Stern's Diatomic Array and Diatomic Sequence

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Pascal's triangle

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & & & 1 & & \\
 & 1 & & & & & & 1 & \\
1 & & & & & & & & 1 \\
 & & & & \vdots & & & &
\end{array}$$

Pascal's triangle

$$\begin{array}{ccccccccc} &&&&1&&&& \\ &&&1&&1&&& \\ &&1&&2&&1&& \\ &1&&&&&&1& \\ 1&&&&&&&&1 \\ &&&&\vdots&&&& \end{array}$$

Pascal's triangle

$$\begin{array}{ccccccccc} &&&&1&&&& \\ &&&1&&1&&& \\ &&1&&2&&1&& \\ &1&&\mathbf{3}&&\mathbf{3}&&1& \\ 1&&&&&&&&1 \\ &&&&\vdots&&&& \end{array}$$

Pascal's triangle

1

1 1

1 2 1

1 3 3 1

1 **4** **6** **4** 1

⋮

Binomial coefficients

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1 \\
 & & & & \vdots & & & &
\end{array}$$

$\boldsymbol{\binom{n}{k}}$: "$n$ choose k," the kth element in row n in Pascal's triangle, beginning with row 0 and with the 0th element in each row. For instance, $\binom{4}{2} = 6$.

Binomial coefficients

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1 \\
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\end{array}$$

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Note. There is a simple formula for $\binom{n}{k}$, namely, $\frac{n!}{k!\,(n-k)!}$, but this is irrelevant here.

Properties of Pascal's triangle

- number of elements in row n: $n + 1$

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$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Each element in row $n-1$ contributes **twice** to row n.

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- generating function for entries in row n:

$$\sum_{k=0}^{n} \binom{n}{k} x^k = (1 + x)^n \quad (\textbf{binomial theorem})$$

Further properties of Pascal's triangle

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$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

- sum of cubes of entries in row n: no simple formula or recurrence relation (similar to Fibonacci recurrence)

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Similar to Pascal's triangle, but we also "bring down" (copy) each number from one row to the next.

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 & & & & 1 & & & & \\
 & & 1 & & & & 1 & & \\
 & 1 & & & & & & 1 & \\
1 & & & & & & & & 1 \\
 & & & & \vdots & & & &
\end{array}$$

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Similar to Pascal's triangle, but we also "bring down" (copy) each number from one row to the next.

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & 1 & & \mathbf{1} & & 1 & & \\ & 1 & & & & & & 1 & \\ 1 & & & & & & & & 1 \\ & & & & \vdots & & & & \end{array}$$

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1

1 1 1

1 2 2 1

1 1

⋮

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$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & 1 & & 1 & & 1 & & \\
 & 1 & \mathbf{1} & 2 & \mathbf{1} & 2 & \mathbf{1} & 1 & \\
1 & & & & & & & & 1 \\
 & & & & \vdots & & & &
\end{array}$$

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Similar to Pascal's triangle, but we also "bring down" (copy) each number from one row to the next.

$$\begin{array}{ccccccccccccccc}
&&&&&&&1&&&&&&& \\
&&&1&&&&1&&&&1&&& \\
&1&&1&&2&&1&&2&&1&&1& \\
1&&\mathbf{2}&&\mathbf{3}&&\mathbf{3}&&\mathbf{3}&&\mathbf{3}&&\mathbf{2}&&1 \\
&&&&&&&\vdots&&&&&&&
\end{array}$$

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Similar to Pascal's triangle, but we also "bring down" (copy) each number from one row to the next.

1

1 1 1

1 1 2 1 2 1 1

1 **1** 2 **1** 3 **2** 3 **1** 3 **2** 3 **1** 2 **1** 1

⋮

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Similar to Pascal's triangle, but we also "bring down" (copy) each number from one row to the next.

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 & & & 1 & & & & 1 & & & & 1 & & & \\
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\end{array}$$

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Similar to Pascal's triangle, but we also "bring down" (copy) each number from one row to the next.

1
1 1 1
1 1 2 1 2 1 1
1 1 2 1 3 2 3 1 3 2 3 1 2 1 1
⋮

(slight and unimportant variant of the original **Stern's diatomic array**)

Moritz Abraham Stern, 1807–1894

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$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & 1 & & & & 1 & & & & 1 & & & \\
 & 1 & & 1 & & 2 & & 1 & & 2 & & 1 & & 1 & \\
1 & 1 & 2 & 1 & 3 & 2 & 3 & 1 & 3 & 2 & 3 & 1 & 2 & 1 & 1 \\
 & & & & & & & \vdots & & & & & & &
\end{array}
$$

$\left\langle {n \atop k} \right\rangle$: kth entry in row n, beginning with row 0 and 0th entry of each row ("Stern analogue" of binomial coefficients). Will also call "***n* choose *k***."

