A Survey of Promotion and Evacuation

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M.I.T.

$$w = a_1 a_2 \cdots a_n \in \mathfrak{S}_n, \quad w \xrightarrow{\text{rsk}} (P, Q)$$

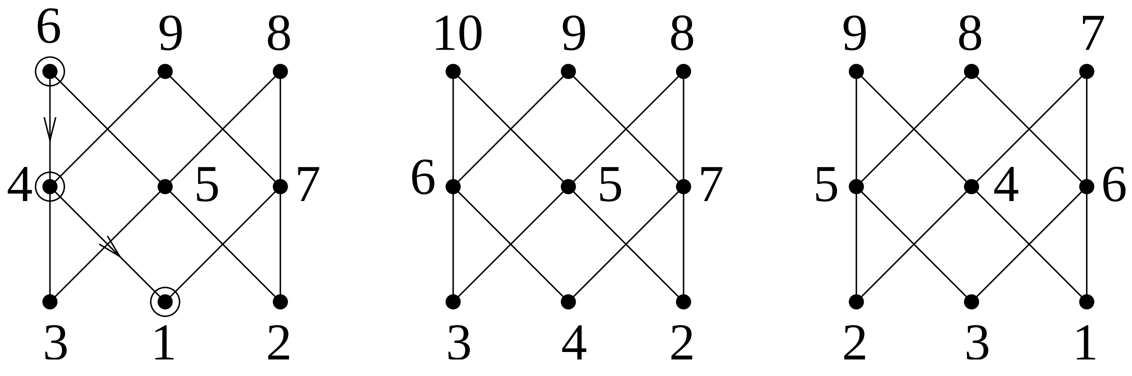
$$w^r := a_n \cdots a_2 a_1 \xrightarrow{\text{rsk}} (P^t, (Q\boldsymbol{\epsilon})^t)$$

Note. $Q\epsilon\epsilon = Q$

Schützenberger (1971): "direct" description of $Q\epsilon$

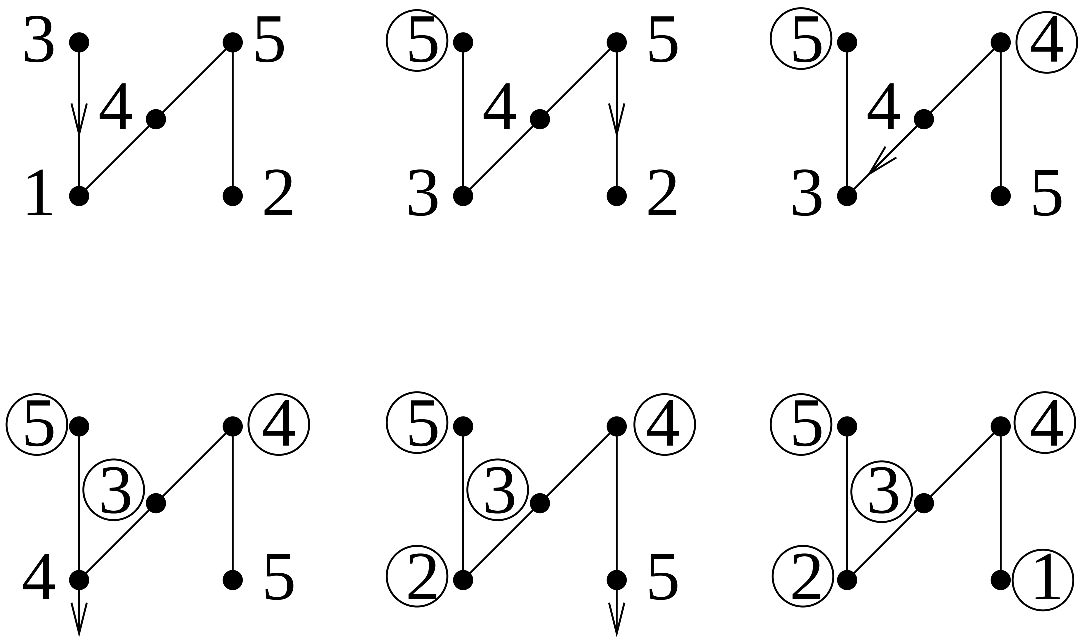
(1972): extended to any linear extension f of a finite poset P

The promotion operator ∂



f $\quad$ $f\partial$

The evacuation operator ϵ



∂ and ϵ are bijections on $\mathcal{L}(P)$, the set of linear extensions of P

Dual promotion $\boldsymbol{\partial^*}$: remove **largest** label and slide up, etc.

Clear: $\partial^{-1} = \partial^*$

Dual evacuation $\boldsymbol{\epsilon^*}$: evacuate from top, etc.

Theorem (Schützenberger).
(a) $\epsilon^2 = 1$

(b) $\partial^p = \epsilon\epsilon^*$, where $\boldsymbol{p} = \#P$

(c) $\partial\epsilon = \epsilon\partial^{-1}$

(d) omitted

Restatement. ϵ and ϵ^* generate a dihedral group D (possibly isomorphic to $\{1\}$, $\mathbb{Z}/2\mathbb{Z}$, or $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$). If not $\epsilon = \epsilon^* = 1$, then $\#D = 2m$, where $m = \mathrm{ord}(\partial^p)$.

$$\boldsymbol{M} = \text{monoid}, \quad \boldsymbol{\tau_1}, \ldots, \boldsymbol{\tau_{p-1}} \in M$$

$$\tau_i^2 = 1, \ 1 \leq i \leq p-1$$

$$\tau_i \tau_j = \tau_j \tau_i, \ \text{if } |j-i| > 1.$$

Define $\boldsymbol{\delta_j}, \boldsymbol{\delta_j^*}, \boldsymbol{\gamma_j}, \boldsymbol{\gamma_j^*} \in M$ by

$$\begin{aligned}
\delta_j &= \tau_1 \tau_2 \cdots \tau_j \\
\delta_j^* &= \tau_j \tau_{j-1} \cdots \tau_1 \ (= \delta_j^{-1}) \\
\gamma_j &= \delta_j \delta_{j-1} \cdots \delta_1 \\
\gamma_j^* &= \delta_j^* \delta_{j-1}^* \cdots \delta_1^*.
\end{aligned}$$

Lemma. *For* $1 \leq j \leq p-1$:

