$G =$ finite group

$\widehat{G} =$ set of (ordinary, complex) irreducible characters of G

$\mathbb{Z}\widehat{G} =$ lattice of virtual characters of G, i.e., $\mathbb{Z}$-linear combinations of characters

$\mathbb{N}\widehat{G} =$ set of characters of G.

Theorem. (Frobenius-Schur) *For $w \in G$, let*

$$r_{2,G}(w) = \#\{u \in G : w = u^2\}.$$

If $\chi \in \widehat{G}$, then

$$\langle r_{2,G}, \chi \rangle = \begin{cases} 1, & \text{if } \chi \text{ is afforded by a real representation} \\ -1, & \text{if } \chi \text{ is real valued but not afforded by a real representation} \\ 0, & \text{if } \chi \text{ is not real valued.} \end{cases}$$

Corollary. (a) $r_{2,G} \in \mathbb{Z}\widehat{G}$

(b) $r_{2,G} \in \mathbb{N}\widehat{G}$ *if and only if every real character is afforded by a real representation.*

(c) $r_{2,G} = \sum_{\chi \in \widehat{G}} \chi$ *if and only if every character is afforded by a real representation.*

Let f be a class function on $\mathfrak{S}_n$. Define the *characteristic map* ch by

$$\mathrm{ch}(f) = \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} f(w) p_{\rho(w)},$$

where $\rho(w)$ is the cycle type of w. If χ^λ is the irreducible character of $\mathfrak{S}_n$ indexed by $\lambda \vdash n$, then $\mathrm{ch}(\chi^\lambda) = s_\lambda$. Note that

$$\mathrm{ch}(r_{2,\mathfrak{S}_n}) = \frac{1}{n!} \sum_{u \in \mathfrak{S}_n} p_{\rho(u^2)}.$$

Corollary. *We have*

$$\frac{1}{n!} \sum_{u \in \mathfrak{S}_n} p_{\rho(u^2)} = \sum_{\lambda \vdash n} s_\lambda.$$

Equivalently,

$$\sum_{n \geq 0} \frac{1}{n!} \sum_{u \in \mathfrak{S}_n} p_{\rho(u^2)}$$

$$= \prod_i (1 - x_i)^{-1} \cdot \prod_{i<j} (1 - x_i x_j)^{-1}.$$

TWO GENERAL THEOREMS

For any $k \geq 1$ and $w \in G$, let

$$r_{k,G}(w) = \#\{u \in G : w = u^k\}.$$

Theorem. $r_{k,G} \in \mathbb{Z}\widehat{G}$

Proof. M. Isaacs, *Character Theory of Finite Groups*, Problem 4.7. $\square$

$$
\begin{aligned}
F_r &= \text{free group on generators } x_1, \ldots, x_r \\
\gamma &= \gamma(x_1, \ldots x_r) \in F_r, \quad w \in G \\
f_{\gamma,G}(w) &= \#\{(u_1, \ldots, u_r) \in G^r : w = \gamma(u_1, \ldots, u_r)\}
\end{aligned}
$$

E.g., $f_{x_1^k,G} = r_{k,G}$.

Let K_G be the set of conjugacy classes of G. For $C \in K_G$ and $w \in G$, let

$$
\chi_C(w) = \begin{cases} 1, & w \in C \\ 0, & \text{otherwise.} \end{cases}
$$

Easy fact:

$$
\chi_C = \frac{|C|}{|G|} \sum_{\chi \in \widehat{G}} \bar{\chi}(C)\chi.
$$

Thus if every character of G is integer-valued (i.e., if $u, v \in G$ are conjugate whenever they generate the same cyclic subgroup), then

$$
\frac{|G|}{|C|}\chi_C \in \mathbb{Z}\widehat{G}.
$$

Theorem. *Let $r \geq 2$. For any G and any $\gamma \in F_r$ we have that $f_{\gamma,G}$ is a $\mathbb{Z}$-combination of the $\frac{|G|}{|C|}\chi_C$'s. In particular, if every character of G is integer-valued, then $f_{\gamma,G} \in \mathbb{Z}\widehat{G}$.*