Stern coefficients

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1 & 1 & 2 & 1 & 3 & 2 & 3 & 1 & 3 & 2 & 3 & 1 & 2 & 1 & 1 \\
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\end{array}$$

$\genfrac{\langle}{\rangle}{0pt}{}{n}{k}$: kth entry in row n, beginning with row 0 and 0th entry of each row ("Stern analogue" of binomial coefficients). Will also call "***n* choose *k***."

$$\genfrac{\langle}{\rangle}{0pt}{}{3}{0} = 1, \quad \genfrac{\langle}{\rangle}{0pt}{}{3}{1} = 1, \quad \genfrac{\langle}{\rangle}{0pt}{}{3}{2} = 2, \quad \genfrac{\langle}{\rangle}{0pt}{}{3}{3} = 1, \quad \genfrac{\langle}{\rangle}{0pt}{}{3}{4} = 3, \ldots$$

Sum of entries in row n

Let $f(n) = \sum_k \left\langle {n \atop k} \right\rangle$, the sum of the entries in row n. Each entry contributes **three** times to the next row:

$$\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & 1 & & & & 1 & & & & 1 & & & \\
 & 1 & & 1 & & 2 & & 1 & & \textcircled{2} & & 1 & & 1 & \\
1 & 1 & 2 & 1 & 3 & 2 & 3 & 1 & \textcircled{3} & \textcircled{2} & \textcircled{3} & 1 & 2 & 1 & 1 \\
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1 & 1 & 2 & 1 & 3 & 2 & 3 & 1 & \textcircled{3} & \textcircled{2} & \textcircled{3} & 1 & 2 & 1 & 1 \\
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Hence $f(n+1) = 3f(n)$. Since $f(0) = 1$, We get $\boxed{f(n) = 3^n}$.

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 & 1 & & 1 & & 2 & & 1 & & \textcircled{2} & & 1 & & 1 & \\
1 & 1 & 2 & 1 & 3 & 2 & 3 & 1 & \textcircled{3} & \textcircled{2} & \textcircled{3} & 1 & 2 & 1 & 1 \\
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Hence $f(n+1) = 3f(n)$. Since $f(0) = 1$, We get $\boxed{f(n) = 3^n}$.

Compare 2^n for Pascal's triangle.

A recurrence

Let $F(x) = 1 + bx + cx^2 + +dx^3 + \cdots$ be any polynomial. A simple computation shows

$$(1+x+x^2)F(x^2) = 1+x+(1+b)x^2+bx^3+(b+c)x^4+cx^5+(c+d)x^6+dx^7+\cdots.$$

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If $1, b, c, \dots$ is a row of Stern's triangle, then the next row by definition is

$$1, 1, 1+b, b, b+c, c, c+d, d, \dots,$$

the coefficients of $(1+x+x^2)F(x^2)$.

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Let $\boldsymbol{F_n(x)} = \sum_k \left\langle {n \atop k} \right\rangle x^k$. Above argument shows

$$\boxed{F_{n+1}(x) = (1 + x + x^2)F_n(x^2).}$$

“Stern analogue” of binomial theorem

previous slide: $F_{n+1}(x) = (1 + x + x^2)F_n(x^2)$.

“Stern analogue” of binomial theorem

previous slide: $\boxed{F_{n+1}(x) = (1+x+x^2)F_n(x^2)}$.

Begin with $F_0(x) = 1$ and iterate:

$$\begin{aligned} F_1(x) &= (1+x+x^2)\cdot 1 &&= 1+x+x^2 \\ F_2(x) &= (1+x+x^2)F_1(x^2) &&= (1+x+x^2)(1+x^2+x^4) \\ F_3(x) &= (1+x+x^2)F_2(x^2) &&= (1+x+x^2)(1+x^2+x^4)(1+x^4+x^8), \end{aligned}$$

Theorem. $F_n(x) = \prod_{i=0}^{n-1}(1+x^{2^i}+x^{2^{i+1}})$.

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Put $x = 1$ to get $\sum_k \left\langle {n \atop k} \right\rangle = 3^n$ (done earlier).

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Put $x = 1$ to get $\sum_k \left\langle {n \atop k} \right\rangle = 3^n$ (done earlier).

Put $x = -1$ to get $\sum_k (-1)^k \left\langle {n \atop k} \right\rangle = 3^{n-1}$, $n \geq 1$ (0 for Pascal's triangle).

Sum of squares

$$\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & 1 & & & & 1 & & & & 1 & & & \\
 & 1 & & 1 & & 2 & & 1 & & 2 & & 1 & & 1 & \\
1 & 1 & 2 & 1 & 3 & 2 & 3 & 1 & 3 & 2 & 3 & 1 & 2 & 1 & 1 \\
 & & & & & & & \vdots & & & & & & &
\end{array}$$

$$\boldsymbol{u_r(n)} = \sum_k \left\langle {n \atop k} \right\rangle^r ,$$

the sum of the rth powers of the entries in row n.

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 & & & 1 & & & & 1 & & & & 1 & & & \\
 & 1 & & 1 & & 2 & & 1 & & 2 & & 1 & & 1 & \\
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 & & & & & & & \vdots & & & & & & &
\end{array}$$

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$$u_2(2) = 1^2 + 1^2 + 2^2 + 1^2 + 2^2 + 1^2 + 1^2 = 13$$

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$$\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & 1 & & & & 1 & & & & 1 & & & \\
 & 1 & & 1 & & 2 & & 1 & & 2 & & 1 & & 1 & \\
1 & 1 & 2 & 1 & 3 & 2 & 3 & 1 & 3 & 2 & 3 & 1 & 2 & 1 & 1 \\
 & & & & & & & \vdots & & & & & & &
\end{array}$$

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$$u_2(2) = 1^2 + 1^2 + 2^2 + 1^2 + 2^2 + 1^2 + 1^2 = 13$$

$$(u_2(0), u_2(1), \ldots, u_2(6)) = (1, 3, 13, 59, 269, 1227, 5597, \ldots)$$

1,3,13,59,269,1227,5597,25531,116461,. . .