(a) $\gamma_j^2 = (\gamma_j^*)^2 = 1$

(b) $\delta_j^{j+1} = \gamma_j \gamma_j^*$

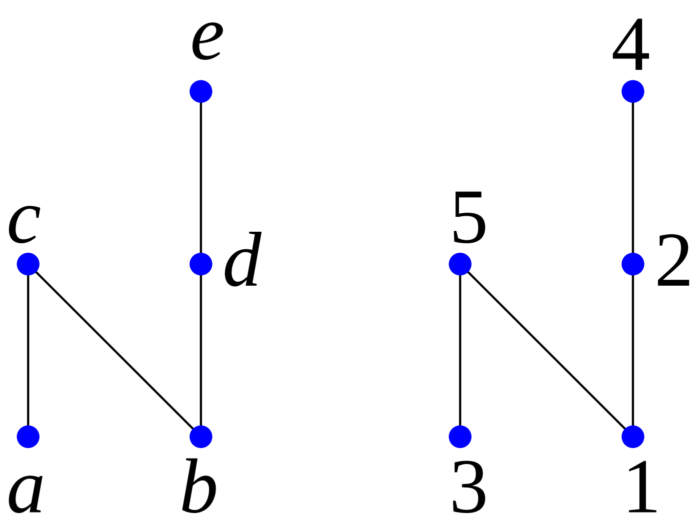
(c) $\delta_j \gamma_j = \gamma_j \delta_j^{-1}$.

Proof. Formal consequence of the relations. E.g., write ***i*** for τ_i. Then

$$\begin{aligned}
\gamma_3^2 &= (123121)^2 \\
&= 1231\underbrace{2\underbrace{11}2}3121 \\
&= 12\underbrace{31}_{13}3121 \\
&= 1\underbrace{2\underbrace{1\underbrace{33}1}2}1 \\
&= \text{id}.
\end{aligned}$$

Malvenuto-Reutenauer:

Regard a linear extension f of P as a **word** $t_1, t_2, \ldots, t_p$, i.e., a permutation of the elements of P.



$b\ d\ a\ e\ c$

For $1 \le i \le p-1$ define operators $\boldsymbol{\tau_i}\colon \mathcal{L}(P) \to \mathcal{L}(P)$ by

$$\tau_i(u_1u_2\cdots u_p)$$
$$= \begin{cases} u_1u_2\cdots u_p, & \text{if } u_i < u_{i+1} \text{ in } P \\ u_1u_2\cdots u_{i+1}u_i\cdots u_p, & \text{if } u_i \,\|\, u_{i+1}. \end{cases}$$

Clear: τ_i is a bijection

$$\tau_i^2 = 1$$
$$\tau_i\tau_j = \tau_j\tau_i, \ |j-i| > 1$$

Proposition.

$$\partial = \delta_{p-1} := \tau_1\tau_2\cdots\tau_{p-1}$$

(so $\gamma = \epsilon$)

Corollary (Schützenberger)

(a) $\epsilon^2 = 1$

(b) $\partial^p = \epsilon^*\epsilon$, where $\boldsymbol{p} = \#P$

(c) $\partial\epsilon = \epsilon\partial^{-1}$

Self-evacuating linear extensions

self-evacuating $f \in \mathcal{L}(P)$:

$$f = f\epsilon$$

order ideal $\boldsymbol{I} \subseteq P$:

$$t \in I,\ s < t \Rightarrow s \in I$$

dual P-domino tableau: chain

$$\emptyset = I_0 \subset I_1 \subset \cdots \subset I_r = P$$

of order ideals I_i such that

$$I_i - I_{i-1} \ = \ \text{2-element chain},\ 2 \leq i \leq r$$

$$I_1 \ = \ \text{1 or 2-element chain},$$

so $r = \lceil p/2 \rceil$.

Let P be a **natural partial order** on $[p]$, i.e.,

$$i \overset{P}{<} j \Rightarrow i \overset{\mathbb{Z}}{<} j.$$

For $f = t_1 \cdots t_p \in \mathcal{L}(P) \subseteq \mathfrak{S}_p$, define

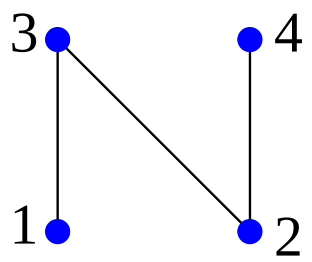
$$\textbf{comaj}(f) = \sum_{i\,:\,t_i > t_{i+1}} (p - i).$$

Theorem. The following are equal.

(1) $\sum_{f\in\mathcal{L}(P)}(-1)^{\mathrm{comaj}(f)}$.

(2) the number of dual P-domino tableaux

(3) the number of self-evacuating linear extensions of P



f	comaj(f)
1 2 3 4	0
2 1 3 4	3
1 2 4 3	1
2 1 4 3	4
2 4 1 3	2

$$(-1)^0+(-1)^3+(-1)^1+(-1)^4+(-1)^2=1$$

$$\text{dual } P\text{-domino tableau} : \emptyset \subset \{2,4\} \subset P$$

$$1243\epsilon = 1243$$

(1) $\sum_{f\in\mathcal{L}(P)}(-1)^{\mathrm{comaj}(f)}$

(2) # dual P-domino tableaux

(3) # self-evacuating $f \in \mathcal{L}(P)$

Idea of proof. (1)=(2) Simple involution argument (2005).

(2)=(3) Follows from: f is a dual domino linear extension if and only if

$$f\tau_1\cdot\tau_3\tau_2\tau_1\cdot\tau_5\tau_4\tau_3\tau_2\tau_1\cdots\tau_m\tau_{m-1}\cdots\tau_1$$

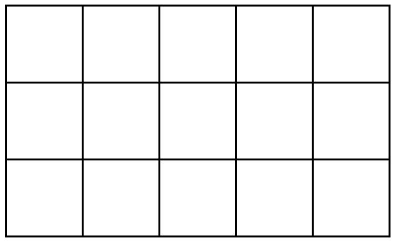
is self-evacuating, where $m = p-1$ if p is even, and $m = p-2$ if p is odd. Proved by an elementary formal argument.

Proved by Stembridge (1996) and Berenstein-Kirillov (2000) for SYT. Above argument follows Berenstein-Kirillov.

When is $D = \langle \epsilon, \epsilon^* \rangle$ “nice”?