Proof. To show:

$$\#\{(u_1, \ldots, u_r) \in G^n : \gamma(u_1, \ldots, u_r) \in C\} \equiv 0 \pmod{|G|}.$$

But this is exactly the special case $m = 1$ of Theorem 1 of L. Solomon, *Arch. Math. (Basel)* **20** (1969), 241–247. Solomon defines an equivalence relation $\sim$ on G^r such that (1) every class has size $|G|$, and (2) if $(u_1, \ldots, u_r) \sim (v_1, \ldots, v_r)$, then $\gamma(u_1, \ldots, u_r)$ and $\gamma(v_1, \ldots, v_r)$ are conjugate. $\square$

Recall $r_k \in \mathbb{Z}\widehat{G}$, where $r_k(w) = \#\{u \in G : u^k = w\}$. What about $G = \mathfrak{S}_n$?

Theorem. (T. Scharf, 1991) *For all k and n, r_k is a character of $\mathfrak{S}_n$.*

In fact: let

$$L_d = \mathrm{ch}\,\mathrm{Lie}_d = \frac{1}{d}\sum_{e|d} \mu(e) p_e^{d/e} = \mathrm{ch}\,\mathrm{ind}_{C_d}^{\mathfrak{S}_d} e^{2\pi i/d},$$

so L_d is Schur-positive. Let

$$\theta_{k,n} = \mathrm{ch}\, r_{k,\mathfrak{S}_n}.$$

Then

$$\sum_{n\geq 0} \theta_{k,n} = \sum_{n\geq 0} h_n \left[\sum_{d|k} L_d\right].$$

Open: If every character of G is rational, is r_k a character of G for all k?

$$\begin{aligned}\dim(r_{k,G}) &= \#\{u \in G : u^k = 1\} \\ &= \#\mathrm{Hom}(\mathbb{Z}/k\mathbb{Z}, G)\end{aligned}$$

$$\sum_{n \geq 0} \dim(r_{k,\mathfrak{S}_n}) \frac{x^n}{n!} = \exp \sum_{d|k} \frac{x^d}{d}$$

Theorem. (Dey 1965, Wohlfahrt 1977) *Let G be a finitely generated group, and let $j_d(G)$ denote the number of subgroups of G of index d. Then*

$$\sum_{n \geq 0} \#\mathrm{Hom}(G, \mathfrak{S}_n) \frac{x^n}{n!} = \exp\left(\sum_{d \geq 1} j_d(G) \frac{x^d}{d}\right).$$

Theorem. *Let*

$$u_d(G) = \frac{1}{d} \sum_{[G:H]=d} [N(H) : H],$$

where H *ranges over all subgroups of* G *of index* d, $N(H)$ *denotes the normalizer of* H *in* G, *and* $[N(H) : H]$ *is the index of* H *in* $N(H)$. *In particular, if every subgroup of* G *of index* d *is normal (e.g., if* G *is abelian) then* $u_d(G) = j_d(G)$. *Then*

$$\sum_{n\geq 0} \#\mathrm{Hom}(G \times \mathbb{Z}, \mathfrak{S}_n) \frac{x^n}{n!} = \prod_{d\geq 1} \left(1 - x^d\right)^{-u_d(G)}.$$

Corollary. *Let $c_m(n)$ be the number of commuting m-tuples $(u_1, \ldots, u_m) \in \mathfrak{S}_n^m$, i.e., $u_i u_j = u_j u_i$ for all i and j. Then*

$$\sum_{n \geq 0} c_m(n) \frac{x^n}{n!} = \prod_{d \geq 1} \left(1 - x^d\right)^{-j_d\left(\mathbb{Z}^{m-1}\right)}.$$

Note (Hermite, Eisenstein, Siegel, Weyl):

$$\sum_{d \geq 1} j_d(\mathbb{Z}^{m-1}) d^{-s} = \zeta(s)\zeta(s-1) \cdots \zeta(s-m+2)$$