To guess the sequence:

1,3,13,59,269,1227,5597,25531,116461,. . .

To guess the sequence:

In the distant past, note $u_2(n+1) \approx 5u_2(n)$, etc.

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Today: use **OEIS** (Online Encyclopedia of Integer Sequences) or various software packages.

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Conjecture. $\boxed{u_2(n+1) = 5u_2(n) - 2u_2(n-1)}$, $n \geq 1$

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Conjecture. $\boxed{u_2(n+1) = 5u_2(n) - 2u_2(n-1)}$, $n \geq 1$

$$\begin{array}{rcl} 13 & = & 5 \cdot 3 - 2 \cdot 1 \\ 59 & = & 5 \cdot 13 - 2 \cdot 3 \\ 269 & = & 5 \cdot 59 - 2 \cdot 13 \\ 1227 & = & 5 \cdot 269 - 2 \cdot 59 \\ & \vdots & \end{array}$$

Aside

If $g(n+1) = 5g(n) - 2g(n-1)$, then there are constants a, b such that

$$g(n) = a \cdot \alpha^n + b \cdot \beta^n,$$

where $x^2 - 5x + 2 = (x-\alpha)(x-\beta)$, and a, b depend on the initial conditions $g(0), g(1)$. Here we get

$$u_2(n) = \frac{17+\sqrt{17}}{34}\left(\frac{5+\sqrt{17}}{2}\right)^n + \frac{17-\sqrt{17}}{34}\left(\frac{5-\sqrt{17}}{2}\right)^n .$$

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where $x^2 - 5x + 2 = (x - \alpha)(x - \beta)$, and a, b depend on the initial conditions $g(0), g(1)$. Here we get

$$u_2(n) = \frac{17 + \sqrt{17}}{34} \left(\frac{5 + \sqrt{17}}{2} \right)^n + \frac{17 - \sqrt{17}}{34} \left(\frac{5 - \sqrt{17}}{2} \right)^n .$$

Since $-1 < \frac{5 - \sqrt{17}}{2} < 1$, a very good approximation to $u_2(n)$ is

$$\begin{aligned} u_2(n) &\approx \frac{17 + \sqrt{17}}{34} \left(\frac{5 + \sqrt{17}}{2} \right)^n \\ &= 0.62126\cdots(4.56155\cdots)^n. \end{aligned}$$

Proof that $u_2(n+1) = 5u_2(n) - 2u_2(n-1)$

$$\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & 1 & & & & 1 & & & & 1 & & & \\
 & 1 & & \color{red}{\mathbf{1}} & & \color{red}{\mathbf{2}} & & 1 & & 2 & & 1 & & 1 & \\
1 & 1 & 2 & 1 & \color{blue}{\mathbf{3}} & 2 & 3 & 1 & 3 & 2 & 3 & 1 & 2 & 1 & 1 \\
 & & & & & & & \vdots & & & & & & &
\end{array}$$

Every two consecutive terms $\left\langle {n \atop k} \right\rangle$ and $\left\langle {n \atop k+1} \right\rangle$ in row n contribute $\left(\left\langle {n \atop k} \right\rangle + \left\langle {n \atop k+1} \right\rangle\right)^2$ to $u_2(n+1)$.

Proof that $u_2(n+1) = 5u_2(n) - 2u_2(n-1)$

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 & & & & & & & 1 & & & & & & & \\
 & & & 1 & & & & 1 & & & & 1 & & & \\
 & 1 & & 1 & & 2 & & 1 & & 2 & & 1 & & 1 & \\
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Every term $\left\langle {n \atop k} \right\rangle$ in row n contributes $\left\langle {n \atop k} \right\rangle^2$ to $u_2(n+1)$.

A recurrence

Therefore

$$\begin{aligned}
u_2(n+1) &= \sum_k \left\langle {n+1 \atop k} \right\rangle^2 \\
&= \sum_k \left[\left(\left\langle {n \atop k} \right\rangle + \left\langle {n \atop k+1} \right\rangle \right)^2 + \left\langle {n \atop k} \right\rangle^2 \right] \\
&= \sum_k \left(\left\langle {n \atop k} \right\rangle^2 + 2 \left\langle {n \atop k} \right\rangle \left\langle {n \atop k+1} \right\rangle + \left\langle {n \atop k+1} \right\rangle^2 + \left\langle {n \atop k} \right\rangle^2 \right) \\
&= 3u_2(n) + 2 \sum_k \left\langle {n \atop k} \right\rangle \left\langle {n \atop k+1} \right\rangle .
\end{aligned}$$

Hence define $\boldsymbol{u_{1,1}(n)} = \sum_k \left\langle {n \atop k} \right\rangle \left\langle {n \atop k+1} \right\rangle$, so

$$u_2(n+1) = 3u_2(n) + 2 \sum_k \left\langle {n \atop k} \right\rangle \left\langle {n \atop k+1} \right\rangle .$$

