Arrangements and Combinatorics

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M.I.T.

The main reference

An introduction to hyperplane arrangements, in *Geometric Combinatorics* (E. Miller, V. Reiner, and B. Sturmfels, eds.), IAS/Park City Mathematics Series, vol. 13, American Mathematical Society, Providence, RI, 2007, pp. 389–496.

math.mit.edu/∼rstan/arrangements/arr.html

Posets

A **poset** (partially ordered set) is a set P and relation $\leq$ satisfying $\forall x, y, z \in P$:

(P1) (**reflexivity**) $x \leq x$

(P2) (**antisymmetry**) If $x \leq y$ and $y \leq x$, then $x = y$.

(P3) (**transitivity**) If $x \leq y$ and $y \leq z$, then $x \leq z$.

Arrangements

$\boldsymbol{K}$: a field

$\boldsymbol{\mathcal{A}}$: a (finite) arrangement in $V = K^n$

rk($\mathcal{A}$) (**rank** of $\mathcal{A}$) : dimension of space spanned by normals to $H \in \mathcal{A}$

Subspaces X, Y, W

$\boldsymbol{Y}$ = any complement to subspace X of K^n spanned by normals to $H \in \mathcal{A}$

$$\boldsymbol{W} = \{v \in V \ : \ v \cdot y = 0 \ \ \forall y \in Y\}.$$

If char$(K) = 0$ can take $W = X$.

Essentialization

$$\operatorname{codim}_W(H \cap W) = 1, \ \forall H \in \mathcal{A}$$

Essentialization of $\mathcal{A}$:

$$\textbf{ess}(\mathcal{A}) = \{H \cap W \ : \ H \in \mathcal{A}\},$$

an arrangment in W.

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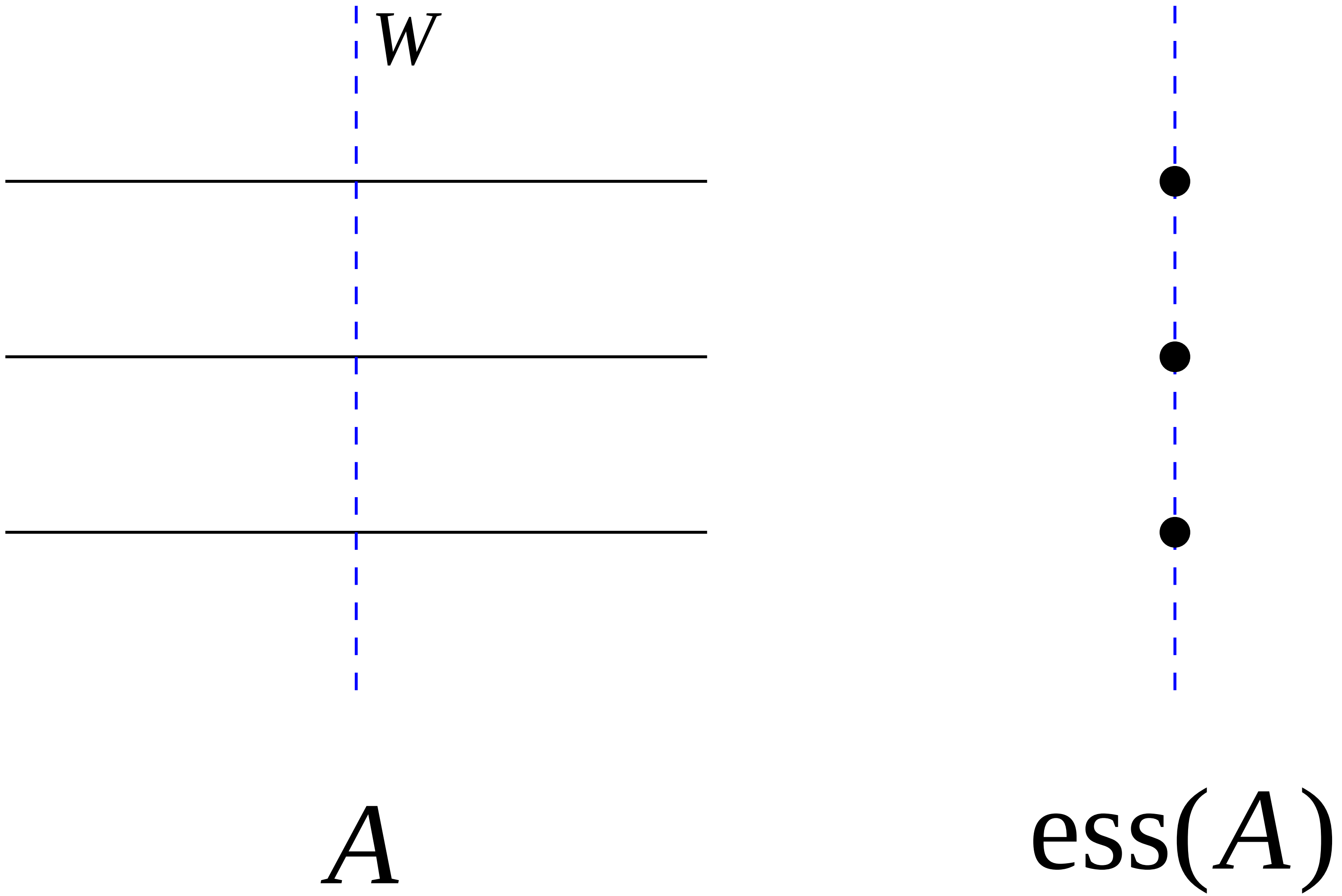
$\mathcal{A}$ is **essential** if ess$(\mathcal{A}) = \mathcal{A}$, i.e., rk$(\mathcal{A}) = \dim(\mathcal{A})$.

Example of essentialization



The intersection poset

$L(\mathcal{A})$: **nonempty** intersections of hyperplanes in $\mathcal{A}$, ordered by **reverse** inclusion

Include V as the bottom element of $L(\mathcal{A})$, denoted $\hat{0}$.

Note. $L(\mathcal{A}) \cong L(\mathrm{ess}(\mathcal{A}))$

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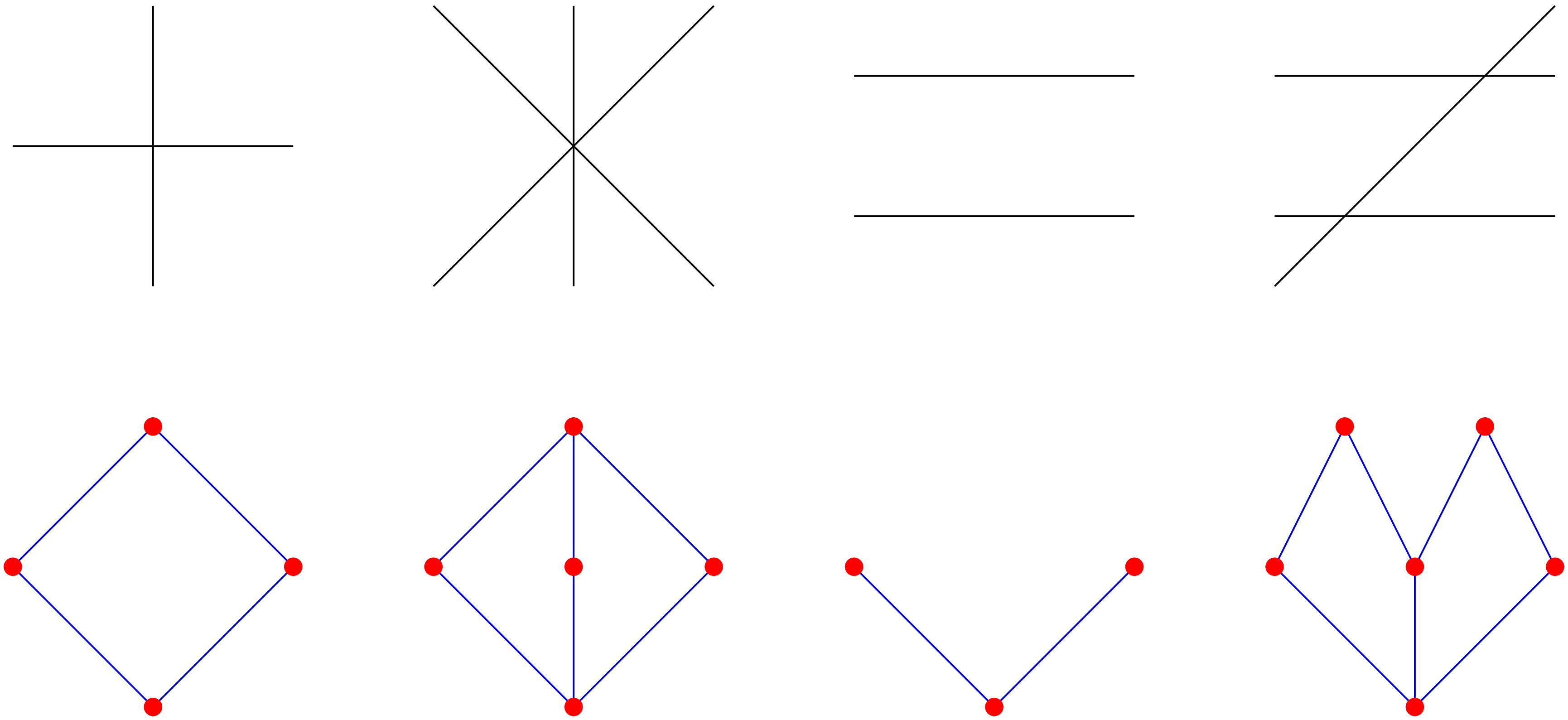
$L(\mathcal{A})$ is the most important combinatorial object associated with $\mathcal{A}$.

Examples of intersection posets



Rank function

Chain of length k: $x_0 < x_1 < \cdots < x_k$

Graded poset of rank n: every maximal chain has length n

Rank function: $\rho(x)$ is the length k of longest chain $x_0 < x_1 < \cdots < x_k = x$.

Rank function on $L(\mathcal{A})$

Proposition. *$L(\mathcal{A})$ is graded of rank equal to $\mathrm{rk}(\mathcal{A})$. Rank function:*

$$\mathrm{rk}(x) = \mathrm{codim}(x) = n - \dim(x),$$

where $\dim(x)$ is the dimension of x as an affine subspace of V.

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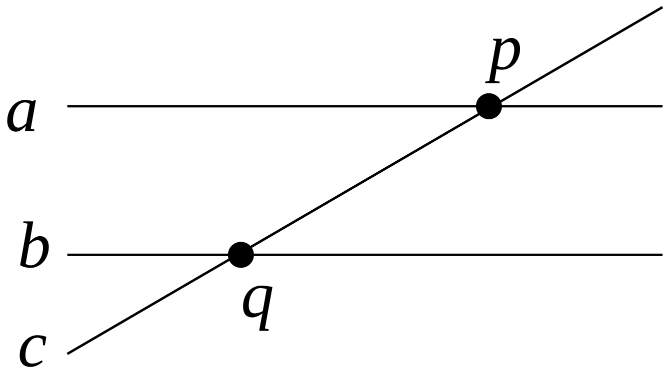
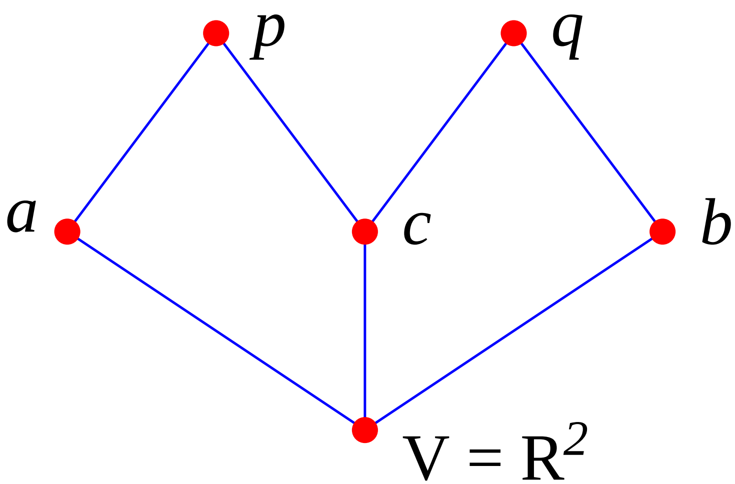
Proof. Straightforward. $\square$

Example of $L(\mathcal{A})$





rank	dim
2	0
1	1
0	2

The Möbius function

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Define $\boldsymbol{\mu} = \boldsymbol{\mu_P} : \mathrm{Int}(P) \to \mathbb{Z}$ (the **Möbius function** of P) by:

$$\begin{aligned} \mu(x,x) &= 1, \text{ for all } x \in P \\ \mu(x,y) &= -\sum_{x \leq z < y} \mu(x,z), \text{ for all } x < y \text{ in } P. \end{aligned}$$