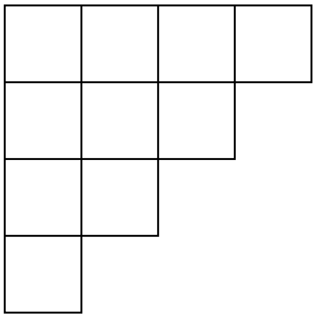
Recall: $\partial^p = \epsilon\epsilon^*$, where $p = \#P$

Schützenberger (1977), Edelman-Greene(1987), Haiman(1992):



$$f\partial^p = f$$

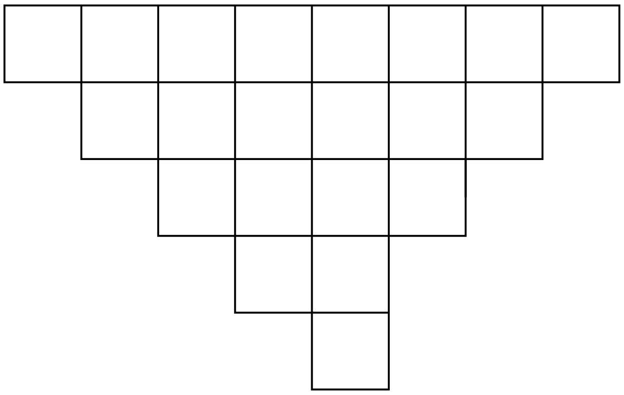
$$D \cong \mathbb{Z}/2\mathbb{Z}$$



$$f\partial^p \;=\; f^t \quad \text{(transpose)}$$

$$D \;\cong\; \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$

$$\begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 6 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array} \;\partial^6 \;=\; \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 6 \\ \cline{1-2} 5 \\ \cline{1-1} \end{array}$$



$$f\partial^p = f$$

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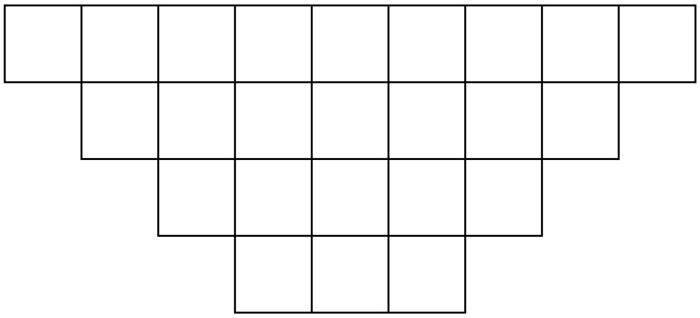


$$f\partial^p = f$$

$$D \cong \mathbb{Z}/2\mathbb{Z}$$

Cyclic sieving

Let $\boldsymbol{p} = mn$, $\boldsymbol{P} = \boldsymbol{m} \times \boldsymbol{n}$ ($m \times n$ rectangle).

$$f = \begin{array}{cccc} \mathbf{1} & 3 & \mathbf{4} & 8 \\ 2 & 5 & \mathbf{6} & \mathbf{11} \\ 7 & 9 & 10 & 12 \end{array}$$

$$\mathbf{maj}(f) = 1 + 4 + 6 + 11 = 22$$

$$\boldsymbol{F(q)} = \sum_{f \in \mathcal{L}(P)} q^{\mathrm{maj}(f)}$$

$$= \frac{q^*(1-q)(1-q^2)\cdots(1-q^p)}{\prod_{t \in P}(1-q^{h(t)})}.$$

$$\zeta = e^{2\pi i/p}$$

Recall for any $f \in \mathcal{L}(P)$: $f\partial^p = f$.

Theorem (Rhoades, 2007) For any $d \in \mathbb{Z}$,

$$\#\{f \in \mathcal{L}(P) \,:\, f = f\partial^d\} = F(\zeta^d).$$

Is there a simpler proof?

Generalizations

order ideal $\boldsymbol{I} \subseteq P$:

$$t \in I,\ s < t \Rightarrow s \in I$$

$\boldsymbol{J(P)}$: poset of order ideals of P, ordered by inclusion (= finite **distributive lattice**)

maximal chain of $J(P)$:

$$\mathfrak{m}: \emptyset = I_0 \subset I_1 \subset \cdots \subset I_p = P$$

corresponds to linear extension $t_1, \ldots, t_p$ *via* $t_i \in I_i - I_{i-1}$.

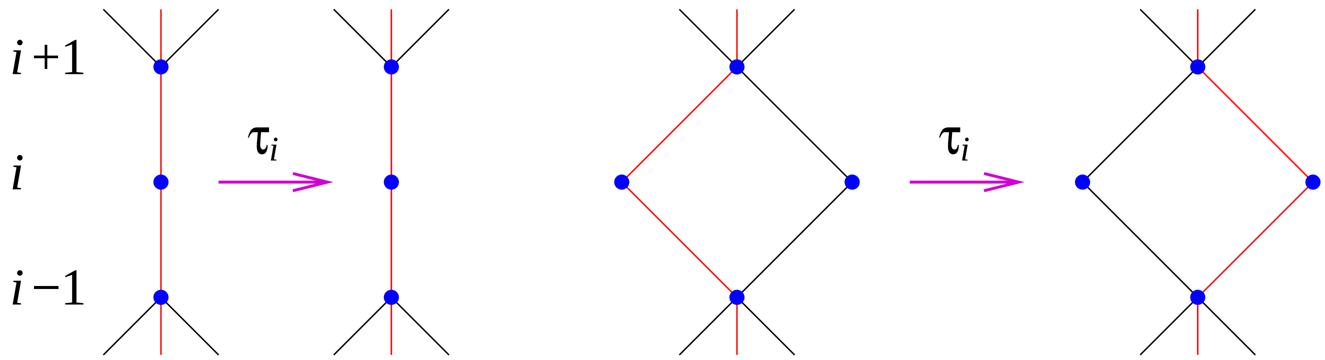
Transferred action of $\tau_i : \mathcal{L}(P) \longrightarrow \mathcal{L}(P)$ on set $\mathfrak{M}(J(P))$ of maximal chains of $J(P)$:

$$\mathfrak{m}\tau_i = \mathfrak{m},$$

if $[I_{i-1}, I_{i+1}] = \{t_{i-1}, t_i, t_{i+1}\}$.

$$\mathfrak{m}\tau_i = \mathfrak{m} \cup \{t_i'\} - \{t_i\},$$

if $[I_{i-1}, I_{i+1}] = \{t_{i-1}, t_i, t_i', t_{i+1}\}$.



$\boldsymbol{P}$: **any** graded poset such that

$$\#[s,t] = 3, 4 \text{ if } \ell(s,t) = 2$$

E.g., Eulerian poset (face lattices of convex polytopes, intervals in Bruhat order, etc.)

