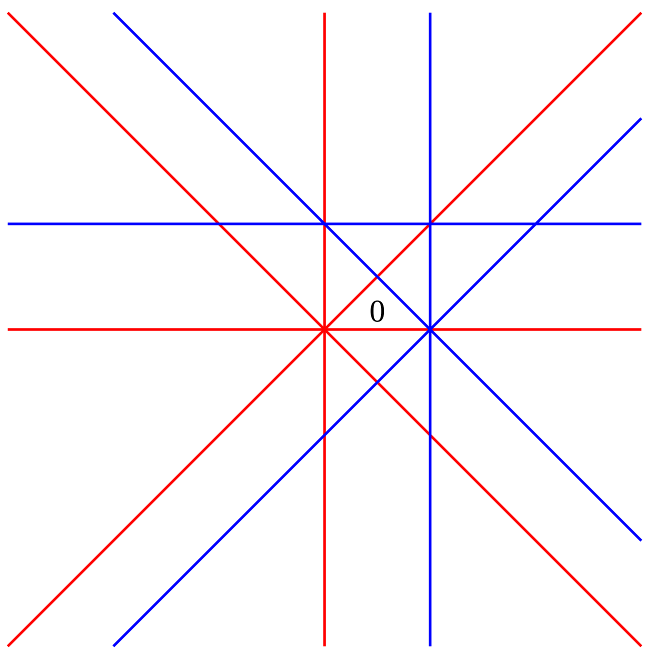
$a \backslash b$	0	1	2	3	4	5	6
0	1	1	2	3	3	3	1
1	3	3	6	7	6	3	
2	5	5	8	9	5		
3	6	7	9	6			
4	5	6	5				
5	3	3					
6	1						

Problem 1. "Determine"

$$D_{\mathcal{S}_n}(q,t) = \sum_{R \in \mathcal{R}(\mathcal{S}_n)} q^{d_1(R_0,R)} t^{d_2(R_0,R)}.$$

Now define the **B_n-Shi arrangement** by

$$\mathcal{S}_n^B : \quad x_i \pm x_j = 0, 1$$
$$x_i = 0, 1.$$



$\mathbf{S_2^B}$

$$D_{\mathcal{S}_2^B}(q) = 1 + 3q + 5q^2 + 8q^3 + 8q^4$$

Theorem (Shi, Headley).

$$\#\mathcal{R}(\mathcal{S}_n^B) = (2n+1)^n$$

Problem 2. "Determine" $D_{\mathcal{S}_n^B}(q)$ (and similarly for other root systems). Is there a nice labeling of the regions?

Combinatorics of real zeros

Let

$$\boldsymbol{P(x)} = a_0 + a_1 x + \cdots + a_n x^n \in \mathbb{R}[x].$$

Set $a_{-k} = 0$ for $k < 0$ or $k > n$. Define

$$\boldsymbol{A_P} = \left[a_{j-i}\right]_{i,j\geq 1},$$

an infinite **Toeplitz matrix**.

Theorem (Aissen-Schoenberg-Whitney, 1952) *TFAE:*

- *Every minor of A_P is ≥ 0, i.e., A_P is* ***totally nonnegative****.*
- *Every zero of $P(x)$ is real and ≤ 0.*

Culture: Edrei-Thoma generalization (conjectured by Schoenberg). Let $P(x) = 1 + a_1 x + \cdots \in \mathbb{R}[[x]]$. TFAE:

- Every minor of A_P is nonnegative.
- $P(x) = e^{\gamma x} \prod_i \frac{1 + r_i x}{1 - s_i x}$, where

$$\gamma, r_i, s_i \geq 0, \ \sum r_i + \sum s_i < \infty.$$

Note.

- A_P easily seen to be t.n. for

$$P(x) = 1 + ax, \ a \geq 0, \text{ or } P(x) = \frac{1}{1 - bx}, \ b \geq 0.$$

- $A_{PQ} = A_P A_Q$
- $e^{\gamma x} = \lim_{n \to \infty} \left(1 + \frac{\gamma x}{n}\right)^n$

Note. Total nonnegativity of A_P involves **infinitely** many inequalities, even for $P(x) = ax^2 + bx + c$. There is also a system of $n-1$ inqualities equivalent to every zero of $P(x) = a_n x^n + \cdots + a_1 x + a_0$ $(a_n \neq 0)$ being real.