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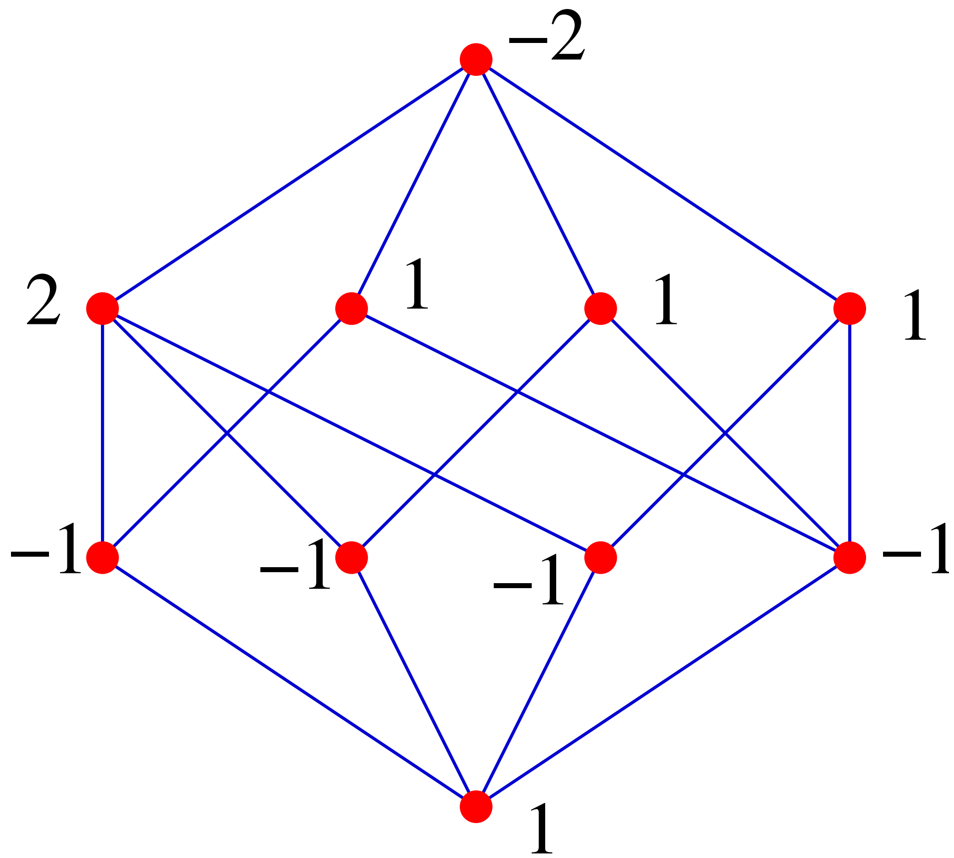
Write $\mu(x) = \mu(\hat{0}, x)$.

Example of Möbius function



Numbers denote $\mu(x)$.

Möbius inversion formula

$P =$ finite poset

$f, g \colon P \to L$ (a field, or even just an abelian group)

Theorem. *Equivalent:*

$$f(x) = \sum_{y \geq x} g(y), \text{ for all } x \in P$$

$$g(x) = \sum_{y \geq x} \mu(x, y) f(y), \text{ for all } x \in P.$$

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Proof. Straightforward. $\square$

The characteristic polynomial

Definition. The *characteristic polynomial* $\chi_{\mathcal{A}}(t)$ of the arrangement $\mathcal{A}$ is defined by

$$\boldsymbol{\chi_{\mathcal{A}}(t)} = \sum_{x \in L(\mathcal{A})} \mu(x) t^{\dim(x)}.$$

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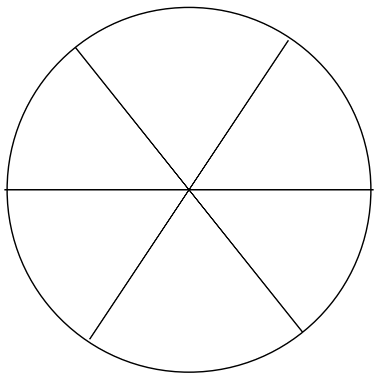
$$\boldsymbol{\chi_{\mathcal{A}}(t)} = \sum_{x \in L(\mathcal{A})} \mu(x) t^{\dim(x)}.$$

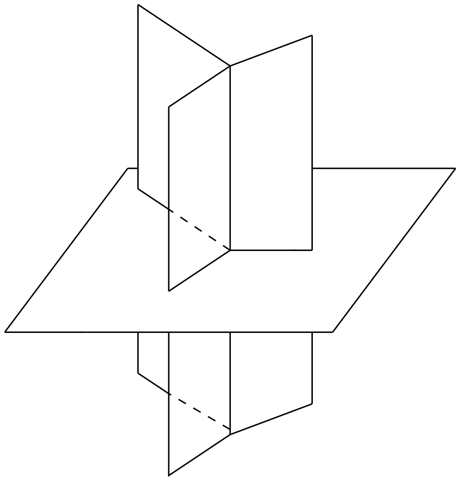
Note. $x = V$ contributes t^n, and each $H \in \mathcal{A}$ contributes $-t^{n-1}$. Hence

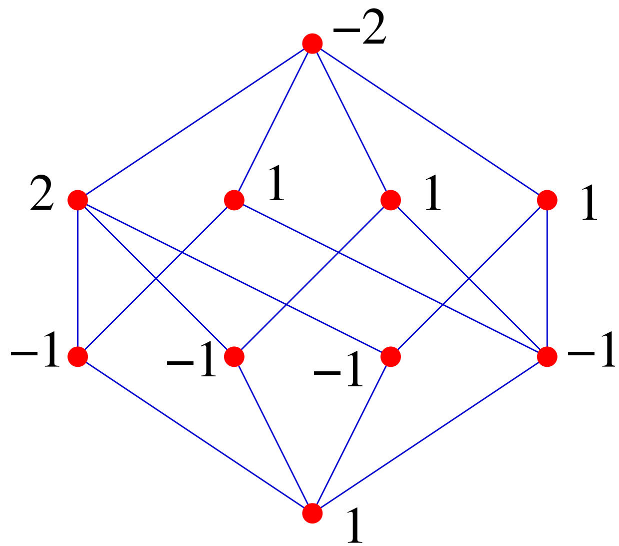
$$\chi_{\mathcal{A}}(t) = t^n - (\#\mathcal{A}) t^{n-1} + \cdots .$$

An example

Example.







$$\chi_{\mathcal{A}}(t) = t^3 - 4t^2 + 5t - 2 = (t-1)^2(t-2).$$

The boolean algebra

Suppose all hyperplanes in $\mathcal{A}$ are linearly independent, and $\#\mathcal{A} = n$. Then all intersections are nonempty and distinct, so

$$L(\mathcal{A}) \cong \boldsymbol{B_n},$$

the **boolean algebra** of all subsets of $[\boldsymbol{n}] = \{1, \ldots, n\}$, ordered by inclusion.

Characteristic polynomial of B_n

Easy induction argument: $\mu(\hat{0}, x) = (-1)^{n-\dim x}$. Hence

$$\chi_{\mathcal{A}}(t) = \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} t^i = (t-1)^n.$$

Regions

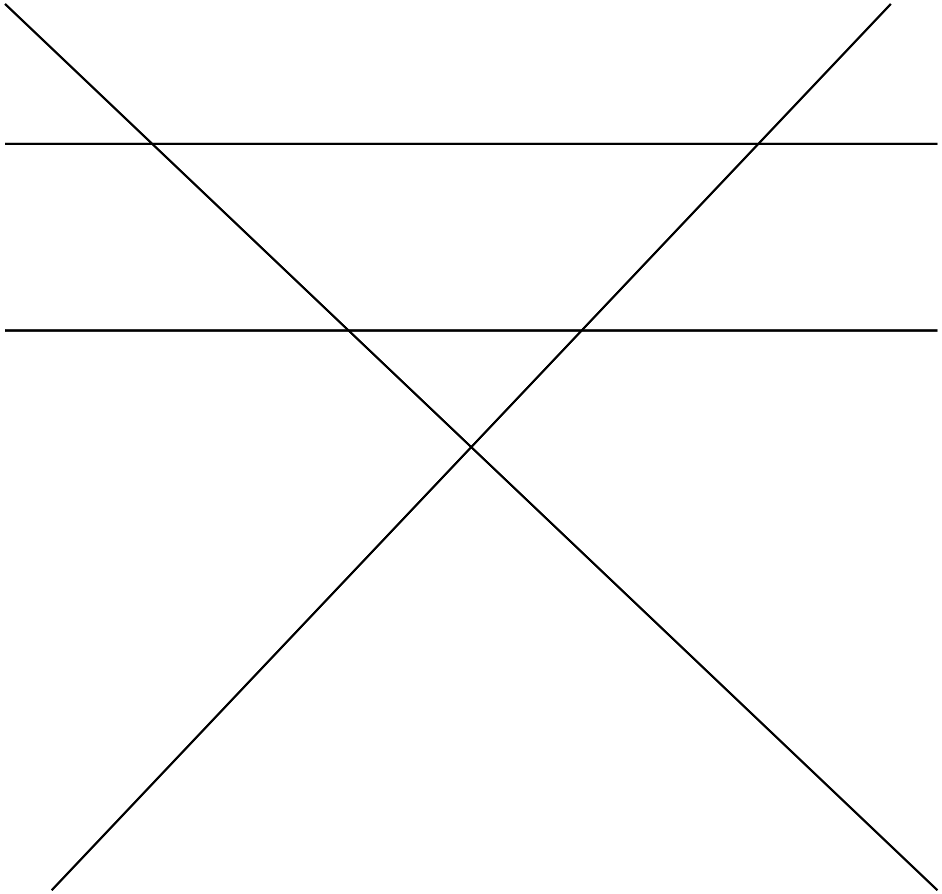
Let $\boldsymbol{K} = \mathbb{R}$. **Region** (or **chamber**) of $\mathcal{A}$: connected component of $\mathbb{R}^n - \bigcup_{H \in \mathcal{A}} H$.

$$\boldsymbol{r(\mathcal{A})} = \text{number of regions of } \mathcal{A}$$

A region R of $\mathcal{A}$ is **relatively bounded** if it becomes bounded in ess$(\mathcal{A})$.

$$\boldsymbol{b(\mathcal{A})} = \text{number of relatively bounded regions of } \mathcal{A}$$

Example of $r(\mathcal{A})$ and $b(\mathcal{A})$



$$r(\mathcal{A}) = 10, \quad b(\mathcal{A}) = 2$$

Zaslavsky's theorem (1975)

Current goal:

Theorem. *Let* $\mathcal{A}$ *be an arrangement of rank* r *in* $\mathbb{R}^n$*. Then*

$$r(\mathcal{A}) = (-1)^n \chi_{\mathcal{A}}(-1)$$

$$b(\mathcal{A}) = (-1)^r \chi_{\mathcal{A}}(1).$$

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Proof will be by induction on $\#\mathcal{A}$ (the number of hyperplanes).

Subarrangements and restrictions

subarrangement of $\mathcal{A}$: a subset $\mathcal{B} \subseteq \mathcal{A}$

For $x \in L(\mathcal{A})$ define

$$\boldsymbol{\mathcal{A}_x} = \{H \in \mathcal{A} : x \subseteq H\} \subseteq \mathcal{A}$$

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Also define the **restriction** of $\mathcal{A}$ to x to be the arrangement in the affine space $\mathcal{A}$:

$$\boldsymbol{\mathcal{A}^x} = \{x \cap H \neq \emptyset \,:\, H \in \mathcal{A} - \mathcal{A}_x\}.$$

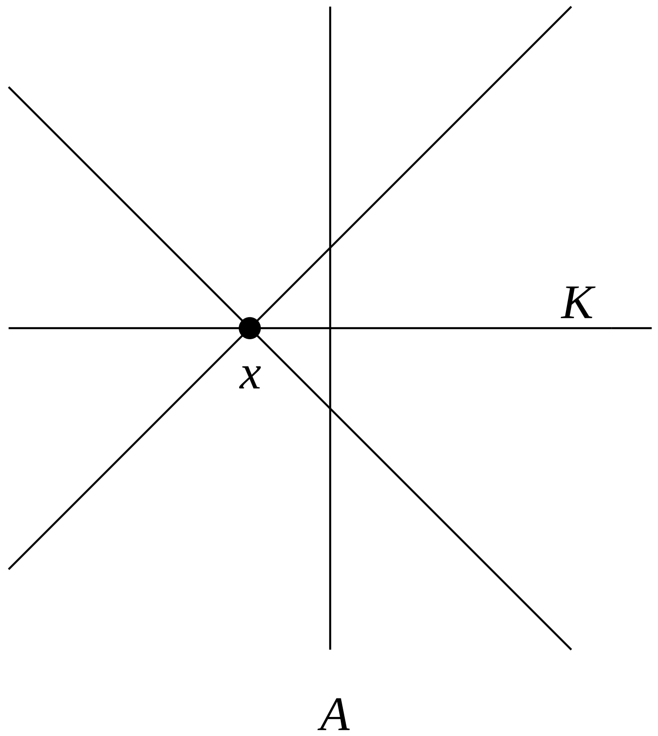
$L(\mathcal{A}_x)$ and $L(\mathcal{A}^x)$