Play the same game with $u_{1,1}(n)$

Previous slide:

$$u_2(n+1) = 3u_2(n) + 2\sum_k \genfrac{\langle}{\rangle}{0pt}{}{n}{k}\genfrac{\langle}{\rangle}{0pt}{}{n}{k+1} = 3u_2(n) + 2u_{1,1}(n).$$

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Previous slide:

$$u_2(n+1) = 3u_2(n) + 2\sum_k \left\langle {n \atop k} \right\rangle \left\langle {n \atop k+1} \right\rangle = 3u_2(n) + 2u_{1,1}(n).$$

Similarly,

$$\begin{aligned} u_{1,1}(n+1) &= \cdots + \left(\left\langle {n \atop k-1} \right\rangle + \left\langle {n \atop k} \right\rangle \right) \left\langle {n \atop k} \right\rangle \\ &\quad + \left\langle {n \atop k} \right\rangle \left(\left\langle {n \atop k} \right\rangle + \left\langle {n \atop k+1} \right\rangle \right) \\ &\quad + \left(\left\langle {n \atop k} \right\rangle + \left\langle {n \atop k+1} \right\rangle \right) \left\langle {n \atop k+1} \right\rangle + \cdots \\ &= 2u_2(n) + 2u_{1,1}(n) \end{aligned}$$

Linear algebra to the rescue!

We have shown that

$$\begin{aligned} u_2(n+1) &= 3u_2(n) + 2u_{1,1}(n) \\ u_{1,1}(n) &= 2u_2(n) + 2u_{1,1}(n). \end{aligned}$$

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Can rewrite in matrix terms:

$$\begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} u_2(n) \\ u_{1,1}(n) \end{bmatrix} = \begin{bmatrix} u_2(n+1) \\ u_{1,1}(n+1) \end{bmatrix}$$

Linear algebra to the rescue!

We have shown that

$$\begin{aligned} u_2(n+1) &= 3u_2(n) + 2u_{1,1}(n) \\ u_{1,1}(n) &= 2u_2(n) + 2u_{1,1}(n). \end{aligned}$$

Can rewrite in matrix terms:

$$\begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} u_2(n) \\ u_{1,1}(n) \end{bmatrix} = \begin{bmatrix} u_2(n+1) \\ u_{1,1}(n+1) \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}^{n+1} \begin{bmatrix} u_2(0) \\ u_{1,1}(0) \end{bmatrix} = \begin{bmatrix} u_2(n+1) \\ u_{1,1}(n+1) \end{bmatrix}.$$

Cayley-Hamilton theorem

Recall: the **characteristic polynomial** $\mathrm{cp}_A(x)$ of an $n \times n$ matrix A is defined by

$$\mathbf{cp}_{\boldsymbol{A}}(\boldsymbol{x}) = \det(xI - A),$$

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Cayley-Hamilton theorem. $\mathrm{cp}_A(A) = 0.$

Let $A = \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}$. Then

$$\mathrm{cp}_A(x) = \det \begin{bmatrix} x-3 & -2 \\ -2 & x-2 \end{bmatrix} = x^2 - 5x + 2.$$

Proof concluded

Therefore $A^2 - 5A + 2I = 0$. Multiply by A^{n-1} and apply to $\begin{bmatrix} u_2(0) \\ u_{1,1}(0) \end{bmatrix}$:

$$(A^{n+1} - 5A^n + 2A^{n-1}) \begin{bmatrix} u_2(0) \\ u_{1,1}(0) \end{bmatrix} = 0.$$

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Since

$$A^m \begin{bmatrix} u_2(0) \\ u_{1,1}(0) \end{bmatrix} = \begin{bmatrix} u_2(m) \\ u_{1,1}(m) \end{bmatrix},$$

we get $u_2(n+1) - 5u_2(n) + 2u_2(n-1) = 0$, as well as the bonus $u_{1,1}(n+1) - 5u_{1,1}(n) + 2u_{1,1}(n-1) = 0$. □

What about $u_r(n) = \sum_k \left\langle {n \atop k} \right\rangle^r$?

Same technique works, but we get a matrix of size $\lfloor \frac{2+r}{2} \rfloor$.

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In other words:

Theorem. *Fix n. Define*

$$\prod_{i=0}^{n-1} (1 + x^{2^i} + x^{2^{i+1}}) = \sum_m a_m x^m.$$

Then $\sum_m a_m^3 = 3 \cdot 7^{n-1}$.

Recurrences for $4 \leq r \leq 7$

$$\begin{aligned}
u_4(n+1) &= 10u_4(n) + 9u_4(n-1) - 2u_4(n-2) \\
u_5(n+1) &= 14u_5(n) + 47u_5(n-1) \\
u_6(n+1) &= 20u_6(n) + 161u_6(n-1) + 40u_6(n-2) - 4u_6(n-3) \\
u_7(n+1) &= 29u_7(n) + 485u_7(n-1) + 327u_7(n-2)
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\end{aligned}$$

Note. The (least) order of the recurrence is known—it is about $r/3$.

