

# A Survey of Euler Numbers and Alternating Permutations

Richard P. Stanley  
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# Definition of Euler numbers

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**Raabe** (1851): introduced the term “Euler numbers”

# Alternating sequences

A sequence  $a_1, a_2, \dots, a_k$  of distinct integers is **alternating** if

$$a_1 > a_2 < a_3 > a_4 < \dots ,$$

and **reverse alternating** if

$$a_1 < a_2 > a_3 < a_4 > \dots .$$

# Euler numbers

$\mathfrak{S}_n$  : symmetric group of all permutations of  $1, 2, \dots, n$

$$\begin{aligned} A_n &= \#\{w \in \mathfrak{S}_n : w \text{ is alternating}\} \\ &= \#\{w \in \mathfrak{S}_n : w \text{ is reverse alternating}\} \end{aligned}$$

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**Note.** No connection with the alternating group (**even** permutations).

# André's theorem

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$$\Rightarrow A_n \sim 2 \left( \frac{2}{\pi} \right)^{n+1} n! \text{ as } n \rightarrow \infty$$

# Naive proof of André's theorem

Let  $y = \sum_{n \geq 0} A_n \frac{x^n}{n!}$ .

Show combinatorially that

$$2A_{n+1} = \sum_{k=0}^n \binom{n}{k} A_k A_{n-k}, \quad n \geq 1.$$

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(There exist more conceptual proofs.)

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$$\tan x = \sum_{n \geq 0} A_{2n+1} \frac{x^{2n+1}}{(2n+1)!}$$

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$\Rightarrow$  combinatorial trigonometry

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Ours is perhaps the worst.

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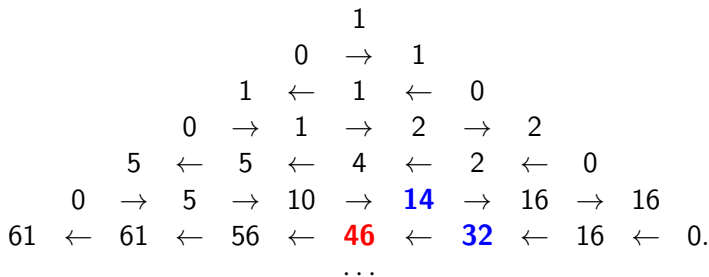
**boustrophedon**: an ancient method of writing in which the lines are inscribed alternately from right to left and from left to right.

From Greek *boustrophēdon* (*βουστροφηδόν*), turning like an ox while plowing: *bous*, ox + *strophē*, a turning (from *strephein*, to turn)

# The boustrophedon array

|    |   |    |   |    |   |    |   |    |   |     |
|----|---|----|---|----|---|----|---|----|---|-----|
|    |   |    |   |    | 1 |    |   |    |   |     |
|    |   |    |   | 0  | → | 1  |   |    |   |     |
|    |   |    | 1 | ←  | 1 | ←  | 0 |    |   |     |
|    |   | 0  | → | 1  | → | 2  | → | 2  |   |     |
|    | 5 | ←  | 5 | ←  | 4 | ←  | 2 | ←  | 0 |     |
| 0  | → | 5  | → | 10 | → | 14 | → | 16 | → | 16  |
| 61 | ← | 61 | ← | 56 | ← | 46 | ← | 32 | ← | 16  |
|    |   |    |   |    |   |    |   |    |   | 0.  |
|    |   |    |   |    |   |    |   |    |   | ... |

# The boustrophedon array



# Boustrophedon entries

- last term in row  $n$ :  $E_{n-1}$
- sum of terms in row  $n$ :  $E_n$
- $k$ th term in row  $n$ : number of alternating permutations in  $\mathfrak{S}_n$  with first term  $k$ , the **Entringer number**  $E_{n-1,k-1}$ .