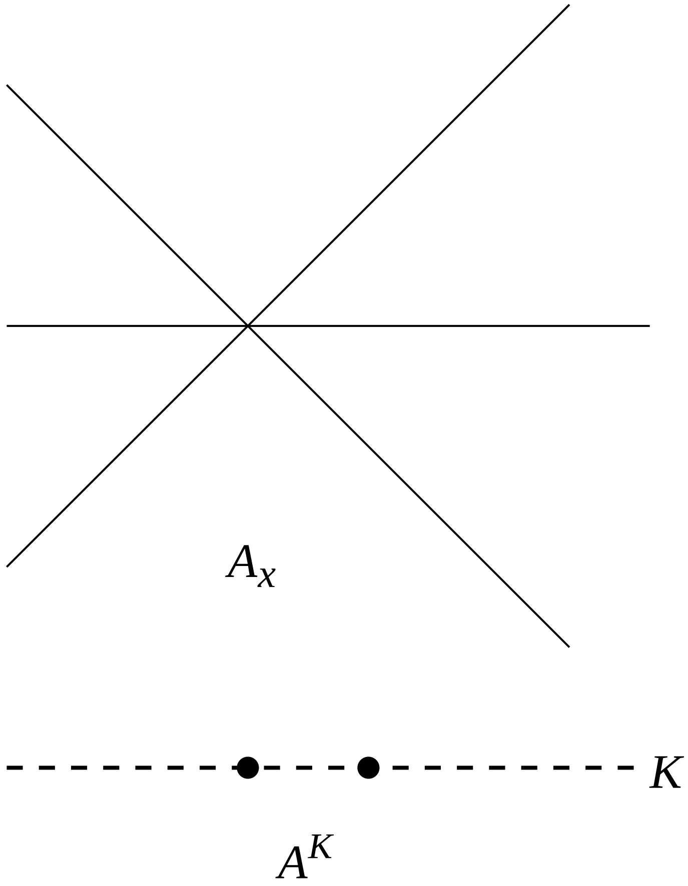
Note that if $x \in L(\mathcal{A})$, then

$$L(\mathcal{A}_x) \cong \Lambda_x \ := \ \{y \in L(\mathcal{A}) \ : \ y \leq x\}$$

$$L(\mathcal{A}^x) \cong V_x \ := \ \{y \in L(\mathcal{A}) \ : \ y \geq x\}.$$

Example of $\mathcal{A}_x$ and $\mathcal{A}^x$





Triple of arrangments

Choose $H_0 \in \mathcal{A}$. Define

$$\mathcal{A}' = \mathcal{A} - \{H_0\}$$

$$\mathcal{A}'' = \mathcal{A}^{H_0}.$$

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$$\begin{aligned} \mathcal{A}' &= \mathcal{A} - \{H_0\} \\ \mathcal{A}'' &= \mathcal{A}^{H_0}. \end{aligned}$$

Call $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ a **triple** of arrangements with **distinguished hyperplane** H_0.

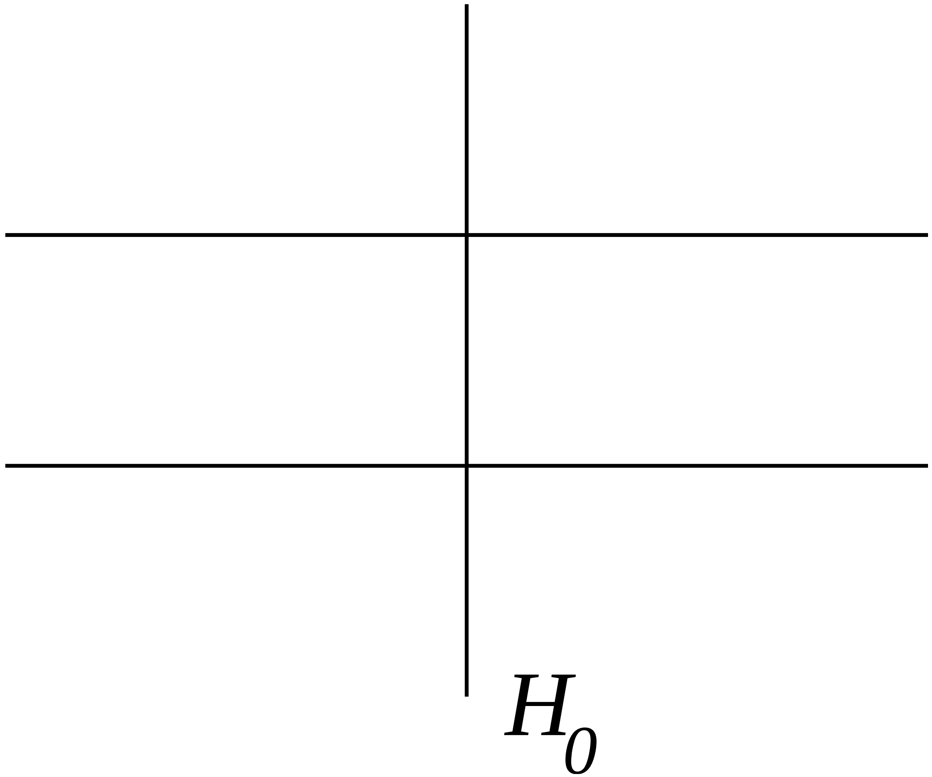
Recurrence for $r(\mathcal{A})$ and $b(\mathcal{A})$

Lemma. *Let* $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ *be a triple of real arrangements with distinguished hyperplane* H_0*. Then*

$$r(\mathcal{A}) = r(\mathcal{A}') + r(\mathcal{A}'')$$

$$b(\mathcal{A}) = \begin{cases} b(\mathcal{A}') + b(\mathcal{A}''), & \text{if } \operatorname{rk}(\mathcal{A}) = \operatorname{rk}(\mathcal{A}') \\ 0, & \text{if } \operatorname{rk}(\mathcal{A}) = \operatorname{rk}(\mathcal{A}') + 1. \end{cases}$$

The case $\mathrm{rk}(\mathcal{A}) = \mathrm{rk}(\mathcal{A}') + 1$



Proof of lemma (sketch)

Note that $r(\mathcal{A})$ equals $r(\mathcal{A}')$ plus the number of regions of $\mathcal{A}'$ cut into two regions by H_0. Easy to give a bijection between regions of $\mathcal{A}'$ cut in two by H_0 and regions of $\mathcal{A}''$, proving

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Proof of recurrence for $b(\mathcal{A})$ analogous. $\square$

The deletion-restriction recurrence

Lemma. Let $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ be a triple of real arrangements. Then

$$\chi_{\mathcal{A}}(t) = \chi_{\mathcal{A}'}(t) - \chi_{\mathcal{A}''}(t).$$

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The proof for $b(\mathcal{A})$ is analogous but a little more complicated.

Whitney's theorem

To prove: $\chi_{\mathcal{A}}(t) = \chi_{\mathcal{A}'}(t) - \chi_{\mathcal{A}''}(t)$.

Basic tool (H. Whitney, 1935, for linear arrangements). A subarrangement $\mathcal{B} \subseteq \mathcal{A}$ is **central** if $\bigcap_{H \in \mathcal{B}} H \neq \emptyset$.

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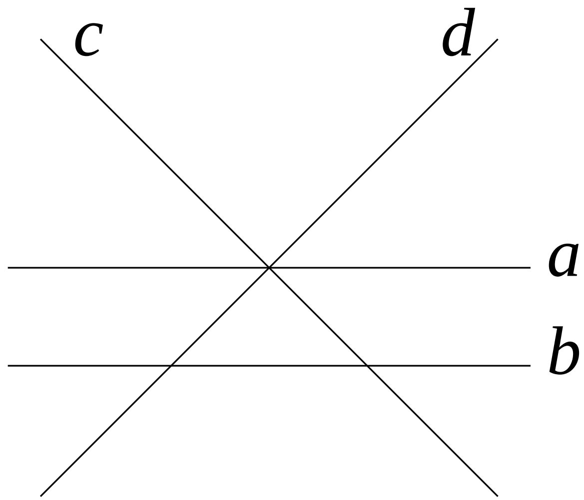
Theorem. Let $\mathcal{A}$ be an arrangement in an n-dimensional vector space. Then

$$\chi_{\mathcal{A}}(t) = \sum_{\substack{\mathcal{B} \subseteq \mathcal{A} \\ \mathcal{B} \text{ central}}} (-1)^{\#\mathcal{B}} t^{n - \mathrm{rk}(\mathcal{B})}.$$

Example of Whitney's theorem



$\mathcal{B}$	$\#\mathcal{B}$	$\mathrm{rk}(\mathcal{B})$
$\emptyset$	0	0
a	1	1
b	1	1
c	1	1
d	1	1
ac	2	2
ad	2	2
bc	2	2
bd	2	2
cd	2	2
acd	3	2

$$\Rightarrow \chi_{\mathcal{A}}(t) = t^2 - 4t + (5-1) = t^2 - 4t + 4.$$

The crosscut theorem

Easy fact: Every interval $[\hat{0}, z]$ of $L(\mathcal{A})$ is a **lattice**, i.e., any two elements x, y have a **meet** (greatest lower bound) $x \wedge y$ and **join** (least upper bound) $x \vee y$.

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Lemma (crosscut theorem for $L(\mathcal{A})$). For all $z \in L(\mathcal{A})$,

$$\mu(z) = \sum_{\substack{\mathcal{B} \subseteq \mathcal{A}_z \\ z = \bigcap_{H \in \mathcal{B}} H}} (-1)^{\#\mathcal{B}}.$$

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Note that $z = \bigcap_{H \in \mathcal{B}} H$ implies that $\mathrm{rk}(\mathcal{B}) = n - \dim z$. Multiply both sides by $t^{\dim(z)}$ and sum over z to obtain

$$\chi_{\mathcal{A}}(t) = \sum_{\substack{\mathcal{B} \subseteq \mathcal{A} \\ \mathcal{B} \text{ central}}} (-1)^{\#\mathcal{B}} t^{n - \mathrm{rk}(\mathcal{B})}. \quad \square$$

Alternative formulation

Later: coefficients of $\chi_{\mathcal{A}}(t)$ **alternate in sign**.
More strongly, if $\mathrm{rk}(x) = i$ then

$$(-1)^i \mu(x) > 0.$$

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$$(-1)^i \mu(x) > 0.$$

Thus:

$$\begin{aligned} r(\mathcal{A}) &= \sum_{x \in L_{\mathcal{A}}} |\mu(x)| \\ b(\mathcal{A}) &= \left| \sum_{x \in L_{\mathcal{A}}} \mu(x) \right| . \end{aligned}$$

A corollary

Corollary. *Let* $\mathcal{A}$ *be a real arrangement. Then* $r(\mathcal{A})$ *and* $b(\mathcal{A})$ *depend only on* $L(\mathcal{A})$*.*

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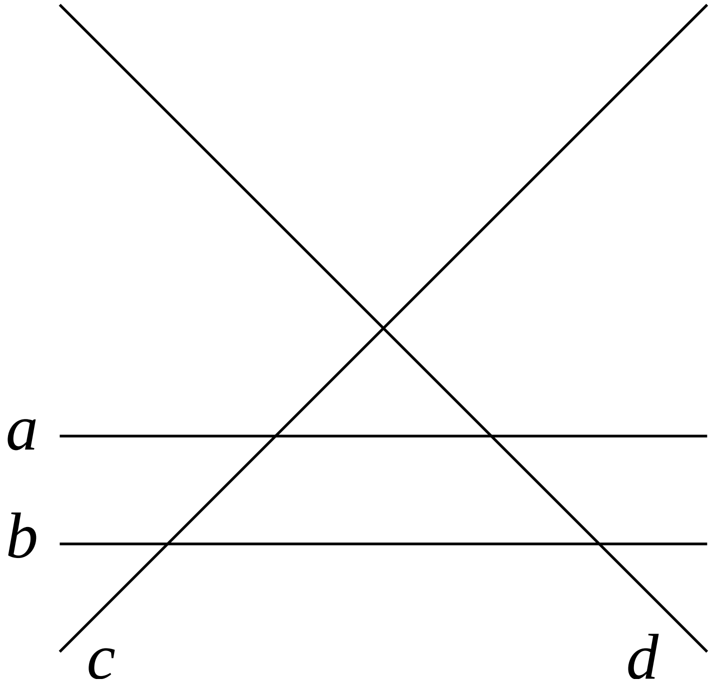

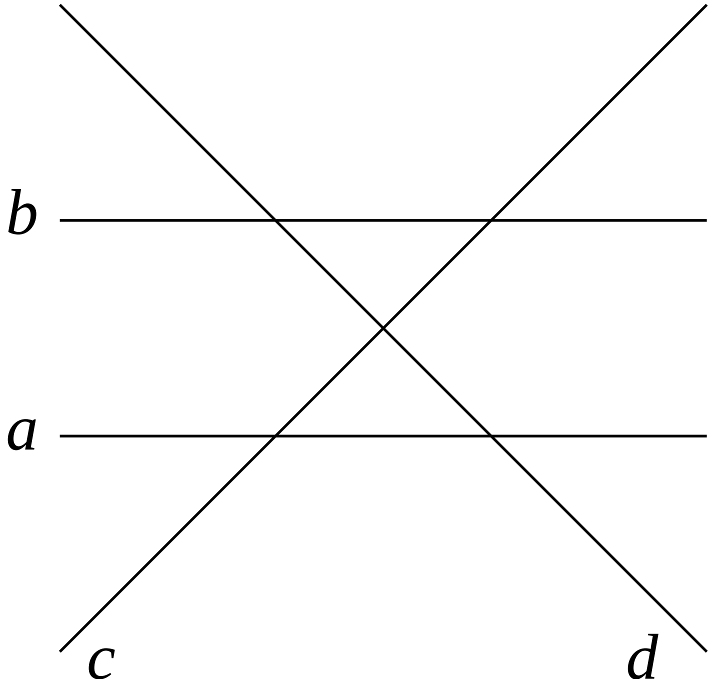

Faces

$\mathcal{R}(\mathcal{A})$: set of regions of $\mathcal{A}$

Definition. A (closed) **face** of a real arrangement $\mathcal{A}$ is a set

$$\emptyset \neq \boldsymbol{F} = \overline{R} \cap x,$$

where $R \in \mathcal{R}(\mathcal{A})$, $x \in L(\mathcal{A})$, and $\overline{\boldsymbol{R}}$ = closure of R.