Stern's diatomic sequence

Recall

$$F_n(x) = \sum_k \left\langle {n \atop k} \right\rangle x^k = \prod_{i=0}^{n-1} \left(1 + x^{2^i} + x^{2^{i+1}}\right).$$

Let $n \to \infty$. We get

$$\begin{aligned} \prod_{i=0}^{\infty} \left(1 + x^{2^i} + x^{2^{i+1}}\right) \quad &:= \quad \sum_{k=0}^{\infty} \boldsymbol{b(k)} x^k \\ &= \quad 1 + x + 2x^2 + x^3 + 3x^4 + 2x^5 + 3x^6 + x^7 + 4x^8 + 3x^9 + 5x^{10} + \cdots. \end{aligned}$$

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$$\begin{aligned}\prod_{i=0}^{\infty}\left(1 + x^{2^i} + x^{2^{i+1}}\right) &:= \sum_{k=0}^{\infty} b(k) x^k \\ &= 1 + x + 2x^2 + x^3 + 3x^4 + 2x^5 + 3x^6 + x^7 + 4x^8 + 3x^9 + 5x^{10} + \cdots.\end{aligned}$$

The sequence $(1, 1, 2, 1, 3, 2, 3, 1, 4, 3, 5 \ldots)$ of coefficients is **Stern's diatomic sequence**. Thus $b(0) = 1, b(1) = 1, b(2) = 2$, etc.

Rephrasing of definition of $b(k)$

$$\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & 1 & & & & 1 & & & & 1 & 1 & & \\
 & 1 & & 1 & & 2 & & 1 & & 2 & & 1 & & 1 & \\
1 & 1 & 2 & 1 & 3 & 2 & 3 & 1 & 3 & 2 & 3 & 1 & 2 & 1 & 1 \\
 & & & & & & & \vdots & & & & & & &
\end{array}$$

$$b(k) = \lim_{n\to\infty} \left\langle {n \atop k} \right\rangle,$$

i.e., the sequence $\left\langle {0 \atop k} \right\rangle, \left\langle {1 \atop k} \right\rangle, \left\langle {2 \atop k} \right\rangle, \ldots$ eventually becomes constant, equal to $b(k)$.

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 & & & & & & & 1 & & & & & & & \\
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 & 1 & & 1 & & \textcircled{2} & & 1 & & 2 & & 1 & & 1 & \\
1 & 1 & \textcircled{2} & 1 & 3 & 2 & 3 & 1 & 3 & 2 & 3 & 1 & 2 & 1 & 1 \\
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Example. $k = 2 : \quad 0, 1, 2, 2, 2, 2, \ldots$, so $b(2) = 2$

Fundamental recurrence for $b(k)$

$$\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & 1 & & & & 1 & & & & 1 & & & \\
 & 1 & & 1 & & 2 & & 1 & & 2 & & 1 & & 1 & \\
1 & 1 & 2 & 1 & 3 & 2 & 3 & 1 & 3 & 2 & 3 & 1 & 2 & 1 & 1 \\
 & & & & & & & \vdots & & & & & & &
\end{array}$$

Theorem. $b(2k+1) = b(k)$ *and* $b(2k+2) = b(k) + b(k+1)$.

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 & & & & & & & 1 & & & & & & & \\
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1 & 1 & 2 & 1 & 3 & 2 & 3 & 1 & 3 & 2 & 3 & 1 & 2 & 1 & 1 \\
 & & & & & & & \vdots & & & & & & &
\end{array}$$

Theorem. *$b(2k+1) = b(k)$ and $b(2k+2) = b(k) + b(k+1)$.*

Proof. The entry in Stern's triangle directly below $\genfrac{\langle}{\rangle}{0pt}{}{n}{k}$ is $\genfrac{\langle}{\rangle}{0pt}{}{n+1}{2k+1}$, so they are equal. Let $n \to \infty$ in $\genfrac{\langle}{\rangle}{0pt}{}{n}{k} = \genfrac{\langle}{\rangle}{0pt}{}{n+1}{2k+1}$, and use

$$b(k) = \lim_{n\to\infty} \genfrac{\langle}{\rangle}{0pt}{}{n}{k}$$

to get $b(2k+1) = b(k)$. Similarly $b(2k+2) = b(k) + b(k+1)$ $\square$.

A startling property of Stern's diatomic sequence

Theorem. *The sequence*

$$\left(\frac{b(1)}{b(2)}, \frac{b(2)}{b(3)}, \frac{b(3)}{b(4)}, \dots\right) = \left(\frac{1}{1}, \frac{2}{1}, \frac{1}{2}, \frac{3}{1}, \frac{2}{3}, \frac{3}{2}, \frac{1}{3}, \frac{4}{1}, \dots\right)$$

contains every positive rational number exactly once, and they are in lowest terms.