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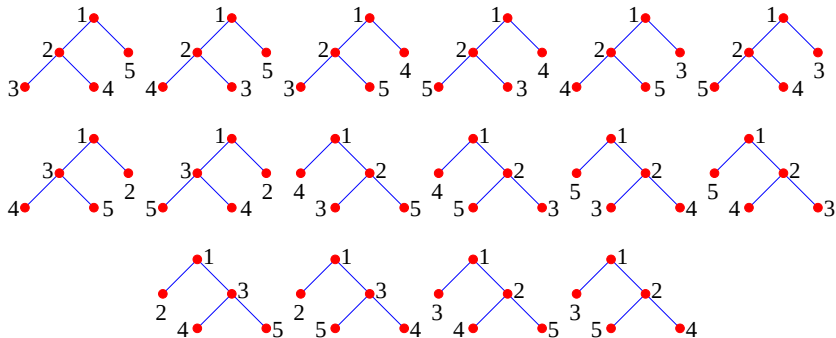
$$\sum_{m \geq 0} \sum_{n \geq 0} E_{m+n, [m, n]} \frac{x^m}{m!} \frac{y^n}{n!} = \frac{\cos x + \sin y}{\cos(x+y)},$$

$$[m, n] = \begin{cases} m, & m+n \text{ odd} \\ n, & m+n \text{ even.} \end{cases}$$

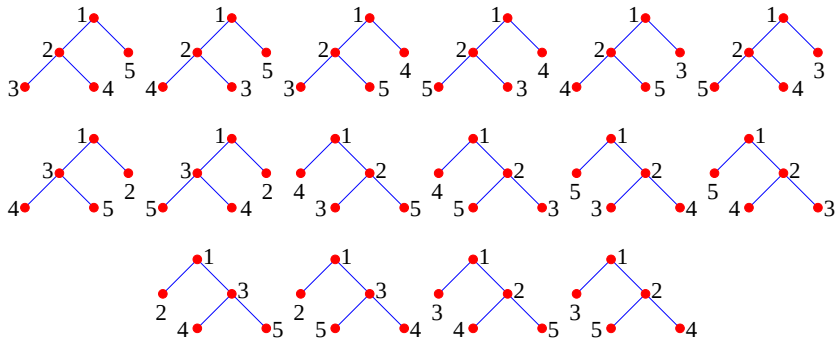
## Some occurrences of $E_n$

(1)  $E_{2n+1}$  is the number of complete increasing binary trees on the vertex set  $[2n+1] = \{1, 2, \dots, 2n+1\}$ .

## Five vertices ( $E_5 = 16$ )



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Slightly more complicated for  $E_{2n}$

## Proof for $2n + 1$

$b_1 b_2 \cdots b_m$  : sequence of distinct integers

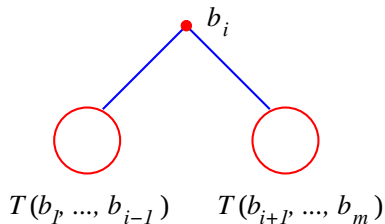
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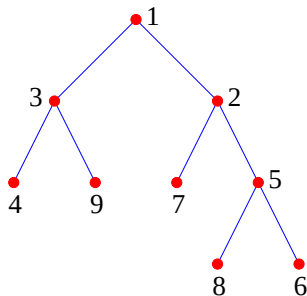
$$b_i = \min\{b_1, \dots, b_m\}$$

Define recursively a binary tree  $T(b_1, \dots, b_m)$  by



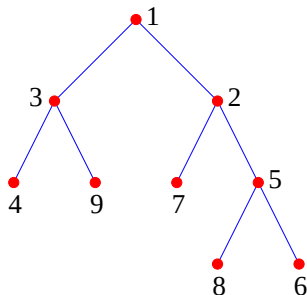
# Completion of proof

Example. 439172856



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Let  $w \in \mathfrak{S}_{2n+1}$ . Then  $T(w)$  is complete if and only if  $w$  is alternating, and the map  $w \mapsto T(w)$  gives the desired bijection.

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Proof omitted.

## Orbit representatives for $n = 5$

12-3-4-5      123-4-5      1234-5

12-3-4-5      123-4-5      123-45

12-3-4-5      12-34-5      125-34

12-3-4-5      12-34-5      12-345

12-3-4-5      12-34-5      1234-5

$$E_4 = 5$$

## Volume of a polytope

(3) Let  $\mathcal{E}_n$  be the convex polytope in  $\mathbb{R}^n$  defined by

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**Theorem.** *The volume of  $\mathcal{E}_n$  is  $E_n/n!$ .*

## Naive proof

$$\text{vol}(\mathcal{E}_n) = \int_{x_1=0}^1 \int_{x_2=0}^{1-x_1} \int_{x_3=0}^{1-x_2} \cdots \int_{x_n=0}^{1-x_{n-1}} dx_1 dx_2 \cdots dx_n$$