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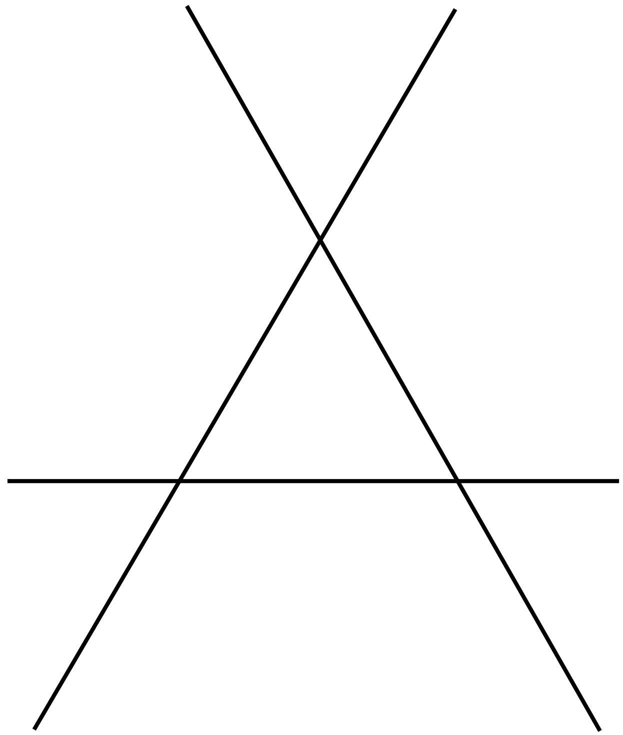
$\boldsymbol{f_k(\mathcal{A})}$: number of k-dimensional faces (**k-faces**) of $\mathcal{A}$

Example of $f_i(\mathcal{A})$



$$f_0(\mathcal{A}) = 3, \;\; f_1(\mathcal{A}) = 9, \;\; f_2(\mathcal{A}) = r(\mathcal{A}) = 7$$

Formula for $f_k(\mathcal{A})$

$$f_k(\mathcal{A}) = \sum_{\substack{x \in L(\mathcal{A}) \\ \mathrm{corank}(x)=k}} \sum_{y \geq x} |\mu(x,y)|$$

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$$f_k(\mathcal{A}) = \sum_{\substack{x \in L(\mathcal{A}) \\ \mathrm{corank}(x)=k}} \sum_{y \geq x} |\mu(x,y)|$$

Proof. Easy consequence of Zaslavsky's formula for $r(\mathcal{A})$. $\square$

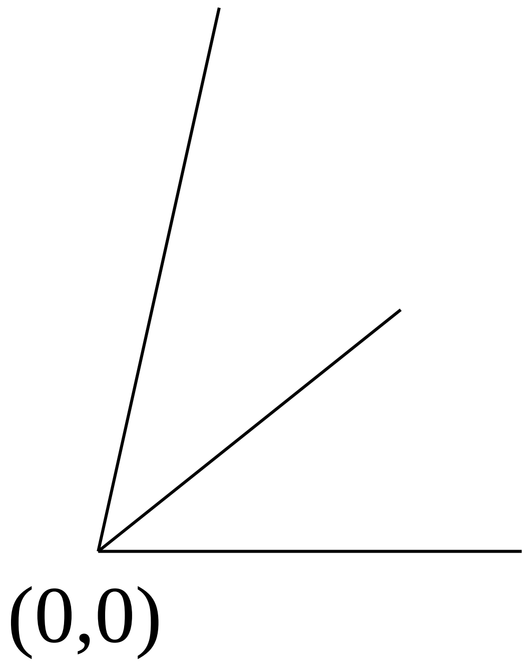
Zonotopes

Let $X, Y \subseteq K^n$

Minkowski sum:
$\boldsymbol{X + Y} = \{x + y \ : \ x \in X, \ y \in Y\}$

zonotope: a Minkowski sum $L_1 + \cdots + L_k$ of line segments in $\mathbb{R}^n$

Example of zonotope



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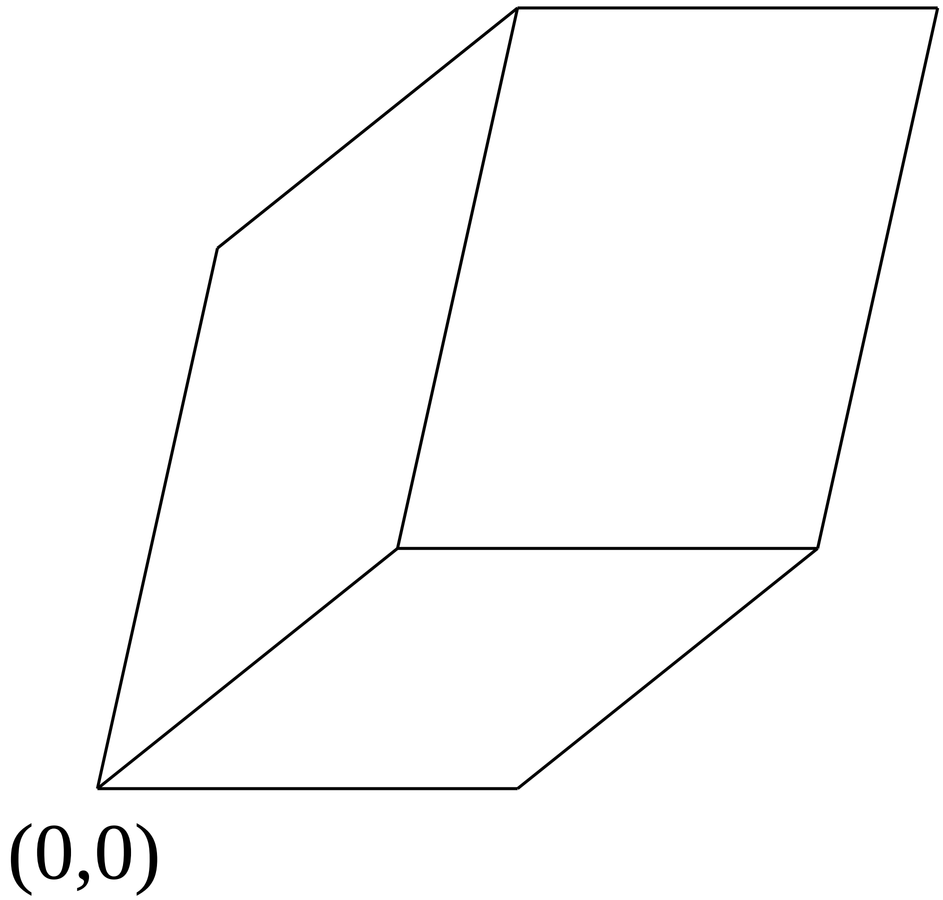

Characterization of zonotopes

Theorem. *Let $\mathcal{P}$ be a convex polytope. The following are equivalent.*

- $\mathcal{P}$ is a zonotope.
- Every face of $\mathcal{P}$ is centrally-symmetric.
- Every 2-dimensional face of $\mathcal{P}$ is centrally-symmetric.

The zonotope of a real arrangement

$\mathcal{A}$: a real central arrangement

$n_1, \ldots, n_k$: normals to $H \in \mathcal{A}$

L_i: line segment from 0 to n_i

$Z(\mathcal{A})$: the zonotope $L_1 + \cdots + L_k$

Number of faces of $Z(\mathcal{A})$

Theorem. *Let* $f_i(Z(\mathcal{A}))$ *denote the number of* i*-dimensional faces of* $Z(\mathcal{A})$*. Then*

$$f_i(Z(\mathcal{A})) = f_{n-i}(\mathcal{A}).$$

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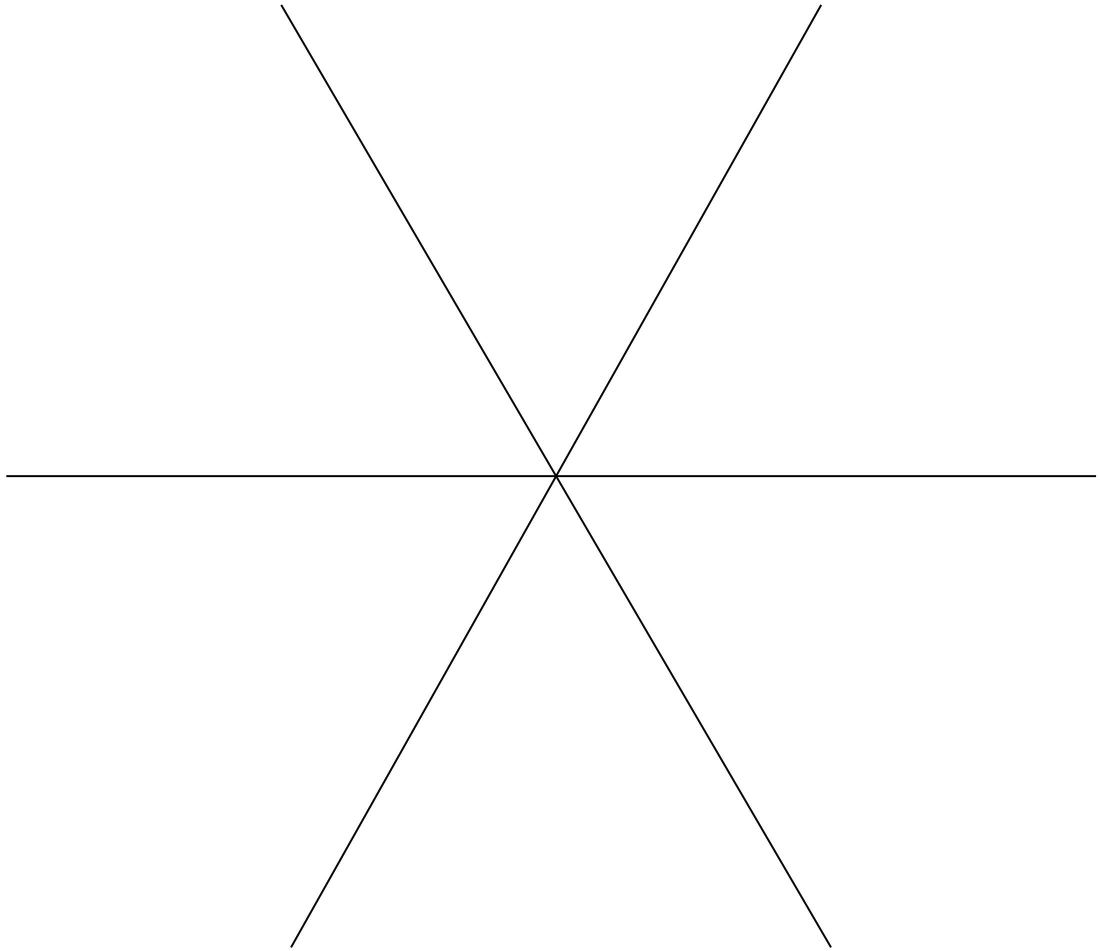
Informally, $Z(\mathcal{A})$ is a “dual object” to $\mathcal{A}$.