Theorem. *Let G be a finite group, and let $f_1, \ldots, f_m$ be class functions on G. Define a class function F by*

$$F(w) = \sum_{u_1 \cdots u_m = w} f_1(u_1) \cdots f_m(u_m).$$

Let χ be an irreducible character of G. Then

$$\langle F, \chi \rangle = \left(\frac{|G|}{\chi(1)} \right)^{m-1} \langle f_1, \chi \rangle \cdots \langle f_m, \chi \rangle.$$

(F is a kind of "Hadamard product" of $f_1, \ldots, f_m$.)

Corollary. *Let* $(a_1, \ldots, a_m) \in \mathbb{Z}^m$*, and define a class function* $h = h_{a_1,\ldots,a_m}$ *on* $\mathfrak{S}_n$ *by*

$$h(w) = \#\{(u_1, \ldots, u_m) \in \mathfrak{S}_n^m : w = u_1^{a_1} \cdots u_m^{a_m}\}.$$

Then h *is a character of* $\mathfrak{S}_n$*.*

Note. Previous theorem for $G = \mathfrak{S}_n$ is equivalent to:

Let $\tilde{s}_\lambda = H_\lambda s_\lambda$, where $H_\lambda =$ product of hook-lengths of λ. Define a bilinear product $\Box$ on Λ by

$$\tilde{s}_\lambda \Box \tilde{s}_\mu = \delta_{\lambda\mu} \tilde{s}_\lambda.$$

Then for $\lambda, \mu \vdash n$,

$$p_\lambda \Box p_\mu = \frac{z_\lambda \, z_\mu}{n!} \sum_{\substack{\rho(u)=\lambda \\ \rho(v)=\mu}} p_{\rho(uv)}.$$

PERSIFICATION

Define a random walk $X = (w_0, w_1, \ldots)$ on $\mathfrak{S}_n$ as follows. Let $w_0 = 1$. Given w_i, choose $u \in \mathfrak{S}_n$ uniformly, and let

$$w_{i+1} = w_i\, u^2.$$

Theorem. (a) *The eigenvalues of this Markov chain are the numbers* $1/f^\lambda$ *with multiplicity* $\left(f^\lambda\right)^2$.

(b) *For fixed* m *we have*

$$\| X^m - \mathcal{U}_{\mathfrak{S}_n} \| = O(n^{-(m-1)}).$$

In particular, X^m *is "nearly uniform" on* $\mathfrak{S}_n$ *for* $m = 2$.

Open: What happens when u^2 is replaced by $\gamma(u_1, \ldots, u_r)$ for $\gamma \in F_r$?

COMMUTATORS

Let

$$f(w) = \#\{(u,v) \in G \times G \,:\, w = uvu^{-1}v^{-1}\}.$$

Theorem. *For all $\chi \in \widehat{G}$, we have*

$$\langle f, \chi \rangle = |G|/\chi(1).$$

Proof (hint):

$$\begin{aligned} uvu^{-1}v^{-1} &= u\left(vu^{-1}v^{-1}\right) \\ &= u \cdot (\text{conjugate of } u^{-1}). \ \square \end{aligned}$$

Open: Find a "natural" G-module affording the character f.

Recall: Let $\lambda = (4,3,3)$.

$$\begin{array}{cccc} 6 & 5 & 4 & 1 \\ 4 & 3 & 2 & \\ 3 & 2 & 1 & \end{array} \qquad \begin{array}{cccc} 0 & 1 & 2 & 3 \\ -1 & 0 & 1 & \\ -2 & -1 & 0 & \end{array} .$$

hook-lengths contents

Corollary. *We have*

$$\frac{1}{n!}\sum_{u,v\in\mathfrak{S}_n} p_{\rho(uvu^{-1}v^{-1})} = \sum_{\lambda\vdash n} H_\lambda s_\lambda,$$

where H_λ is the product of the hook-lengths of λ.