Example. Every zero of $x^3 + bx^2 + cx + d$ is real $\iff$

$$-27d^2 + 18bcd - 4c^3 - 4b^3 d + b^2 c^2 \geq 0$$
$$b^2 - 3c \geq 0$$

Rephrasing of A-S-W theorem.

Let $P(x) \in \mathbb{R}[x]$, $P(0) = 1$. Define

$$F_P(\boldsymbol{x}) = P(x_1)P(x_2)\cdots,$$

a symmetric formal series in $\boldsymbol{x} = (x_1, x_2, \ldots)$. TFAE:

- Every zero of $P(x)$ is real and < 0.
- $F_P(\boldsymbol{x})$ is **s-positive**, i.e., a nonnegative linear combination of Schur functions s_λ.
- $F_P(\boldsymbol{x})$ is **e-positive**, i.e., a nonnegative linear combination of elementary symmetric functions e_λ.

Example.

$$P(x) = \frac{A_5(x)}{x} = 1+26x+66x^2+26x^3+x^4$$

$$\begin{aligned} F_P &= 1 + 26s_1 + (66s_2 + 610s_{11}) \\ &\quad + (26s_3 + 1690s_{21} + 14170s_{111}) + \cdots \\ &= 1 + 26e_1 + (544e_2 + 66e_{11}) \\ &\quad + (12506e_3 + 1638e_{21} + 26e_{111}) + \cdots \end{aligned}$$

Eulerian polynomial:

$$\boldsymbol{A_n(x)} = \sum_{w \in \mathfrak{S}_n} x^{\mathrm{des}(w)+1},$$

where

$$\mathbf{des(w)} = \#\{i : w(i) > w(i+1)\}.$$

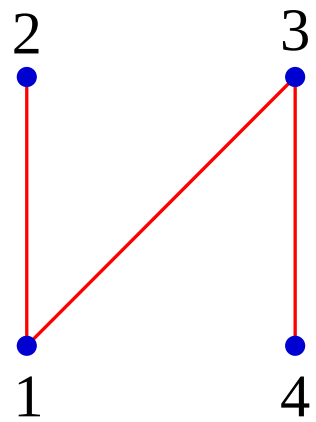
Problem 3. (a) Let $P(x) = A_n(x)/x$. Find a combinatorial interpretation for the coefficients of the expansion of $F_P(\boldsymbol{x})$ in terms of s_λ's or e_λ's, thereby showing they are nonnegative.

(b) Generalize to other polynomials $P(x)$.

Let P be a partial ordering of $1, \ldots, n$. Let

$$\mathcal{L}_P = \{w = w_1 \cdots w_n \in \mathfrak{S}_n : i \overset{P}{<} j \Rightarrow w^{-1}(i) < w^{-1}(j)\}.$$

$$W_P(x) = \sum_{w \in \mathcal{L}_P} x^{\mathrm{des}(w)}.$$



w	des(w)
1423	1
4123	1
1432	2
4132	2
1243	1

$$W_P(x) = 3x + 2x^2 : \quad \text{all zeros real!}$$

Poset Conjecture. For any poset P on $1, \ldots, n$, all zeros of $W_P(x)$ are real.

Let Q be a finite poset.

chain polynomial: $C_Q(x) = \sum_\sigma x^{\#\sigma}$,

where σ ranges over all chains of Q.

Special case (open). Let L be a finite distributive lattice. Then all zeros of $C_L(x)$ are real.

Also open: All zeros of $C_L(x)$ are real if L is a finite **modular** lattice.

Theorem (Gasharov, Skandera) *Suppose P has no induced $\mathbf{3}+\mathbf{1}$. Then every zero of $C_P(x)$ is real.*

Jack symmetric function:

$$\boldsymbol{J_\lambda = J_\lambda(x;\alpha)}$$

$$J_\lambda(\boldsymbol{x};1) = s_\lambda(\boldsymbol{x})$$

Example.

$$P(x) = \frac{A_5(x)}{x} = 1+26x+66x^2+26x^3+x^4$$

$$\begin{aligned} F_P \;=\; & 1 + 26J_1 + \left(66J_2 + \frac{4(169\alpha+136)}{\alpha+1}J_{11}\right) \\ & + \left(26J_3 + \frac{78(44\alpha+21)}{2\alpha+1}J_{21}\right. \\ & \left. + \frac{52(338\alpha^2+816\alpha+481)}{(\alpha+1)(\alpha+2)}J_{111}\right) + \cdots \end{aligned}$$