An example of $Z(\mathcal{A})$



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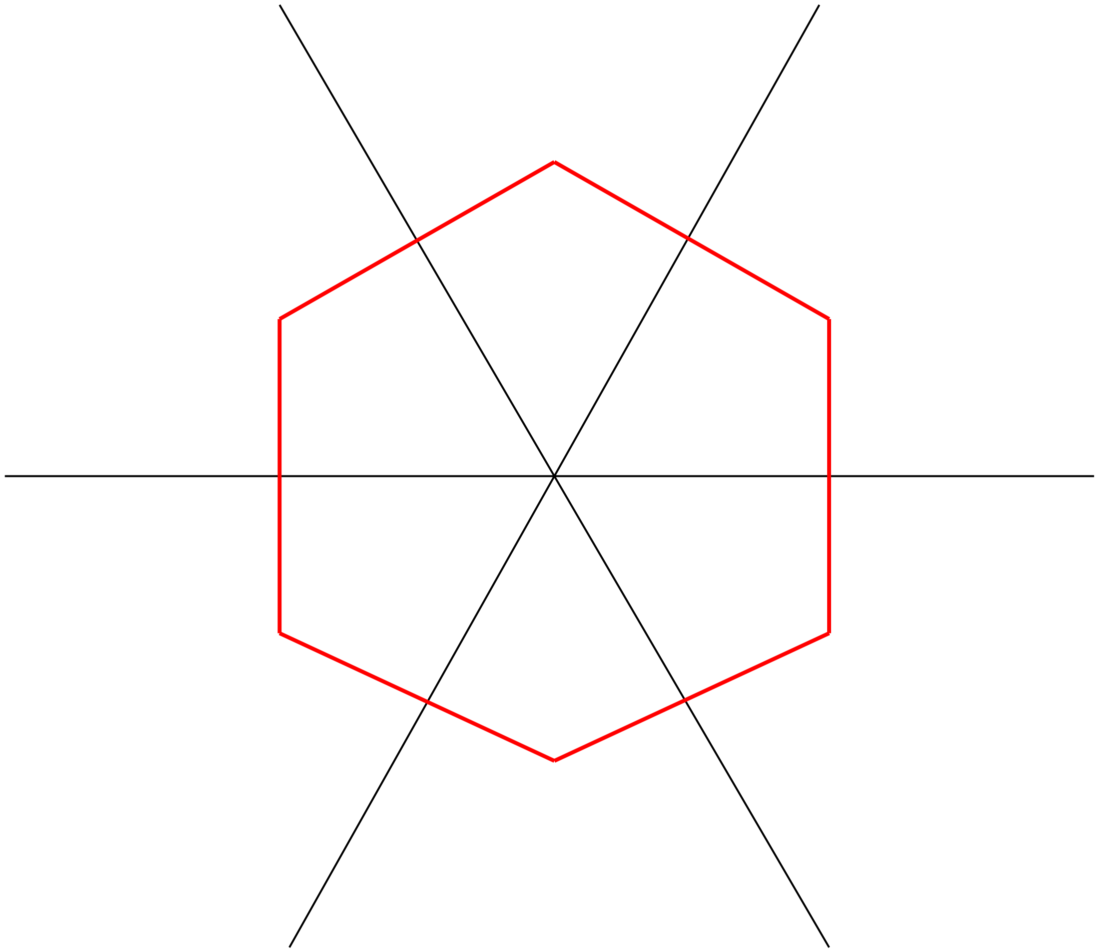

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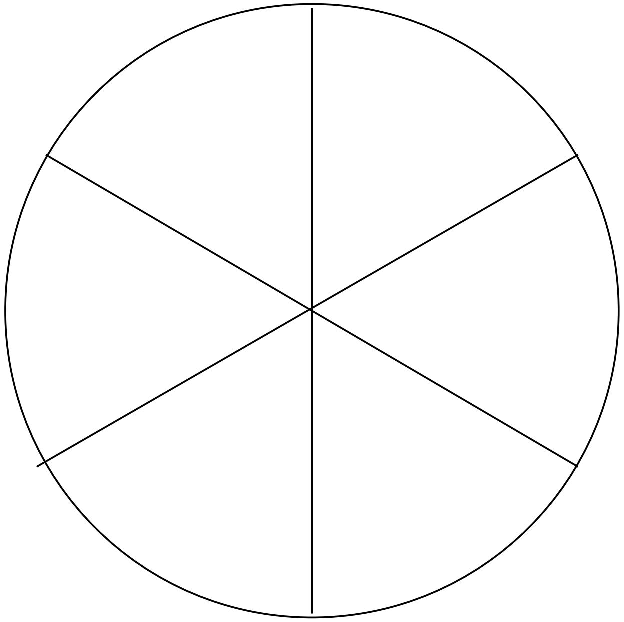

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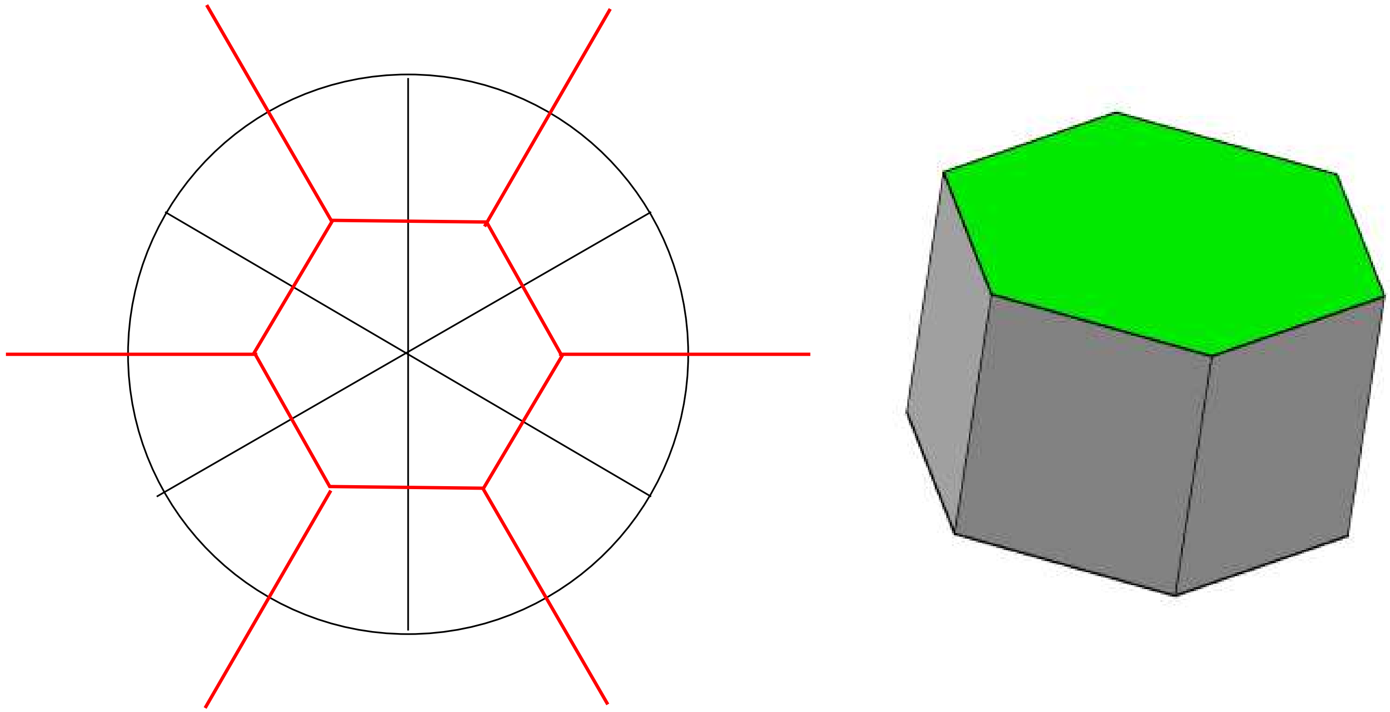
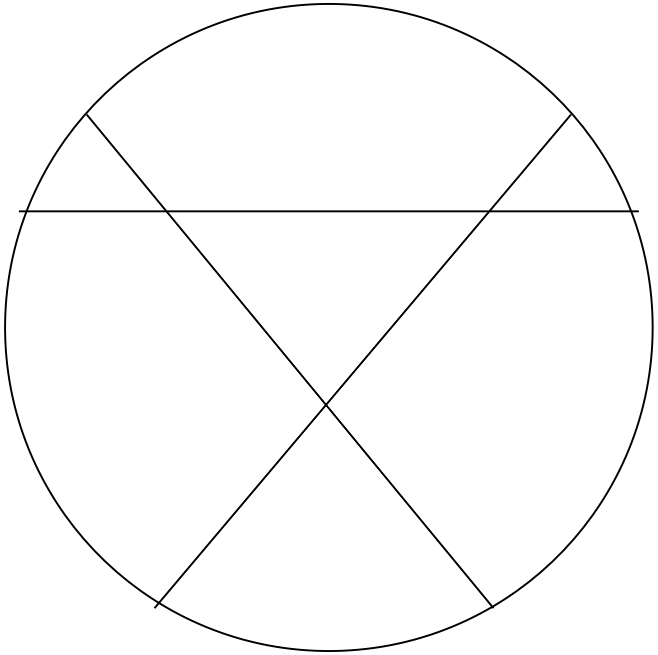


hexagonal prism

Another example



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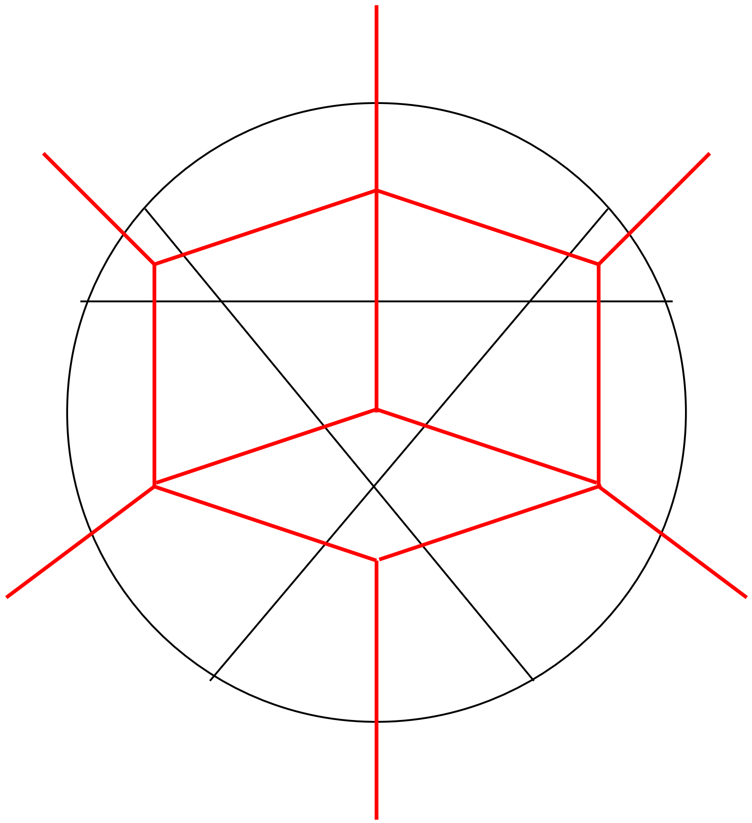

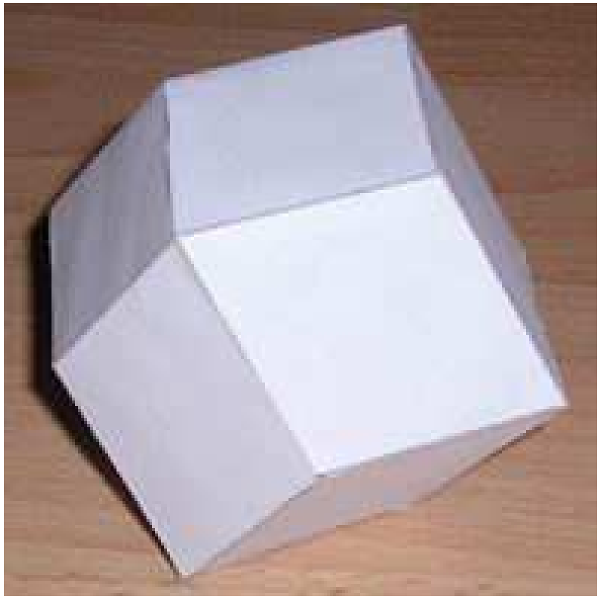

rhombic dodecahedron

Graphical arrangements

G: graph on vertex set $[n]$ (no loops or multiple edges)

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If $G = K_n$, the **complete graph** on $[n]$, then $\mathcal{A}_{K_n}$ is the **braid arrangement** $\boldsymbol{\mathcal{B}_n}$.

Set partitions

partition of a finite set S: $\boldsymbol{\pi} = \{B_1, \ldots, B_k\}$, such that

$$B_i \neq \emptyset, \ \ \bigcup B_i = S, \ \ B_i \cap B_j = \emptyset \ (i \neq j)$$

B_i is a **block** of π.

$\boldsymbol{\Pi_S}$: set of partitions of S

Let $\pi, \sigma \in \Pi_S$. Then π is a **refinement** of σ, written $\boldsymbol{\pi \leq \sigma}$, if every block of π is contained in a block of σ.