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$$\begin{aligned} f'_n(t) &= \int_{x_2=0}^{1-t} \int_{x_3=0}^{1-x_2} \cdots \int_{x_n=0}^{1-x_{n-1}} dx_2 dx_3 \cdots dx_n \\ &= f_{n-1}(1-t). \end{aligned}$$

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$$\mathbf{F(y)} = \sum_{n \geq 0} f_n(t) y^n$$

$$\Rightarrow \frac{\partial^2}{\partial t^2} F(y) = -y^2 F(y)$$

$$\Rightarrow F(y) = (\sec y)(\cos(t-1)y + \sin ty)$$

$$\Rightarrow F(y)|_{t=1} = \sec y + \tan y.$$

# Tridiagonal matrices

An  $n \times n$  matrix  $\mathbf{M} = (m_{ij})$  is **tridiagonal** if  $m_{ij} = 0$  whenever  $|i - j| \geq 2$ .

**doubly-stochastic**:  $m_{ij} \geq 0$ , row and column sums equal 1

$\mathcal{T}_n$ : set of  $n \times n$  tridiagonal doubly stochastic matrices

# Polytope structure of $\mathcal{T}_n$

**Easy fact:** the map

$$\begin{aligned}\mathcal{T}_n &\rightarrow \mathbb{R}^{n-1} \\ M &\mapsto (m_{12}, m_{23}, \dots, m_{n-1,n})\end{aligned}$$

is a (linear) bijection from  $\mathcal{T}$  to  $\mathcal{E}_{n-1}$ .

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**Application** (**Diaconis** et al.): random doubly stochastic tridiagonal matrices and random walks on  $\mathcal{T}_n$

## Prelude: distribution of $\text{is}(w)$

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Vershik-Kerov, Logan-Shepp:

$$\begin{aligned} E(n) &:= \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \text{is}(w) \\ &\sim 2\sqrt{n} \end{aligned}$$

# Limiting distribution of $\text{is}(w)$

## Baik-Deift-Johansson:

For fixed  $t \in \mathbb{R}$ ,

$$\lim_{n \rightarrow \infty} \text{Prob} \left( \frac{\text{is}_n(w) - 2\sqrt{n}}{n^{1/6}} \leq t \right) = F(t),$$

the **Tracy-Widom distribution**.

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How do  $E(n)$  and  $D(n)$  compare?

We expect  $D(n) > E(n)$  because it's easier for a subsequence to be alternating than increasing. But how much bigger?

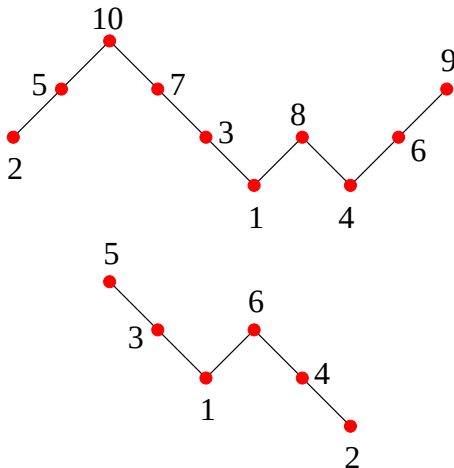
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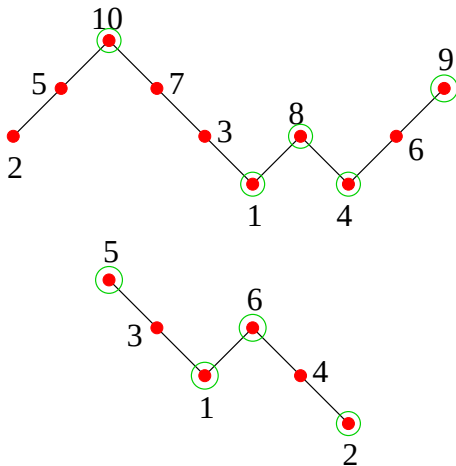
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**Proof.**