$\mathfrak{M}(P) = \{\text{maximal chains}\}$

Define

$$\boldsymbol{\tau_i} \colon \mathfrak{M}(P) \to \mathfrak{M}(P)$$

as above, viz., if $\mathfrak{m} : t_0 < t_1 < \cdots < t_m$ is a maximal chain and $1 \le i \le m-1$, then

$$\mathfrak{m}\tau_i = \mathfrak{m}, \text{ if } [t_{i-1}, t_{i+1}] = \{t_{i-1}, t_i, t_{i+1}\}.$$

$$\mathfrak{m}\tau_i = \mathfrak{m} - \{t_i\} \cup \{t_i'\}, \\ \text{if } [t_{i-1}, t_{i+1}] = \{t_{i-1}, t_i, t_i', t_{i+1}\}.$$

Define as before

$$\begin{aligned} \boldsymbol{\partial_j} &= \tau_1\tau_2\cdots\tau_j \\ \boldsymbol{\partial_j^*} &= \tau_j\tau_{j-1}\cdots\tau_1 \; (= \partial_j^{-1}) \\ \boldsymbol{\epsilon_j} &= \partial_j\partial_{j-1}\cdots\partial_1 \\ \boldsymbol{\epsilon_j^*} &= \partial_j^*\partial_{j-1}^*\cdots\partial_1^*. \end{aligned}$$

Since

$$\tau_i^2 = 1,\ 1 \le i \le p-1$$

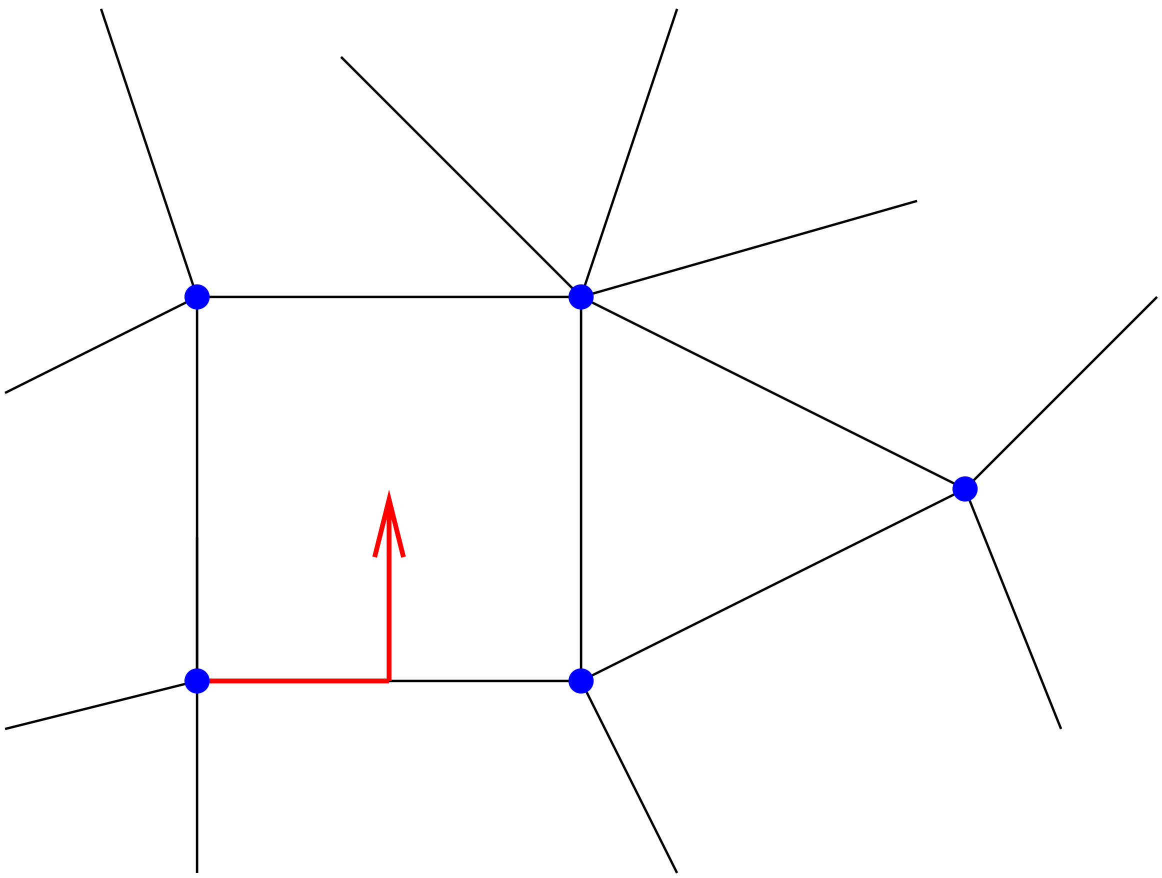
$$\tau_i \tau_j = \tau_j \tau_i, \text{ if } |j-i| > 1,$$