Recall: Let $\kappa(w)$ = number of cycles of w. Then

$$\begin{aligned} p_{\rho(w)}(1^q) &= q^{\kappa(w)} \\ s_\lambda(1^q) &= H_\lambda^{-1}\prod_{t\in\lambda}(q+c(t)). \end{aligned}$$

Corollary. *We have*

$$\frac{1}{n!}\sum_{u,v\in\mathfrak{S}_n} q^{\kappa(uvu^{-1}v^{-1})} = \sum_{\lambda\vdash n}\prod_{t\in\lambda}(q+c(t)).$$

Corollary. *If u, v are chosen at random (uniformly, independently) from $\mathfrak{S}_n$, then the expected number E_n of cycles of $uvu^{-1}v^{-1}$ is*

$$E_n = H_n + \frac{1}{n!}\left[\sum_{\substack{1\leq i\leq n\\ i \text{ odd}}} \frac{i!\,(n-i)!}{n-i+1} + \frac{(-1)^n}{2}\sum_{i=0}^{\lfloor (n-1)/2\rfloor} i!^2\,(n-1-2i)!\right],$$

where

$$\begin{aligned} H_n &= 1+\frac{1}{2}+\cdots+\frac{1}{n} \\ &= \text{expected number of cycles of } w\in\mathfrak{S}_n. \end{aligned}$$

Proof (sketch). We have

$$E_n = \frac{1}{n!}\frac{d}{dq}\sum_{\lambda\vdash n}\prod_{t\in\lambda}(q+c(t))\Bigg|_{q=1}.$$

Three cases: (i) No content of λ is equal to -1. Then $\lambda = (n)$ and the contribution of λ to E_n is H_n.

(ii) λ has exactly one content equal to -1. Then λ has the form $\langle a, b, 1^k\rangle$, where $a \geq b > 0$, $k \geq 0$, and $a+b+k=n$. In this case the contribution of λ to E_n is

$$\frac{1}{n!}\prod_{\substack{t\in\lambda\\ t\neq(2,1)}}(1+c(t)) = (-1)^k a!\,(b-1)!\,k!.$$

(iii) λ has more than one content equal to -1. Then the contribution of λ to E_n is 0.

Sum (ii) over all a, b, k and simplify. $\square$

Corollary. *If u, v are chosen at random (uniformly, independently) from $\mathfrak{S}_n$, then the expected number e_{nj} of j-cycles of $uvu^{-1}v^{-1}$ is given by*

$$e_{nj} = \frac{1}{j}\left[1 + \frac{1}{\binom{n}{j}} \sum_{i=0}^{j-1}{}' \frac{(-1)^i}{\binom{j-1}{i}} \frac{n-j+i+1}{n-2j+i+1}\right],$$

where $\sum'$ indicates that we are to omit the term $i = 2j - n - 1$ when $2j > n$.

Proof (sketch). Let Γ_j be the functional on symmetric functions defined by

$$\Gamma_j(f) = \frac{\partial}{\partial p_j} f \bigg|_{p_i=1},$$

where $g|_{p_i=1}$ indicates that we are to expand g as a polynomial in the p_i's and then set each $p_i = 1$. Then

$$e_{nj} = \frac{1}{n!} \Gamma_j \left(\sum_{\lambda \vdash n} H_\lambda s_\lambda \right).$$

One shows

$$\Gamma_j(s_\lambda) = \begin{cases} \frac{1}{j}(-1)^{\mathrm{ht}(B)}, & \text{if } \lambda/(n-j) \text{ is a border strip } B \\ 0, & \text{otherwise,} \end{cases}$$

etc. $\square$

A BASIC OPEN QUESTION

Let $\gamma \in F_r$. Recall

$$f_{\gamma,G}(w) = \#\{(u_1, \ldots, u_r) \in G^r : \gamma(u_1, \ldots, u_r) = w\}.$$

- For what γ is $f_{\gamma,\mathfrak{S}_n}$ a character of $\mathfrak{S}_n$ for all $n \geq 1$?

- For what γ is $f_{\gamma,G}$ a character of G for all finite groups G?