## Proof (cont.)

$$w = a_1 a_2 \cdots a_n$$

$$\text{Prob}(a_1 > a_2) = 1/2$$

$$\text{Prob}(a_i \text{ is a peak or valley}) = 2/3, \quad 2 \leq i \leq n-1$$

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$$\begin{aligned} \Rightarrow D(n) &= \frac{1}{2} + (n-2)\frac{2}{3} + 1 \\ &= \frac{4n+1}{6} \end{aligned}$$

## Variance of $\text{as}(w)$

$$V(n) = \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \left( \text{as}(w) - \frac{4n+1}{6} \right)^2, \quad n \geq 2$$

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similar results for higher moments

## A new distribution?

$$P(t) = \lim_{n \rightarrow \infty} \text{Prob}_{w \in \mathfrak{S}_n} \left( \frac{\text{as}(w) - 2n/3}{\sqrt{n}} \leq t \right)$$

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Stanley distribution?

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**Theorem** (Pemantle, Widom, (Wilf)).

$$\lim_{n \rightarrow \infty} \text{Prob}_{w \in \mathfrak{S}_n} \left( \frac{\text{as}(w) - 2n/3}{\sqrt{n}} \leq t \right) \\ = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{t\sqrt{45}/4} e^{-s^2} ds$$

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Allows proofs by induction.

# Umbral enumeration

**Umbral formula:** involves  $E^k$ , where  $E$  is an indeterminate (the **umbra**). Replace  $E^k$  with the Euler number  $E_k$ . (Technique from 19th century, modernized by **Rota** et al.)

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**Example.**

$$\begin{aligned}(1 + E^2)^3 &= 1 + 3E^2 + 3E^4 + E^6 \\ &= 1 + 3E_2 + 3E_4 + E_6 \\ &= 1 + 3 \cdot 1 + 3 \cdot 5 + 61 \\ &= 80\end{aligned}$$

## Another example

$$\begin{aligned}(1+t)^E &= 1 + Et + \binom{E}{2}t^2 + \binom{E}{3}t^3 + \dots \\&= 1 + Et + \frac{1}{2}E(E-1)t^2 + \dots \\&= 1 + Et + \frac{1}{2}(E^2 - E)t^2 + \dots \\&= 1 + E_1t + \frac{1}{2}(E_2 - E_1)t^2 + \dots \\&= 1 + t + \frac{1}{2}(1 - 1)t^2 + \dots \\&= 1 + t + O(t^3).\end{aligned}$$

## Alt. fixed-point free involutions

**fixed point free involution**  $w \in \mathfrak{S}_{2n}$ : all cycles of length two  
(number =  $1 \cdot 3 \cdot 5 \cdots (2n - 1)$ )

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$$n = 3 : \quad 214365 = (1, 2)(3, 4)(5, 6)$$

$$645231 = (1, 6)(2, 4)(3, 5)$$

$$f(3) = 2$$

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## Proof idea

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Now use known results on combinatorial properties of characters of  $\mathfrak{S}_n$ .

## Entry 16 of Ramanujan's second notebook

As  $x$  tends to  $0+$ ,

$$2 \sum_{n=0}^{\infty} (-1)^n \left( \frac{1-x}{1+x} \right)^{n(n+1)} \sim 1 + x + x^2 + 2x^3 + 5x^4 + 17x^5 + \dots$$

LHS is a mock theta function. This is an **analytic** (nonformal identity).

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Computed first 50 coefficients and noticed they were all positive integers. **Berndt** showed positivity (easy) and **Galway** (1997) integrality by a difficult argument.

## Connection with alternating permutations

**Recall:**  $f(n)$ : number of alternating fixed-point free involutions in  $\mathfrak{S}_{2n}$

$$\begin{aligned} F(x) &:= \sum_{n \geq 0} f(n)x^n \\ &= \left( \frac{1+x}{1-x} \right)^{(E^2+1)/4} \\ &= \left( \frac{1+x}{1-x} \right)^{1/4} \exp \left( \frac{E^2}{4} \log \frac{1+x}{1-x} \right) \\ &= \left( \frac{1+x}{1-x} \right)^{1/4} \sum_{n=0}^{\infty} \frac{E_{2n}}{2^{2n}n!} \log^n \left( \frac{1+x}{1-x} \right), \end{aligned}$$

the series of Berndt.

# Generalizations?

What can replace  $\frac{1+x}{1-x}$  in

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What about  $\frac{1+ax}{1-bx}$ ?

# The final slide

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