A surprise actor

Calkin-Wilf tree: an infinite binary tree with vertices labelled by positive rational numbers a/b

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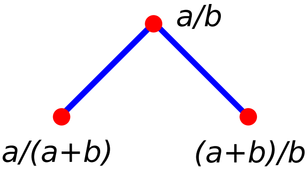
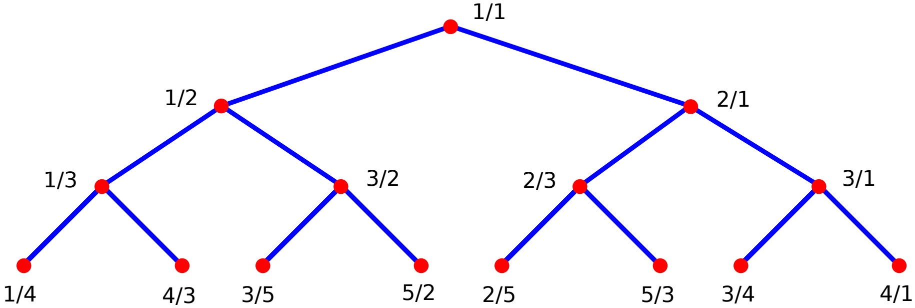
Other vertices labelled recursively by the rule:



The Calkin-Wilf tree



History lesson

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- a similar tree defined by **Johannes Kepler**, 1619

Vertex labels

Theorem. *In the Calkin-Wilf tree, every positive rational number occurs exactly once as a vertex label, and it is in lowest terms.*

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Proof by example. From any rational $a/b > 0$, where $\gcd(a, b) = 1$, we can uniquely work our way up to $1/1$. For instance,

$$\frac{11}{7} \to \frac{4}{7} \to \frac{4}{3} \to \frac{1}{3} \to \frac{1}{2} \to \frac{1}{1}.$$

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More specifically

$$\frac{a}{b} \to \begin{cases} \frac{a-b}{b}, & a > b \\ \frac{a}{b-a}, & b > a. \end{cases}$$

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$$\frac{a}{b} \to \begin{cases} \frac{a-b}{b}, & a > b \\ \frac{a}{b-a}, & b > a. \end{cases}$$

Finally, each fraction is in lowest terms since if $\gcd(a, b) = 1$, then $\gcd(a + b, b) = 1$ and $\gcd(a, a + b) = 1$. $\square$

Aside: Euclidean algorithm

The process

$$\frac{11}{7} \to \frac{4}{7} \to \frac{4}{3} \to \frac{1}{3} \to \frac{1}{2} \to \frac{1}{1}$$

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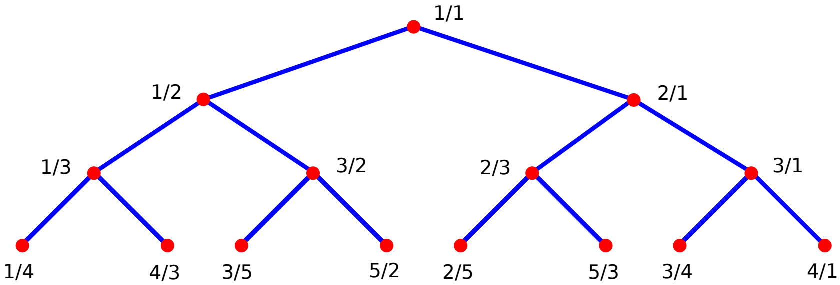
Alas, no time to pursue here.

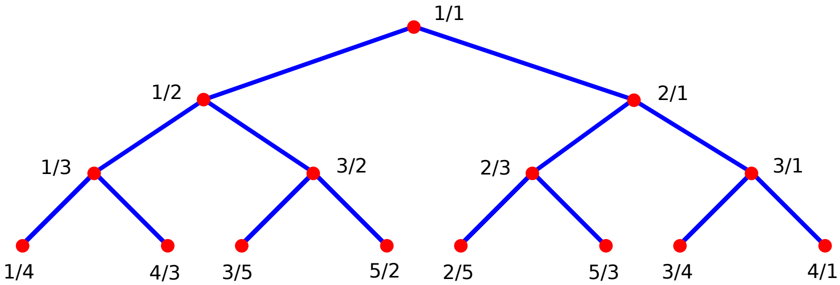
Connection with Stern's diatomic sequence



Read the labels in breadth-first order (usual reading order):

$$\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{3}{2}, \frac{2}{3}, \frac{3}{1}, \frac{1}{4}, \dots .$$

Connection with Stern's diatomic sequence



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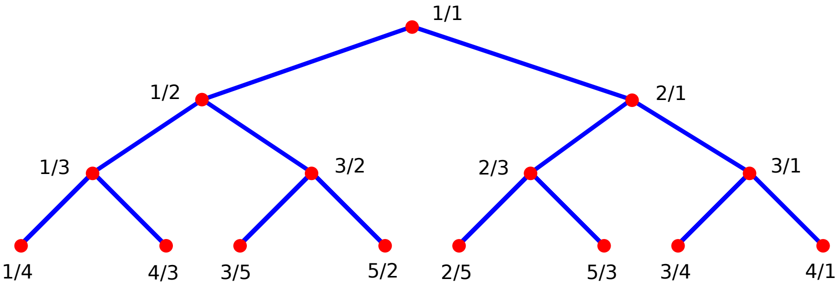
$$\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{3}{2}, \frac{2}{3}, \frac{3}{1}, \frac{1}{4}, \ldots.$$

