Magic Squares and Syzygies

Richard P. Stanley

Traditional magic squares

Many elegant, ingenious, and beautiful constructions, but no general theory involving all of them.

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$$n = 3, \quad r = 7$$

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$$n = 3, \ \ r = 7 \qquad n = 3, \ \ r = 6$$

A basic question

How many $n \times n$ magic squares have line sum r?

Call this number $\boldsymbol{H_n(r)}$.

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The cases $n = 1, n = 2$

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What about $n = 2$? Let $0 \le i \le r$.

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$$\begin{bmatrix} \boldsymbol{i} & r - i \\ r - i & i \end{bmatrix}$$

Hence $H_2(r) = \boldsymbol{r + 1}$ ($r + 1$ choices for $0 \le i \le r$).

The case $r = 1$

If every row and column of an $n \times n$ magic square M sums to 1, then M has one 1 in every row and column and 0's elsewhere, i.e., M is a **permutation matrix**.

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Then $n - 1$ choices for 1 in the second row.

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Thus $H_n(1) = n(n-1)(n-2)\cdots 1 = \boldsymbol{n!}$.

Binomial coefficients

Let $0 \le k \le n$. Define

$$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}.$$

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Examples. $\binom{n}{2} = \frac{n(n-1)}{2} = \frac{1}{2}(n^2 - n)$

$$\binom{n+2}{3} = \frac{(n+2)(n+1)n}{6} = \frac{1}{6}(n^3 + 3n^2 + 2n).$$

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In general, $\binom{n+j}{k}$ is a **polynomial** in n, degree k.

Let's get serious

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$$\begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

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If all these sums were different, then from **Combinatorics 101** we would have

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However, the sums are **not** all different!

A relation

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Syzygies

Such a relation is called a **syzygy**.

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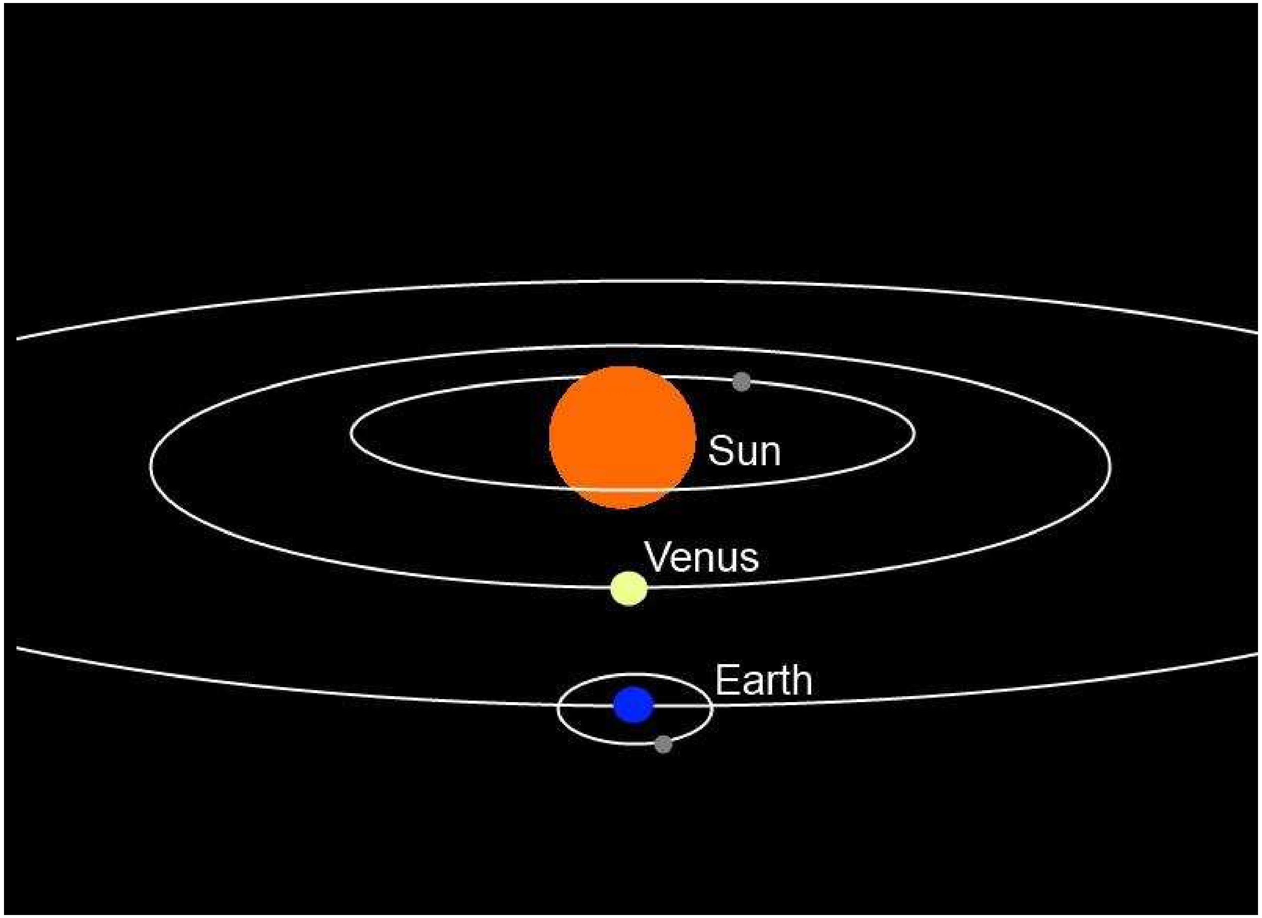
from **suzugos** ($\sigma\acute{\upsilon}\zeta\upsilon\gamma o\zeta$), “yoked together”