Schützenberger's results hold, i.e.,

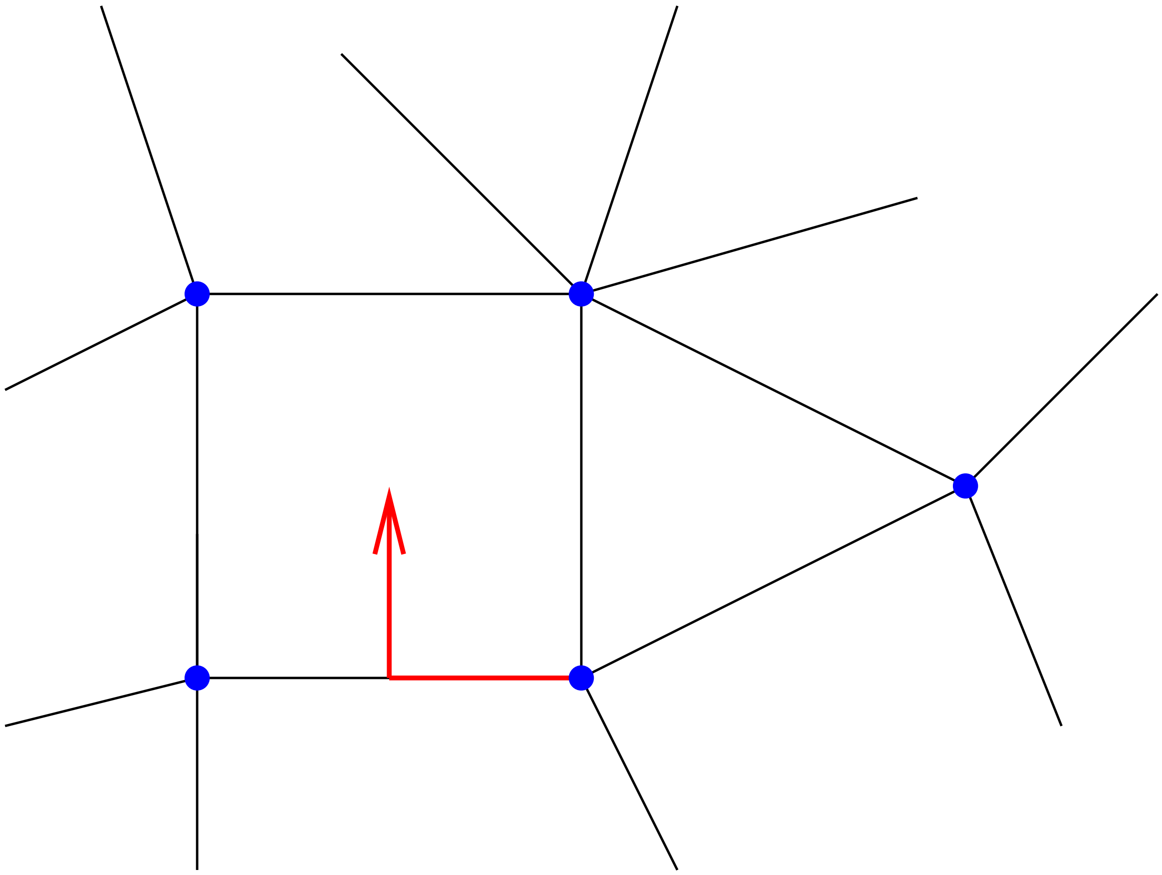
(a) $\epsilon^2 = 1$

(b) $\partial^p = \epsilon\epsilon^*$, where $\boldsymbol{p} = \ell(P)$

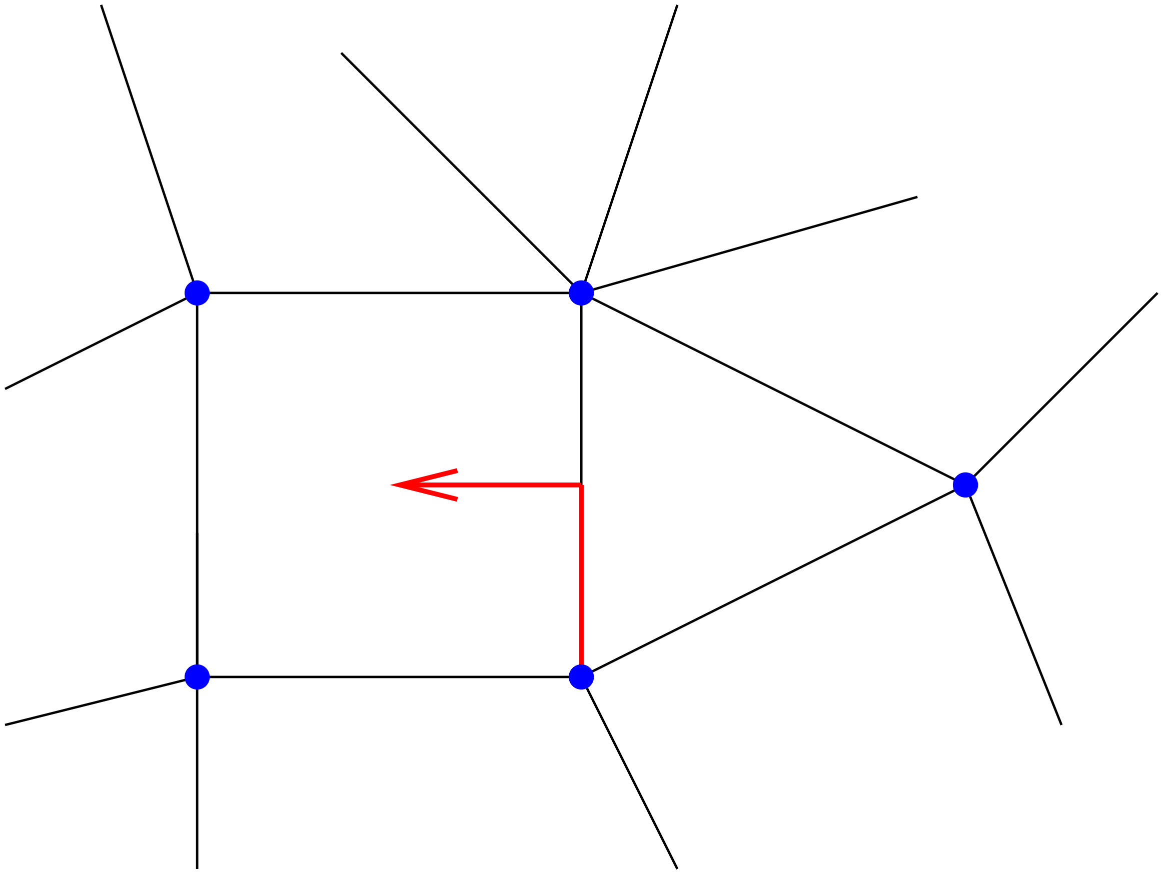
(c) $\partial\epsilon = \epsilon\partial^{-1}$