numerators: $1, 1, 2, 1, 3, 2, 3, 1, \ldots$

denominators: $1, 2, 1, 3, 2, 3, 1, 4, \ldots$

Connection with Stern's diatomic sequence



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numerators: $1, 1, 2, 1, 3, 2, 3, 1, \ldots$
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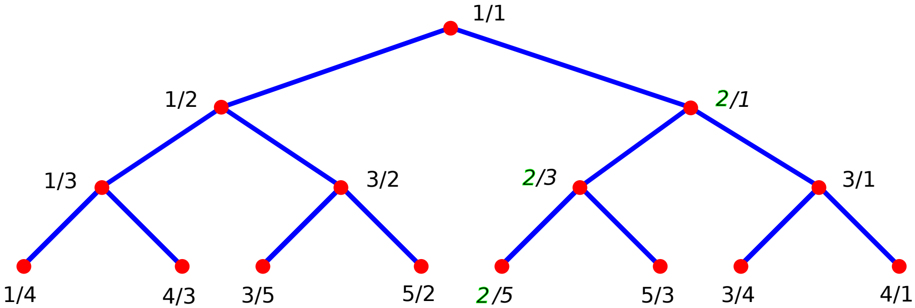
familiar?

Why?

Let the terms of the numerator sequence be $c(0), c(1), \ldots$.



The left child of any vertex v has the same numerator as v. If we write this in terms of $c(k)$, it becomes

$$c(2k+1) = c(k).$$

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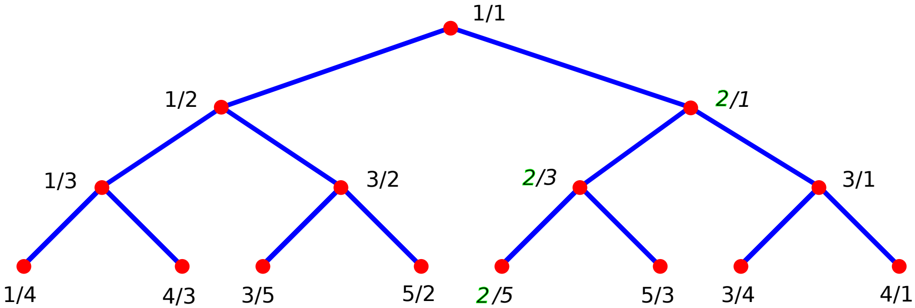

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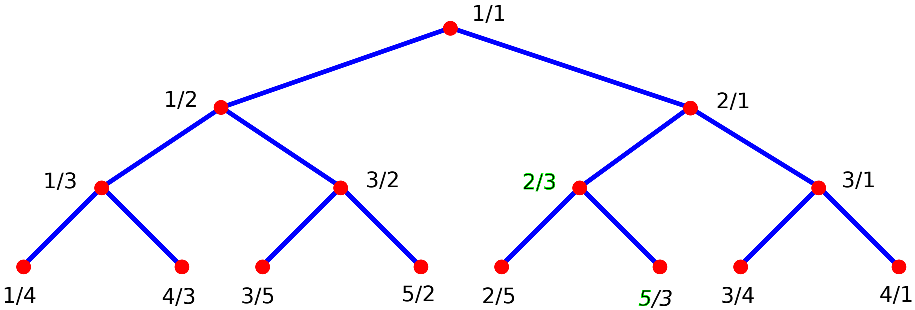
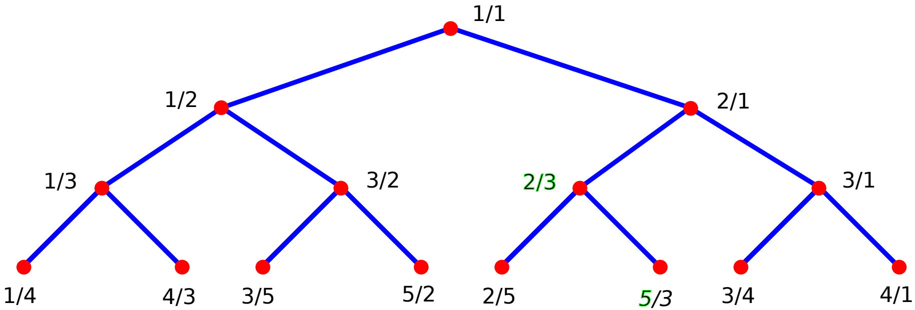
E.g., the green numbers above are $2 = c(2) = c(5) = c(11) = \cdots$.

$c(2k+2)$



- $2 = c(5)$, $3 = c(6)$

$c(2k+2)$



- $2 = c(5)$, $3 = c(6)$
- $5 = 2 + 3$ is numerator of $c(12)$.

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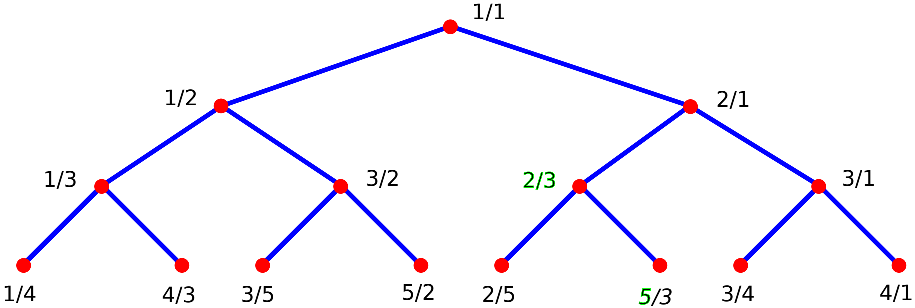

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- Hence $c(12) = c(5) + c(6)$.