Astronomical significance



Correction term

The relation (syzygy) between the six 3×3 permutation matrices means that our first approximation $H_3(r) = \binom{r+5}{5}$ is an overcount. Get a “second approximation”

$$
\begin{aligned}
H_3(r) &= \binom{r+5}{5} - \binom{r+2}{5} \\
&= \frac{1}{8}(r^4 + 6r^3 + 15r^2 + 18r + 8) \\
&= \frac{1}{8}(r+1)(r+2)(r^2+3r+4).
\end{aligned}
$$

No further corrections!

No further syzygies, so this is correct!

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First proved by **Percy Alexander MacMahon** (1854–1929) c. 1916, as part of his **syzygetic method**. First real result on $H_n(r)$.

Percy Alexander MacMahon

Anand-Dumir-Gupta (1966)

Conjecture (1966). For any n, $H_n(r)$ is a polynomial in r of degree $(n-1)^2$. Moreover,

$$H_n(-1) = H_n(-2) = \cdots = H_n(-n+1) = 0$$

$$H_n(r) = (-1)^{n-1} H_n(-n-r).$$

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Example. $H_3(r) = \frac{1}{8}(r+1)(r+2)(r^2+3r+4)$

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Now, however, there are **178** “independent” syzygies with varying numbers of terms.

Not the end of the story · · ·

These **178** syzygies have relations among them, called **second-order syzygies**.

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Then **7416** third-order, **16440** fourth order, **25144**, **35562**, **42204**, **35562**, **25144**, **16440**, **7416**, **1837**, **178**, and finally ending in **one** 14th order syzygy!

Generalities

To define syzygies rigorously requires **commutative algebra**.

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For $n \times n$ magic squares, guarantees that the syzygy process will come to an end in at most $n!$ steps.

This implies that $H_n(r)$ is indeed a polynomial in r, at least for r sufficiently large. Extra tweaking needed for all r.

More refined results

More sophisticated algebra (Cohen-Macaulay rings, Auslander-Buchsbaum theorem): the chain of syzygies ends in exactly $n! - n^2 + 2n - 2$ steps.

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Number of syzygies at each step for $n = 4$:

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Note symmetry! (Sequence is palindromic.)

Symmetry

Key reason for symmetry: let ***J*** be the $n \times n$ all 1's matrix. Then M is an $n \times n$ magic square with row and column sums r if and only if $M + J$ is an $n \times n$ magic square with **positive** entries and row and column sums $r + n$.