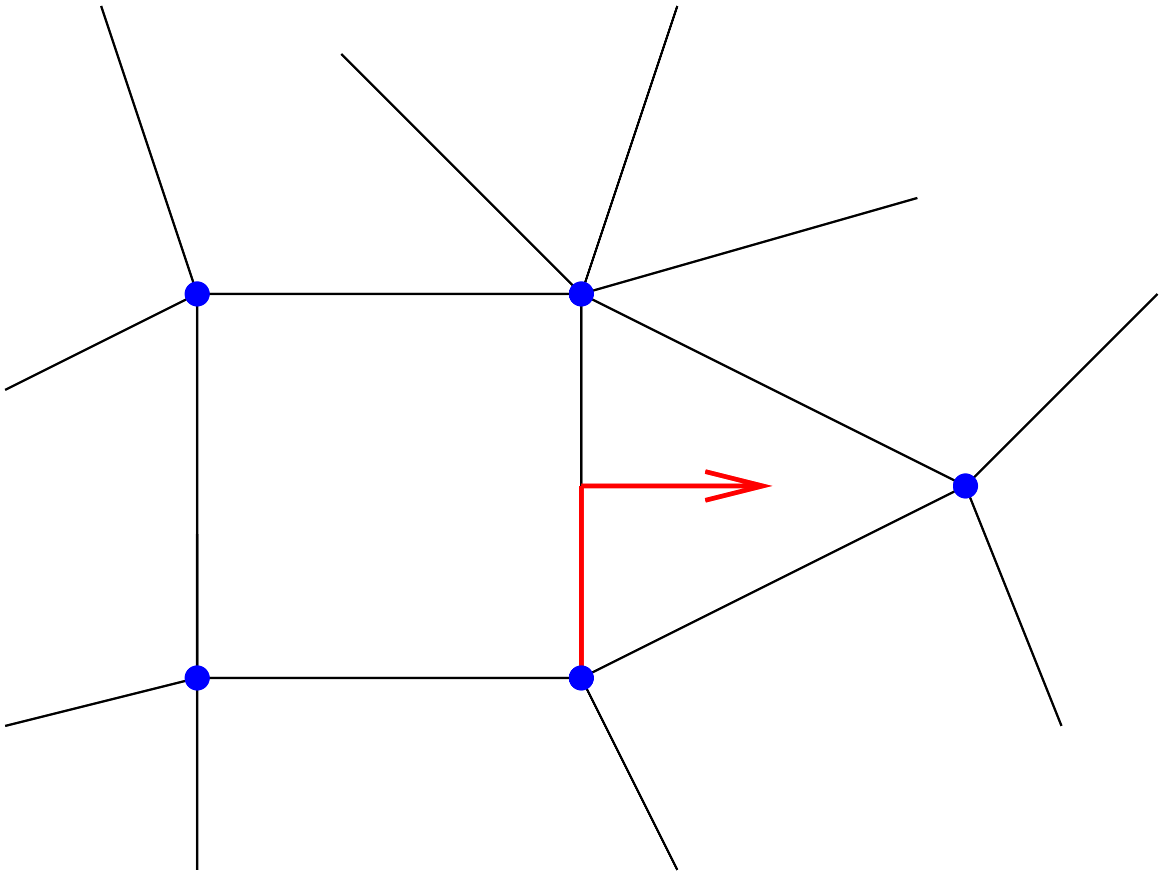
f



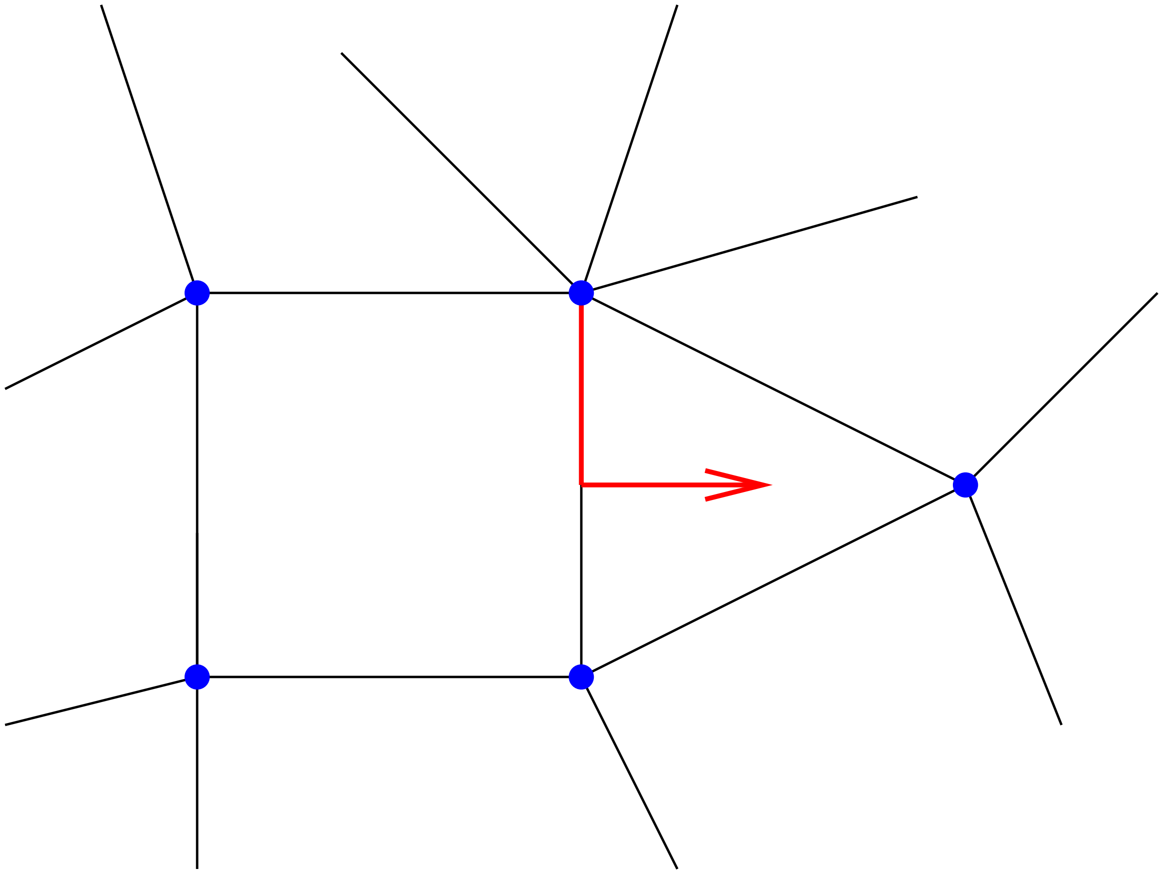
$$f\tau_1$$



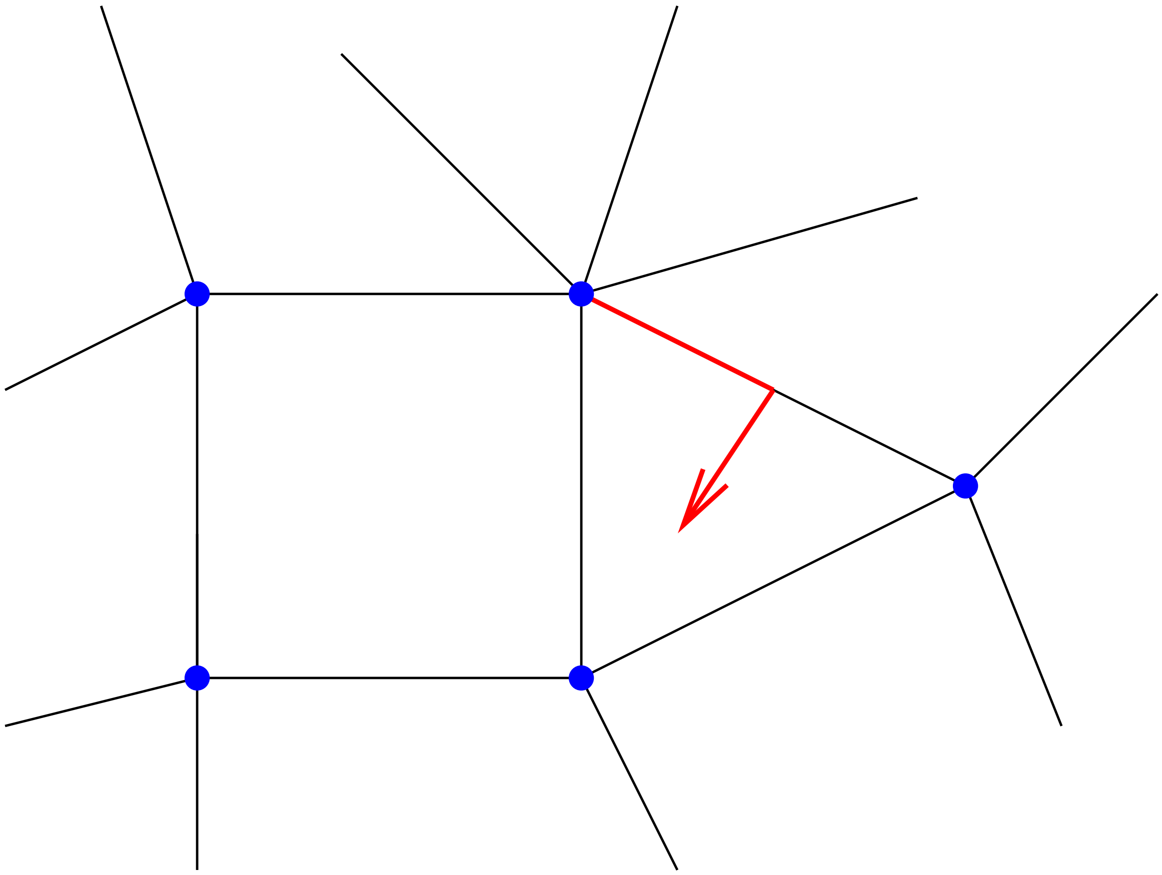
$$f\tau_1\tau_2$$



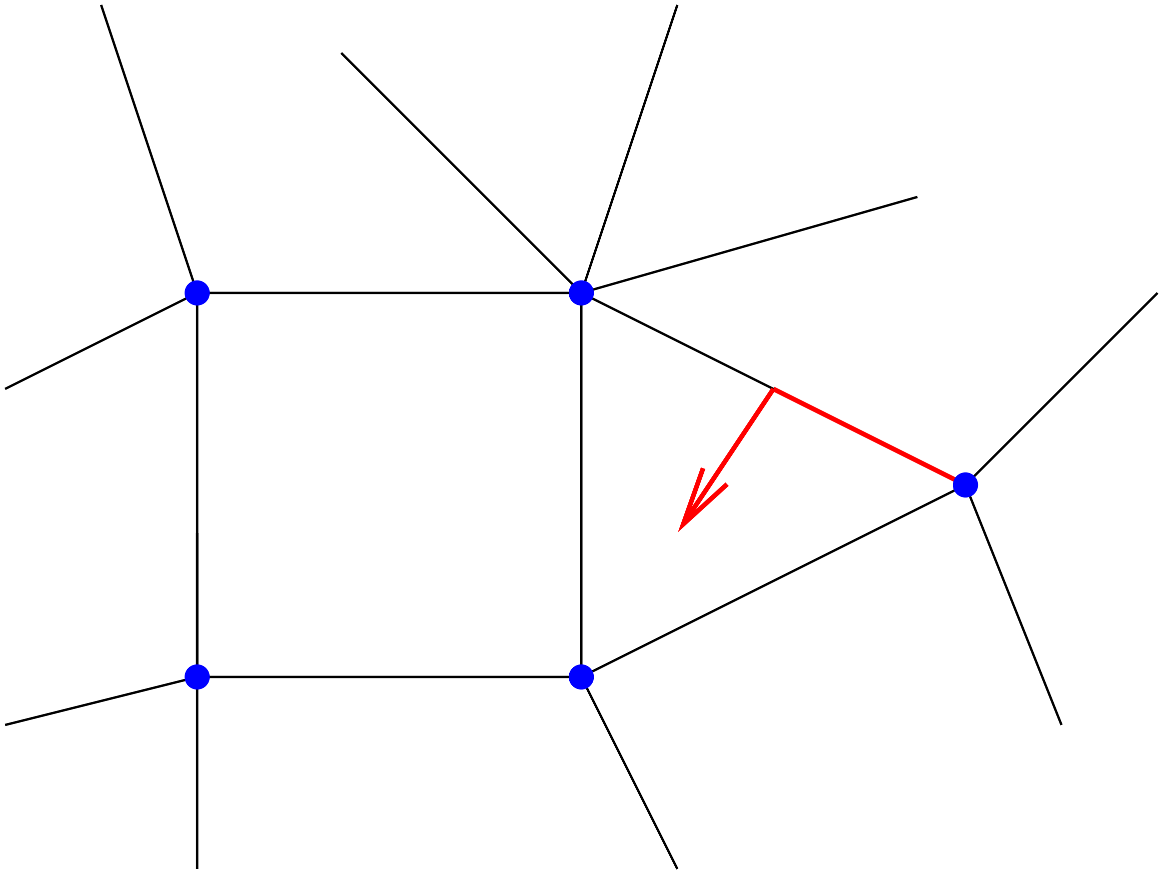
$$f\tau_1\tau_2\tau_3 = f\partial$$



$$f\tau_1\tau_2\tau_3\tau_1$$

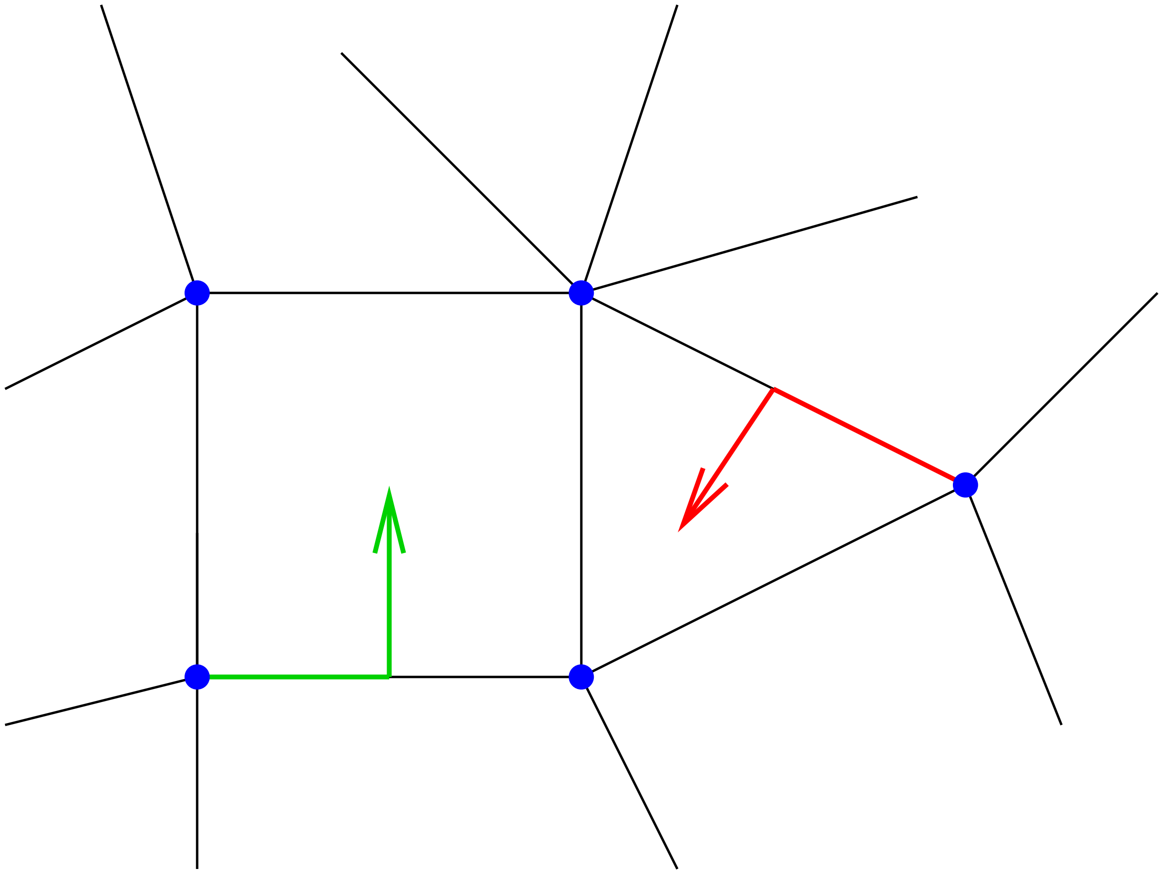


$$f\tau_1\tau_2\tau_3\tau_1\tau_2$$



$$f\tau_1\tau_2\tau_3\tau_1\tau_2\tau_1 = f\epsilon$$

$$\epsilon^2 = 1$$



$$f\tau_1\tau_2\tau_3 = \tau_1\tau_2\tau_1 = f\epsilon$$

$$\epsilon^2 = 1$$

For the face lattice of the n-cube, a maximal chain corresponds to a **signed permutation** $a_1a_2\cdots a_n$, e.g., $4\overline{25}3\overline{1}$. Then

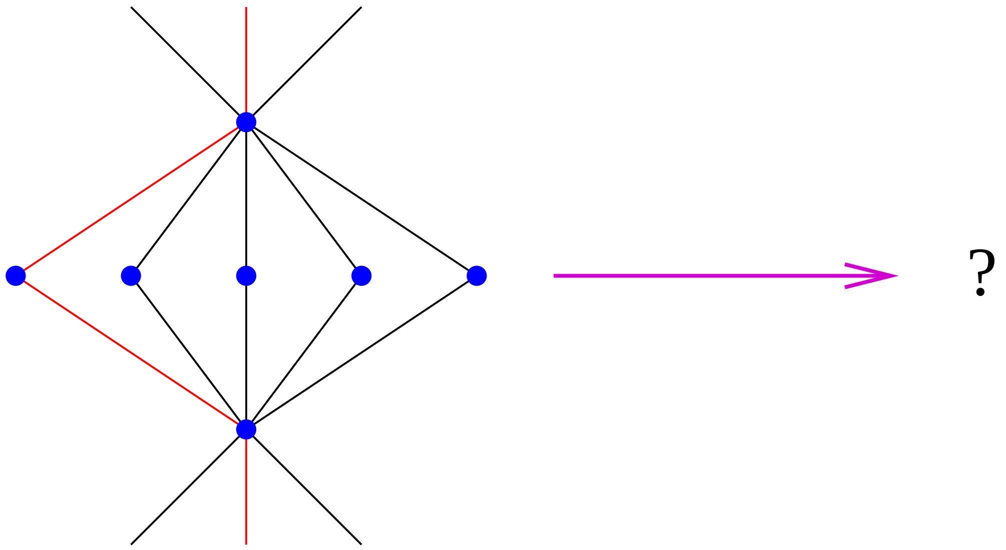
$$a_1a_2\cdots a_n\epsilon = \overline{a_n}\cdots\overline{21}$$

$$a_1a_2\cdots a_n\epsilon^* = a_{n-1}\cdots a_2a_1\overline{a_n}$$