Random observations:

- $f_{xyx^2y,G}(w) = |G|$.

  Proof. Let $a = uvu$ and $b = uv$. Then $u = b^{-1}a$ and $v = a^{-1}b^2$, so as (u, v) ranges over $G \times G$ so does (a, b). But $uvu^2v = ab$, which is clearly equidistributed over G. Thus we get

  $$\#\{(u, v) \in G \times G \, : \, w = uvu^2v\} = |G|. \ \square$$

  Key point: The homomorphism $\varphi : F_2 \to F_2$ defined by $\varphi(x) = x^{-1}y$ and $\varphi(y) = x^{-1}y^2$ is an *automorphism* of F_2. All automorphisms of F_r have been classified, and there are no "surprises."

- Let $\gamma = x_1^6 x_2^5 x_3^6 x_4^3 x_5^5 x_6^3 x_7^6 x_8^6$. Then $f_{\gamma,G}$ is a character for all G.

  Proof. Let $\chi \in \widehat{G}$. Then

  $$\langle f, \chi \rangle = \left(\frac{|G|}{\chi(1)} \right)^7 \langle r_3, \chi \rangle^2 \cdot \langle r_5, \chi \rangle^2 \cdot \langle r_6, \chi \rangle^4 \in \mathbb{N}. \ \square$$

- Let $\gamma = xy^k xy^{-k}$. Then $f_{\gamma,\mathfrak{S}_n} \in \mathbb{N}\widehat{\mathfrak{S}}_n$.

  Proof. Let $a = xy^k$ and $b = y^{-1}$. This is invertible, and $\gamma = a^2 b^{2k}$. But $f_{a^2b^{2k},\mathfrak{S}_n}$ is a character. $\square$

- Let $\gamma = xy^k x^{-1} y^k$. Then $f_{\gamma,\mathfrak{S}_n} \in \mathbb{N}\widehat{\mathfrak{S}}_n$.

  Proof (hint). Show that for any $\chi \in \widehat{G}$,

$$\langle f_{\gamma,G}, \chi \rangle = \frac{|G|}{\chi(1)} \langle r_k, \chi\bar{\chi} \rangle.$$

  But r_k is a character of $\mathfrak{S}_n$. $\square$

- Suppose that every character of G is real (i.e., every element of G is conjugate to its inverse). Let $k(G)$ be the number of conjugacy classes of G. Then

  $$s(G) := \#\{(u,v) \in G{\times}G \,:\, u^2 = v^2\} = k(G){\cdot}|G|.$$

  Proof. Let $a = u$ and $b = uv^{-1}$ (invertible). Then $u^2v^{-2} = (aba^{-1})b$. But $\{(aba^{-1})b\} = \{(ab^{-1}a^{-1})b\} = \{b^{-1}a^{-1}ba\}$. Let $k(a)$ be the number of conjugates of a. Then

  $$\begin{aligned} s(G) &= \#\{(a,b) \in G \times G \,:\, ab = ba\} \\ &= \sum_{a \in G} \#C(a) \\ &= \sum_{a \in G} \frac{|G|}{k(a)} \\ &= k(G) \cdot |G|. \ \square \end{aligned}$$

- If γ is any of

$$x^2\, y^2\, x^2\, y^2$$

$$x^2\, y^3\, x^2\, y^{-3}$$

$$x^2\, y^2\, x^2\, y^3,$$

then we don't know whether $f_{\gamma,\mathfrak{S}_n}$ is a character for all n. (The case $x^2\, y^2\, x^2\, y^2$ has been checked for $n \leq 16$, and the other two for $n \leq 7$.)

- If γ is any of

$$x\, y^{-1}\, x^2\, y$$

$$x^2\, y^3\, x^{-2}\, y^{-3}$$

$$x^2\, y^3\, x^5\, y^4,$$

then for some n, $f_{\gamma,\mathfrak{S}_n}$ is *not* a character.

Reference:

http://www-math.mit.edu/~rstan/ec/ec.html