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Symmetry implies the remainder of the Anand-Dumir-Gupta conjecture:

$$H_n(-1) = H_n(-2) = \cdots = H_n(-n+1) = 0$$

$$H_n(r) = (-1)^{n-1} H_n(-n-r).$$

5×5

5×5

over 5.7×10^{34} syzygies!

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Mankind may never know the exact number.

Another approach

Every 3×3 magic square has exactly one of the forms:

$$\begin{bmatrix} a+e & b+d & c \\ c+d & a & b+e \\ b & c+e & a+d \end{bmatrix}, \begin{bmatrix} a & b+d & c+e+1 \\ c+d & a+e+1 & b \\ b+e+1 & c & a+d \end{bmatrix},$$

$$\begin{bmatrix} a+d+1 & b & c+e+1 \\ c & a+e+1 & b+d+1 \\ b+e+1 & c+d+1 & a \end{bmatrix},$$

where $a, b, c, d, e \geq 0$.

Further formulas

This gives:

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Regard an $n \times n$ magic square as lying in the n^2-dimensional space $\mathbb{R}^{n^2}$.

4×4

For 4×4 magic squares we get

$$\mathbf{1}\binom{r+9}{9}+\mathbf{14}\binom{r+8}{9}+\mathbf{87}\binom{r+7}{9}+\mathbf{148}\binom{r+6}{9}$$

$$+\mathbf{87}\binom{r+5}{9}+\mathbf{14}\binom{r+4}{9}+\mathbf{1}\binom{r+3}{9}$$

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Note symmetry of $(1, 14, 87, 148, 87, 14, 1)$.
Equivalent to previous symmetry phenomena.

Computation

Symmetry property useful for computation.

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Example. $H_4(r)$ is a polynomial of degree 9. Ten values are needed to compute $H_4(r)$, say $H_4(0) = 1, H_4(1) = 24, H_4(2), \ldots, H_4(9)$.

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Example. $H_4(r)$ is a polynomial of degree 9. Ten values are needed to compute $H_4(r)$, say $H_4(0) = 1, H_4(1) = 24, H_4(2), \ldots, H_4(9)$.

But $H_4(-1) = H_4(-2) = H_4(-3) = 0$, $H_4(0) = -H_4(-4), H_4(1) = -H_4(-5), H_4(2) = -H_4(-6), H_4(3) = -H_4(-7)$.

Thus only need to compute $H_4(0) = 1, H_4(1) = 24, H_4(2), H_4(3)$.

Unimodality

$$(1, 14, 87, 148, 87, 14, 1)$$

Note $1 \le 14 \le 87 \le 148$ (**unimodality**).

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True for any n.

Very deep proof: toric varieties, intersection homology, hard Lefschetz theorem, $\ldots$.

Zeros of $H_n(r)$

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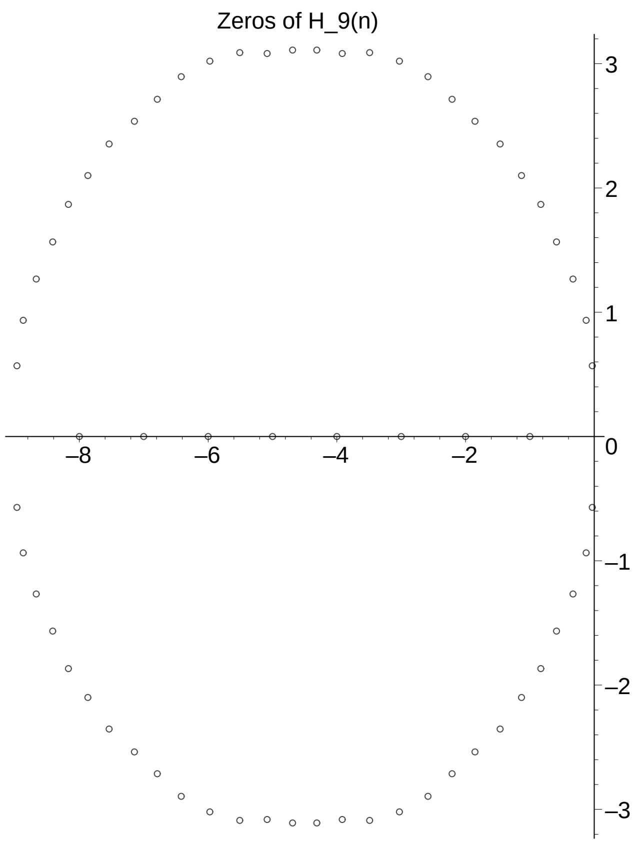