$$\begin{aligned} a_1a_2\cdots a_n\epsilon\epsilon^* &= a_1a_2\cdots a_n\partial^{n+1} \\ &= \overline{a_2a_3\cdots a_{n-1}}a_1 \end{aligned}$$

$$\mathrm{ord}(\epsilon\epsilon^*) = \begin{cases} 2n, & n \text{ even} \\ n, & n \text{ odd.} \end{cases}$$

What about more general posets?



Need to work in $\mathbb{Q}\mathfrak{M}(P)$, the $\mathbb{Q}$-vector space with basis $\mathfrak{M}(P)$. Let

$$\mathfrak{m} : t_0 < t_1 < \cdots < t_p.$$

Want

$$\mathfrak{m}\tau_i = \alpha\mathfrak{m} + \beta \sum_{\substack{\mathfrak{m}' \neq \mathfrak{m} \\ \mathfrak{m} \cap \mathfrak{m}' = \mathfrak{m} - \{t_i\}}} \mathfrak{m}'.$$

Let

$$q + 1 = \#\{t : t_{i-1} < t < t_{i+1}\}.$$

$$\tau_i^2 = 1 \Rightarrow \beta = 0 \quad \text{(trivial)}$$
$$\text{or } \alpha = \pm\frac{q-1}{q+1}, \quad \beta = \pm\frac{2}{q+1}.$$