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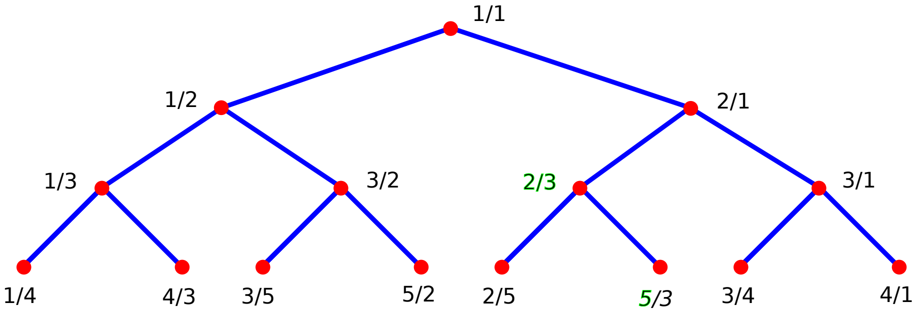

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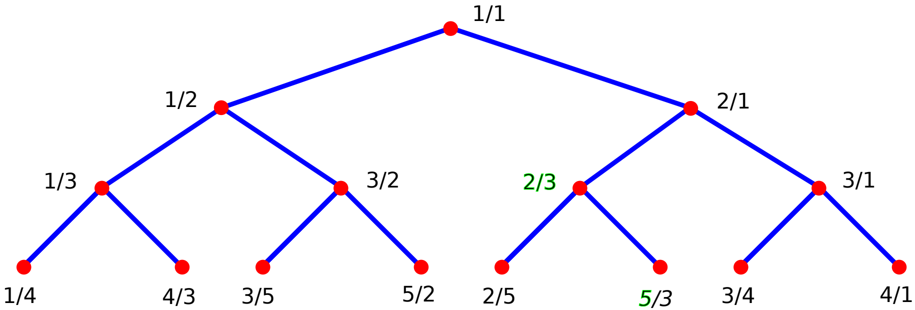

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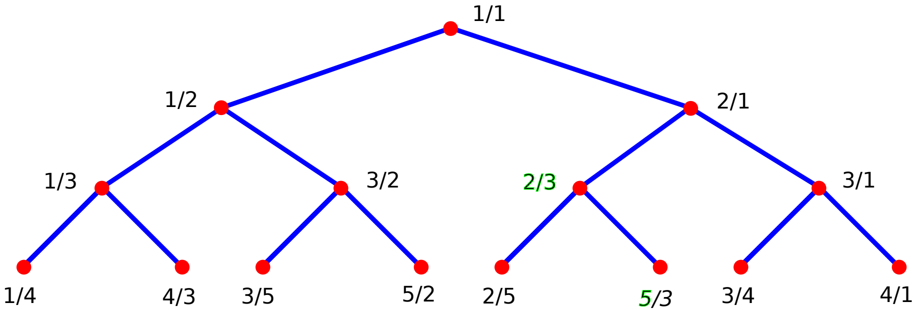
- In general, $c(2k+2) = c(k) + c(k+1)$.
- Thus $c(k)$ satisfies the same recurrence as $b(k)$, as well as the initial conditions $c(0) = b(0) = 1$ and $c(1) = b(1) = 1$.

$c(2k+2)$



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- Thus $c(k)$ satisfies the same recurrence as $b(k)$, as well as the initial conditions $c(0) = b(0) = 1$ and $c(1) = b(1) = 1$.
- $\Rightarrow c(k) = b(k)$ for all $k \geq 0$.

Conclusion

We have proved:

Theorem. *The sequence*

$$\left(\frac{b(1)}{b(2)}, \frac{b(2)}{b(3)}, \frac{b(3)}{b(4)}, \dots\right) = \left(\frac{1}{1}, \frac{2}{1}, \frac{1}{2}, \frac{3}{1}, \frac{2}{3}, \frac{3}{2}, \frac{1}{3}, \frac{4}{1}, \dots\right)$$

contains every positive rational number exactly once, and they are in lowest terms.

The final slide

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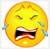

The final slide 😫

Exercises

1. Number of (nonzero) entries in row n of Stern's triangle is $2^{n+1} - 1$.
2. The largest entry in row n of Stern's triangle is the Fibonacci number F_{n+1}.
3. What is $\sum_k \left(\left\langle {n \atop 3k} \right\rangle - \left\langle {n \atop 3k+1} \right\rangle \right)$?
4. $b(k)$ is the number of ways to write k as a sum of powers of 2 (without regard to order), where each power of 2 can be used no more than twice. For instance $b(6) = 3$ since $4 + 2 = 4 + 1 + 1 = 2 + 2 + 1 + 1$. What if each power of 2 can be used no more than once?
5. $b(k)$ is the number of odd binomial coefficients of the form $\binom{k-r}{r}$, $0 \le r \le k$. When $k = 6$ we have three such binomial coefficients: $\binom{6}{0}, \binom{5}{1}, \binom{3}{3}$.

A startling property of “startling”

startling
starling or starting
staring
string
sting
sing
sin
in
I