First variation

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Example.

$$\begin{bmatrix} * & * & 0 & * \\ * & * & 0 & * \\ 0 & 0 & * & * \\ * & * & * & * \end{bmatrix}$$

$H_{n,S}(r)$

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Example. $\begin{bmatrix} 0 & * & * \\ * & 0 & * \\ * & * & 0 \end{bmatrix}$

$$0 \leq i \leq r : \begin{bmatrix} 0 & i & r-i \\ r-i & 0 & i \\ i & r-i & 0 \end{bmatrix},$$

so $H_{3,S}(r) = r + 1$.

Polynomiality

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Thus Birkhoff-von Neumann, syzygy, and shellability arguments still apply:

Theorem. *$H_{n,S}(r)$ is a polynomial in r.*

An example

$$M = \begin{bmatrix} * & * & 0 & * \\ * & * & 0 & * \\ 0 & 0 & * & * \\ * & * & * & * \end{bmatrix}$$

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NOTE. The sequence $(1,2,2)$ is **not** symmetric!

Minimal positive magic squares

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There are **two** minimal positive magic squares:

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 1 & 0 & 1 \\ 0 & 0 & 3 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 2 & 1 & 0 & 1 \\ 1 & 2 & 0 & 1 \\ 0 & 0 & 3 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Reciprocity

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Reciprocity

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$(1,2,2)$ and $(2,2,1)$ are reverses!

Second variation

$S_n(r)$: number of $n \times n$ **symmetric** magic squares with line sums equal to r

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$$\begin{bmatrix} 1 & 2 & 0 & \mathbf{5} \\ 2 & 2 & \mathbf{3} & 1 \\ 0 & \mathbf{3} & 4 & 1 \\ \mathbf{5} & 1 & 1 & 1 \end{bmatrix}$$

Not a polynomial!

$$S_3(r) = \begin{cases} \frac{1}{8}(2r^3 + 9r^2 + 14r + \mathbf{8}), & r \text{ even} \\ \frac{1}{8}(2r^3 + 9r^2 + 14r + \mathbf{7}), & r \text{ odd} \end{cases}$$

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Theorem. *For any* $n \geq 1$ *there exist polynomials* $P_n(r)$ *and* $Q_n(r)$ *such that*

$$S_n(r) = \begin{cases} P_n(r), & r \text{ even} \\ Q_n(r), & r \text{ odd.} \end{cases}$$

Birkhoff-von Neumann fails

False that every symmetric magic square M is a sum of symmetric permutation matrices:

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = ?$$

Why r even and r odd

However $2M$ is sum of symmetric magic squares of row and column sum **two**:

$$M = P_1 + \cdots + P_k \ \ \text{(permutation matrices)}$$

$$\begin{aligned} \Rightarrow 2M &= M + M^t \\ &= (P_1 + P_1^t) + \cdots + (P_k + P_k^t). \end{aligned}$$

Third variation

$I_n(r)$: number of $n \times n \times n$ **magic cubes** (rows, columns, and pillars sum to r)

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No analogue of Birkhoff-von Neumann theorem. Very general results give:

Theorem. *For every $n \geq 1$ there is a p (depending on n) and polynomials $T_{n,0}(r), \ldots, T_{n,p-1}(r)$ such that*

$$I_n(r) = T_{n,i}(r) \text{ if } r \equiv i \pmod{p}.$$

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Value of p not known in general.

$3 \times 3 \times 3$: $p = 2$

$$I_3(r) = \frac{1}{4480}\Big(18r^8 + 216r^7 + 1218r^6 + 4158r^5$$

$$+ 9387r^4 + 14364r^3 + 14612r^2 + 10032r + 4480\Big)\,,$$

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$4 \times 4 \times 4$: $p = ?$

The last slide

The last slide

The last slide