Special case. $P = \boldsymbol{B_p(q)}$, the lattice of subspaces of $\mathbb{F}_q^p$.

$q = 1, P = B_p$:

maximal chain $\leftrightarrow w = a_1 \cdots a_p \in \mathfrak{S}_p$

$$w\epsilon = a_p \cdots a_1.$$

For $B_p(q)$, equivalent to expanding

$$E_1E_2 \cdots E_{p-1}E_1E_2 \cdots E_{p-2} \cdots E_1E_2E_1$$
$$= \sum_{w \in \mathfrak{S}_p} c_w(q) T_w,$$

where

$$\boldsymbol{E_i} = \frac{1}{q+1}(q - 1 - 2T_i)$$

in the Hecke algebra $\mathcal{H}_p(q)$ of $\mathfrak{S}_p$.

Generators: of $\mathcal{H}_p(q)$: $T_1, T_2, \ldots, T_{p-1}$

Relations:

$$
\begin{aligned}
(T_i - 1)(T_i + q) &= 0 \\
T_i T_j &= T_j T_i, \quad |i-j| \geq 2 \\
T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1}
\end{aligned}
$$

In general, $c_w(q)$ is not "nice,", though many values are nice.

Theorem. $c_{\mathrm{id}}(q) = \left(\frac{1-q}{1+q}\right)^{\lfloor p/2 \rfloor}$