The bond lattice of G

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connected partition of $[n]$: a partition of $[n]$ for which each block induces a connected subgraph of G

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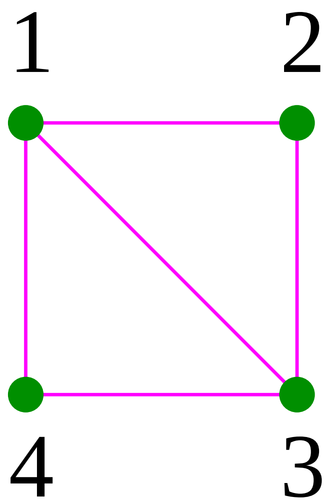
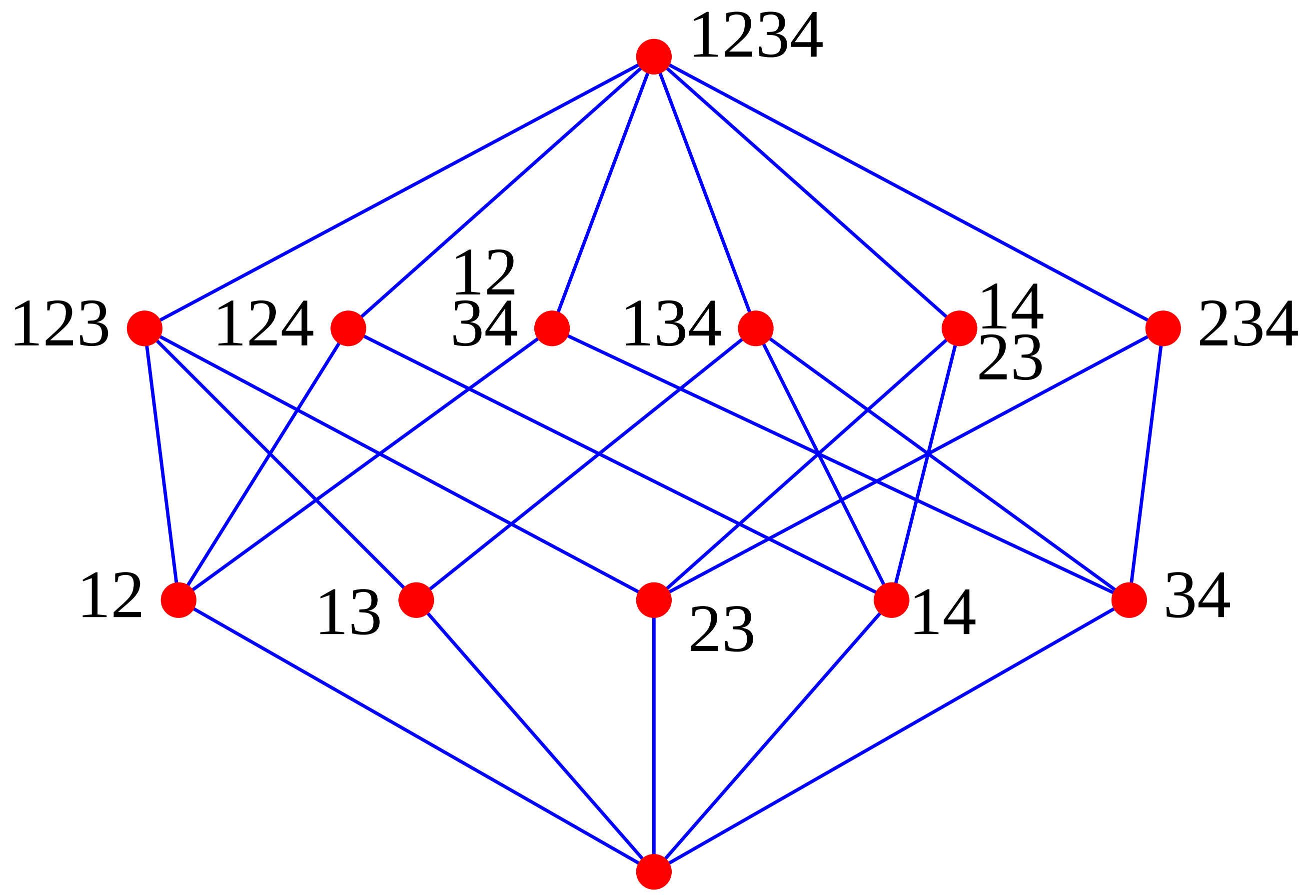
bond lattice $L(G)$ of G: set of connected partitions of $[n]$, ordered by refinement

Example of bond lattice



G



$L(G)$

Bond lattices and intersection posets

G: graph with bond lattice $L(G)$

$\mathcal{A}_G$: graphical arrangement

Theorem. $L(G) \cong L(\mathcal{A}(G))$

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Proof. Let H_{ij} be the hyperplane defined by $x_i = x_j$, $ij \in E(G)$. Let $x \in L(\mathcal{A})$. Define vertices $i \sim j$ if $x \subseteq H_{ij}$. Then $\sim$ is an equivalence relation whose equivalence classes form a connected partition of $[n]$, etc. $\square$

Chromatic polynomial of G

coloring of G is $\kappa\colon [n] \to \mathbb{P} = \{1, 2, \ldots\}$

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Easy fact: $\chi_G(q) \in \mathbb{Z}[q]$

$\chi_{\mathcal{A}(G)}(t)$

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Proof. Let $\sigma \in L(G)$.

$\chi_\sigma(q) =$ number of $f : [n] \to [q]$ such that:

- a, b in same block of $\sigma \Rightarrow f(a) = f(b)$
- a, b in different blocks, $ab \in E \Rightarrow f(a) \neq f(b)$.

Continuation of proof

Given **any** $f : [n] \to [q]$, there is a unique $\sigma \in L(G)$ such that f is enumerated by $\chi_\sigma(q)$. Hence $\forall \pi \in L(G)$,

$$q^{\#\pi} = \sum_{\sigma \geq \pi} \chi_\sigma(q).$$

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Note $\chi_{\hat{0}}(q) = \chi_G(q)$. $\square$

Characteristic polynomial of $\mathcal{B}_n$

Recall: $\mathcal{B}_n = \mathcal{A}(K_n)$ (braid arrangement)

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the lattice of **all** partitions of $[n]$ (ordered by refinement)

Clearly $\chi_{K_n}(q) = q(q-1)\cdots(q-n+1)$.

$$\Rightarrow \chi_{\mathcal{B}_n}(t) = t(t-1)\cdots(t-n+1).$$

Chordal graphs

A graph G is **chordal** (**triangulated, rigid circuit**) if the vertices can be ordered $v_1, \ldots, v_n$ so that for all i, v_i is connected to a **clique** (complete subgraph) of the restriction of G to $\{v_1, \ldots, v_{i-1}\}$.

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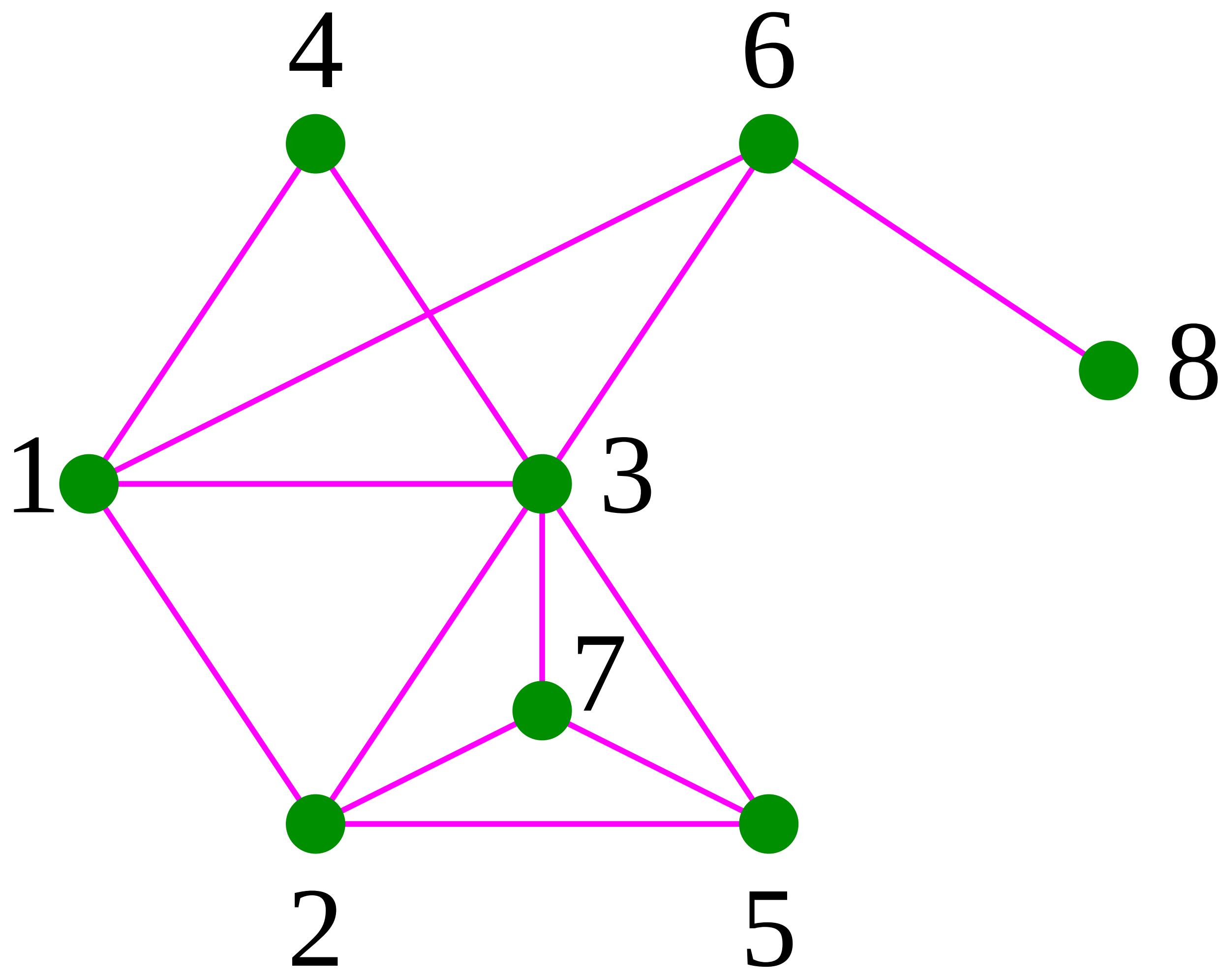
Known fact: G is chordal if and only if every cycle of length at least four has a chord.

Example of a chordal graph



Chordal graph coloring

Let $v_1, \ldots, v_n$ be a vertex ordering so that for all i, v_i is connected to a clique of the restriction $\boldsymbol{G_{i-1}}$ of G to $\{v_1, \ldots, v_{i-1}\}$.

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$$\chi_G(q) = (q - a_1)(q - a_2) \cdots (q - a_n).$$

Acyclic orientations

Orientation of G: assignment $\mathfrak{o}$ of a direction $i \to j$ or $j \to i$ to each edge.

Acyclic orientation: an orientation with no directed cycles

$\chi_G(-1)$

Given $\mathfrak{o}$, define

$$R_{\mathfrak{o}} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \,:\, x_i < x_j \text{ whenever } i \to j \text{ in } \mathfrak{o}\}.$$

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This proof is due to Greene (1977).

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For simplicity we deal only with hyperplane arrangements, though the “right” level of generality is **matroid theory**.

Broken circuits

$\mathcal{A}$: **central** arrangement

circuit: a minimal linearly dependent subset of $\mathcal{A}$

$H_1, H_2, \ldots, H_m$: ordering of $\mathcal{A}$

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broken circuit complex:

$$\mathrm{BC}(\mathcal{A}) = \{F \subseteq \mathcal{A} : F \text{ contains no broken circuit}\}$$

An example

Note: $\mathrm{BC}(\mathcal{A})$ is a **simplicial complex**, i.e.,
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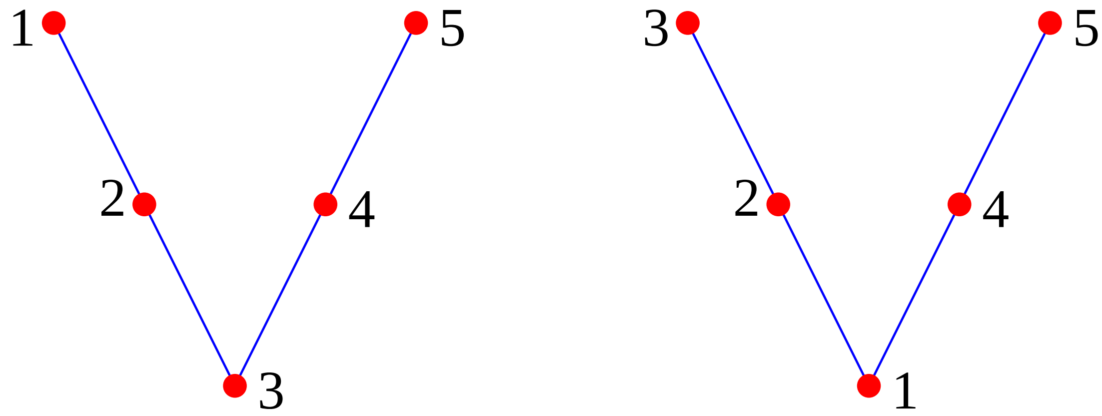

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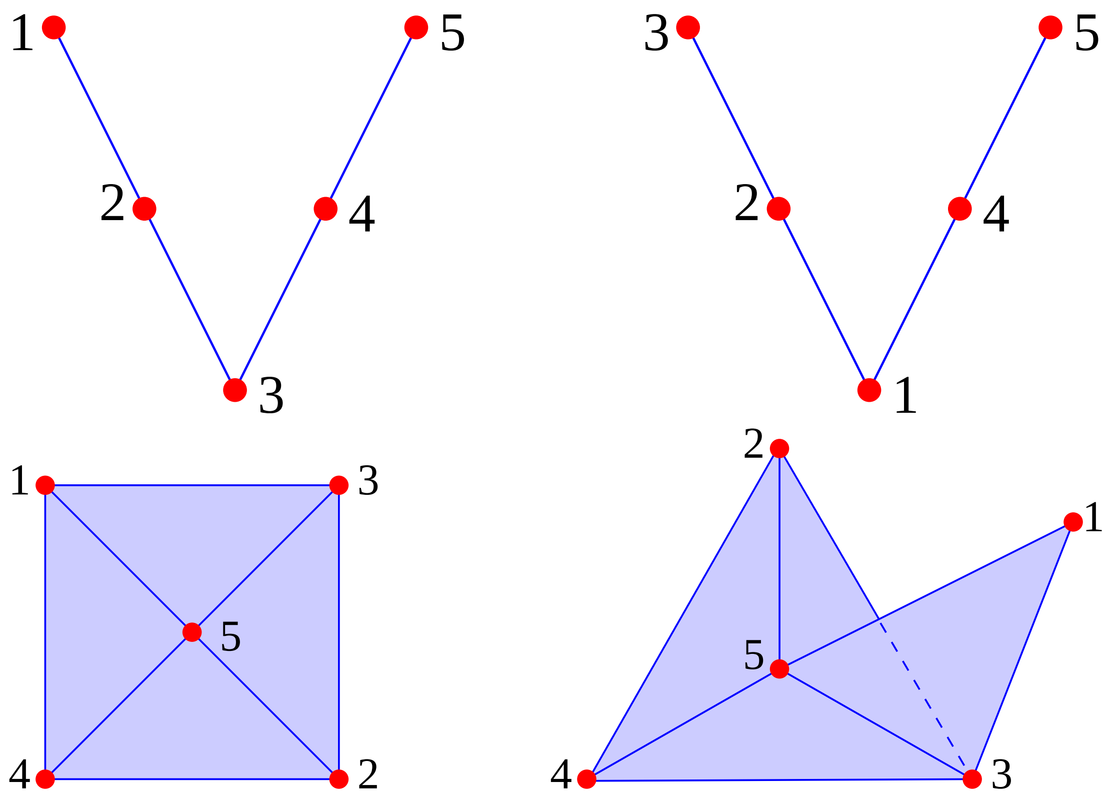
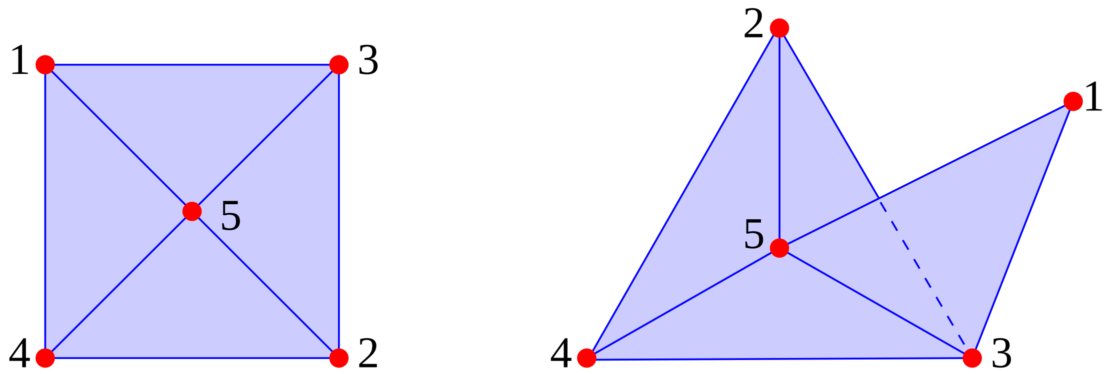


Example (continued)



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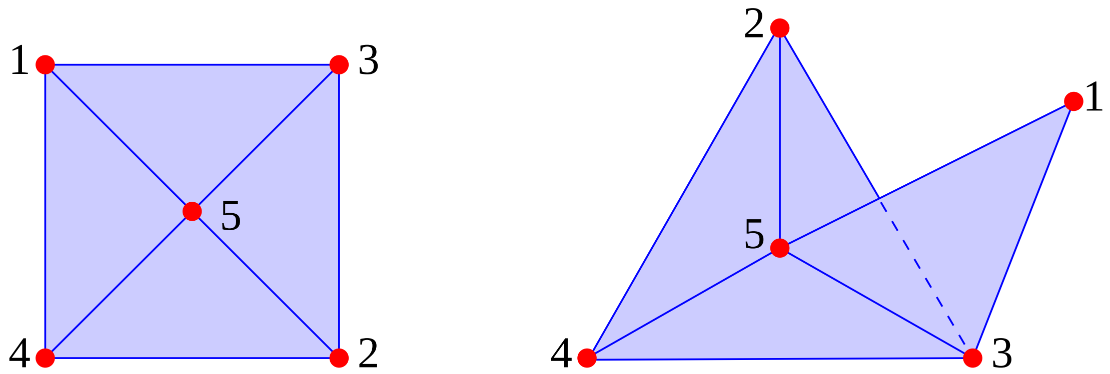
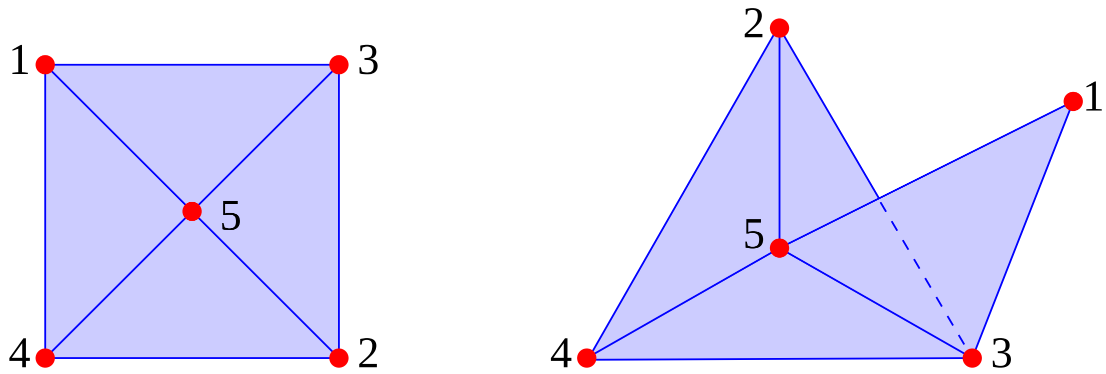


$\boldsymbol{f_i = f_i(\mathrm{BC}(\mathcal{A}))}$: # i-dim. faces of $\mathrm{BC}(\mathcal{A})$

$$f_{-1} = 1, \quad f_0 = 5, \quad f_1 = 8, \quad f_2 = 4$$

Example (continued)



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$$\chi_{\mathcal{A}}(t) = t^3 - 5t^2 + 8t - 4$$

Covers

$\boldsymbol{L} = L_{\mathcal{A}}$

y **covers** x in L: $x < y$, $\nexists x < z < y$

$\boldsymbol{\mathcal{E}(L)}$: edges of Hasse diagram of L, i.e,

$$\mathcal{E}(L) = \{(x, y) \ : \ y \text{ covers } x\}$$

Labelings

$\boldsymbol{\lambda}\colon \mathcal{E}(L) \to \mathbb{P}$ is a **labeling** of L

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If $\boldsymbol{C}$: $x = x_0 < x_1 < \cdots < x_k = y$ is a saturated chain from x to y (i.e., each x_{i+1} **covers** x_i), define

$$\boldsymbol{\lambda(C)} = (\lambda(x_0, x_1), \lambda(x_1, x_2), \ldots, \lambda(x_{k-1}, x_k))$$

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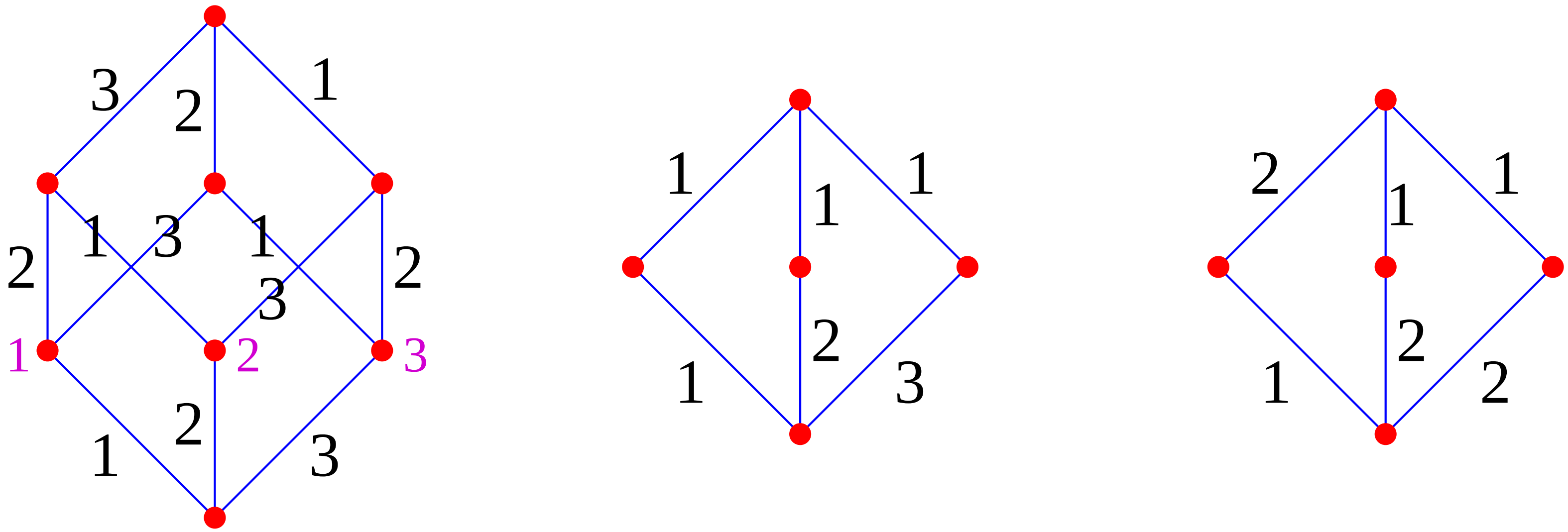
C is **increasing** if

$$\lambda(x_0, x_1) \leq \lambda(x_1, x_2) \leq \cdots \leq \lambda(x_{k-1}, x_k).$$

E-labelings



(a) (b) (c)

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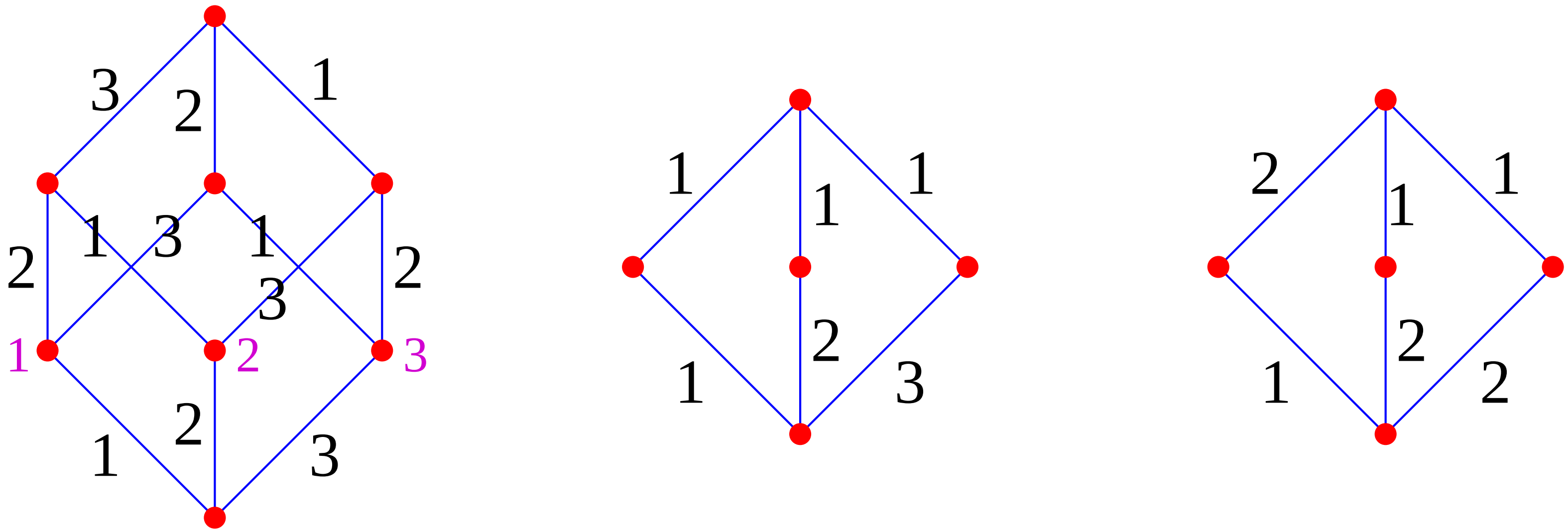


(a) (b) (c)

E-labeling: a labeling for which every interval $[x, y]$ has a **unique** increasing chain.

Labeling and Möbius functions

Theorem. Let λ be an E-labeling of L, and let $x \leq y$ in L, $\operatorname{rank}(x, y) = k$. Then $(-1)^k \mu(x, y)$ is equal to the number of strictly decreasing saturated chains from x to y, i.e.,

$$(-1)^k \mu(x, y) = \#\{x = x_0 < x_1 < \cdots < x_k = y :$$

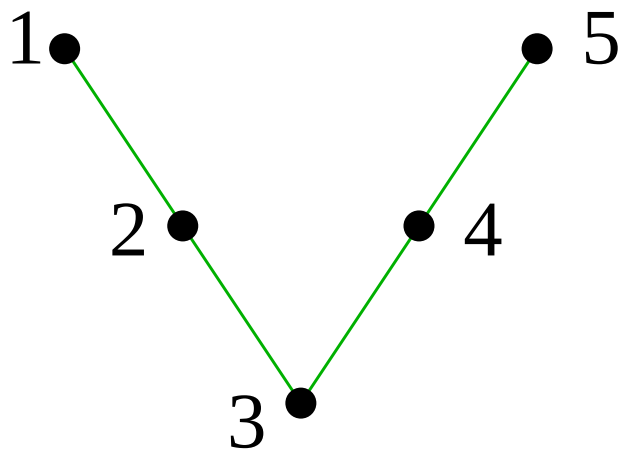
$$\lambda(x_0, x_1) > \lambda(x_1, x_2) > \cdots > \lambda(x_{k-1}, x_k)\}.$$

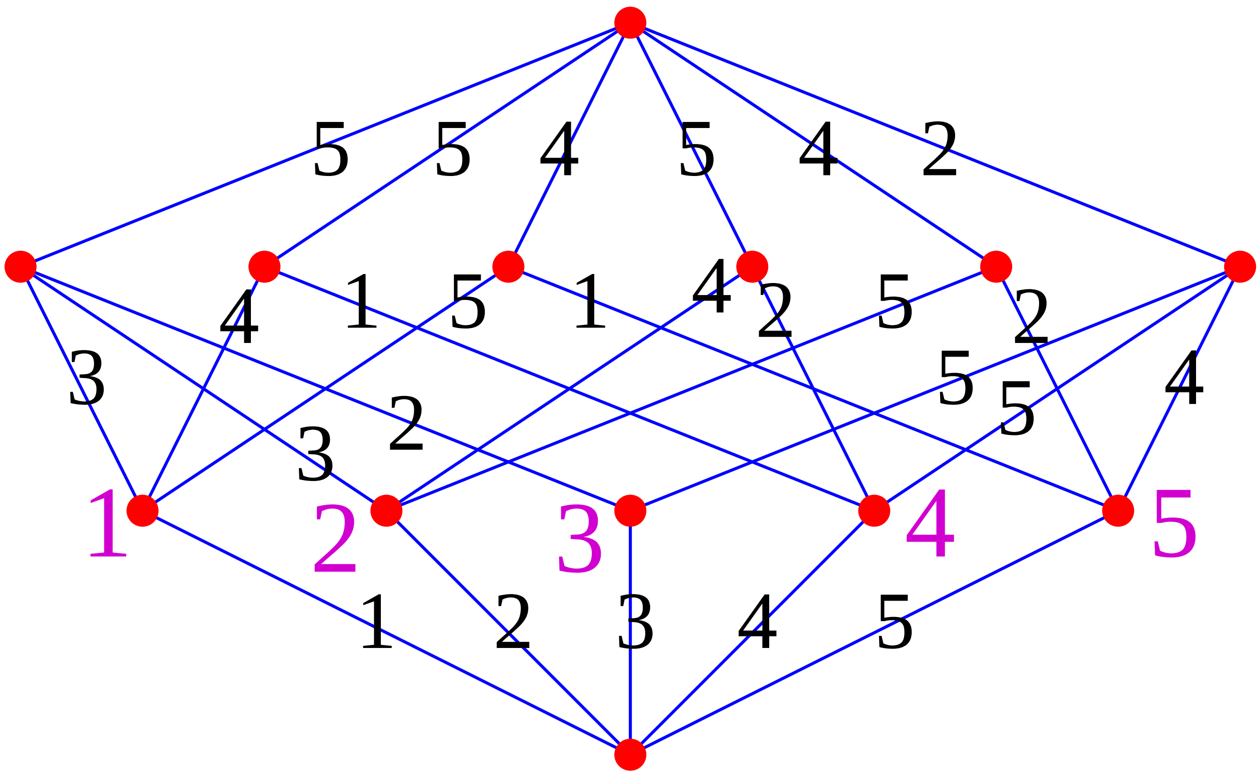
Labeling $L(\mathcal{A})$

$\boldsymbol{H_1, \ldots, H_m}$: ordering of $\mathcal{A}$ (as before)
If y covers x in $L(\mathcal{A})$ then define

$$\tilde{\boldsymbol{\lambda}}\boldsymbol{(x, y)} = \max\{i \,:\, x \vee H_i = y\}.$$

Example of λ





Properties of $\boldsymbol{\lambda}$

Claim 1. Define $\boldsymbol{\lambda}\colon \mathcal{E}(L(\mathcal{A})) \to \mathbb{P}$ by

$$\boldsymbol{\lambda(x, y)} = m + 1 - \tilde{\lambda}(x, y).$$

Then λ is an E-labeling.

Properties of λ

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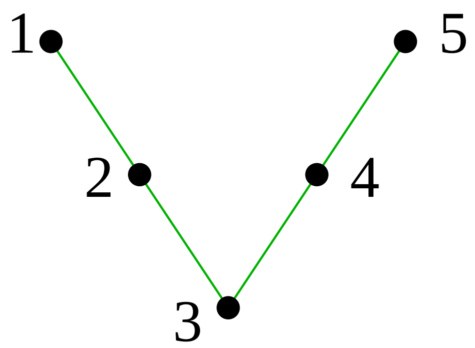
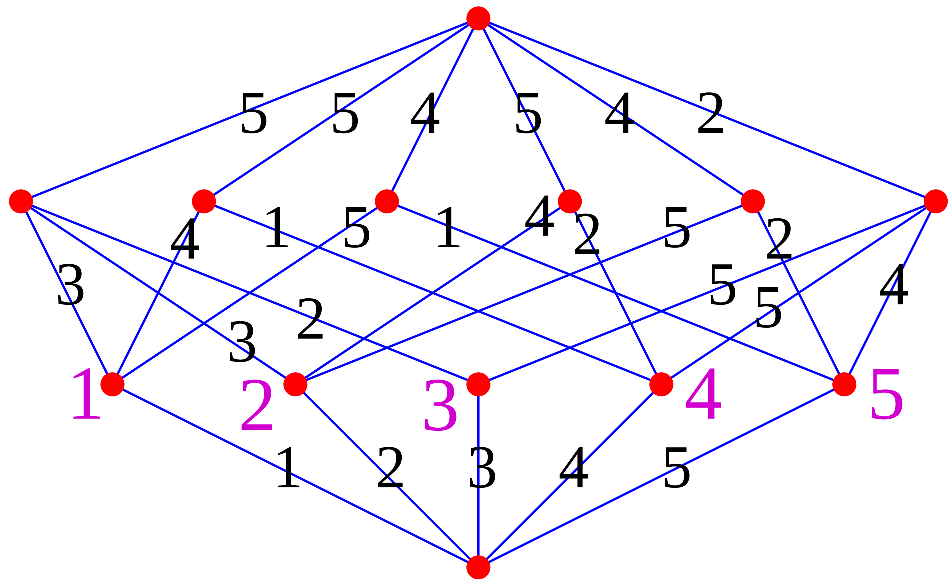
Claim 2. The broken circuit complex $\mathrm{BC}(M)$ consists of all chain labels $\tilde{\lambda}(C)$ (regarded as a **set**), where C is an increasing saturated chain from $\hat{0}$ to some $x \in L(M)$. Moreover, all such $\tilde{\lambda}(C)$ are distinct.

Example of Claim 2.





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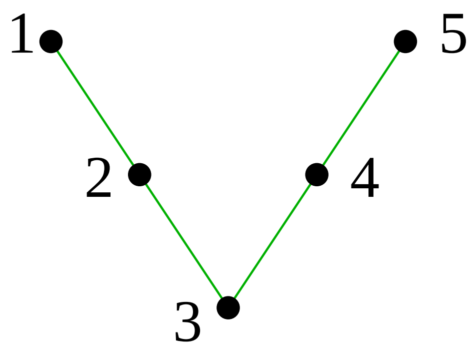

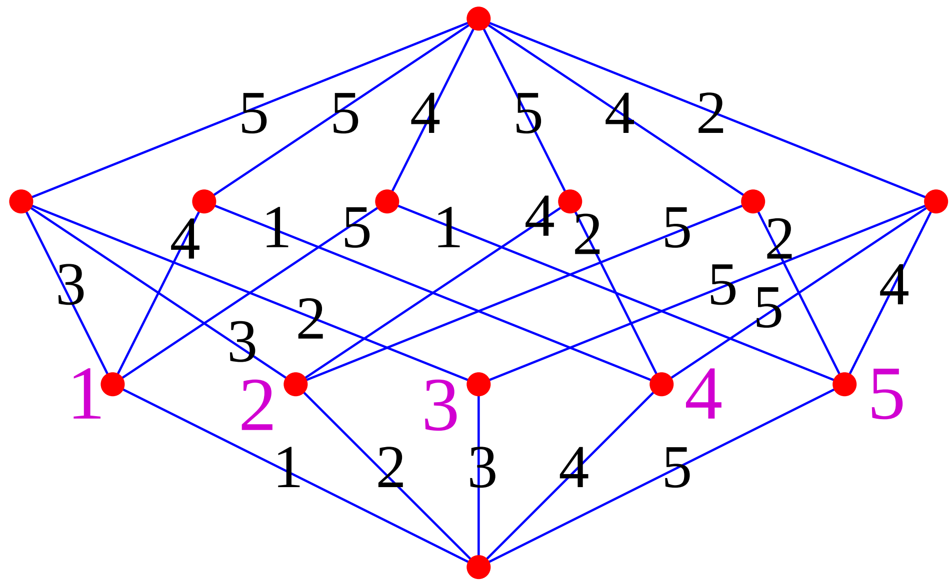

$$\text{broken circuits}: \ 12, 34, 124$$

$$\mathrm{BC}(\mathcal{A}) = \{\emptyset, 1, 2, 3, 4, 5, 13, 14, 15, 23, 24, 25, 35, 45,$$
$$135, 145, 235, 245\}$$

Broken circuit theorem

Immediate consequence of Claims 1 and 2:

Theorem. $$\chi_{\mathcal{A}}(t) = \sum_{F \in \mathrm{BC}(\mathcal{A})} (-1)^{\#F} t^{n-\#F}$$

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Corollary. *The coefficients of $\chi_{\mathcal{A}}(t)$* ***alternate in sign****, i.e., $\chi_{\mathcal{A}}(t) = t^n - a_1 t^{n-1} + a_2 t^{n-2} - \cdots$, where $a_i \geq 0$. In fact*

$$(-1)^i \mu(x, y) > 0, \quad \text{where } i = \mathrm{rank}(x, y).$$

A glimpse of topology

$[\boldsymbol{x}, \boldsymbol{y}]$: (finite) interval in a poset P

$\boldsymbol{c_i}$: number of chains $x = x_0 < x_1 < \cdots < x_i = y$

Note. $c_0 = 0$ unless $x = y$.

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Philip Hall's theorem (1936).
$\mu(x, y) = c_0 - c_1 + c_2 - \cdots$

The order complex

$\boldsymbol{P}$: a poset

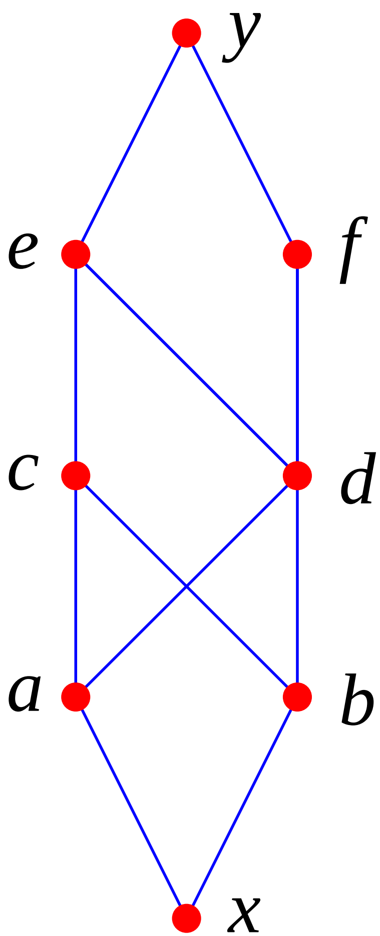
order complex of P:

$$\boldsymbol{\Delta(P)} = \{\text{chains of } P\},$$

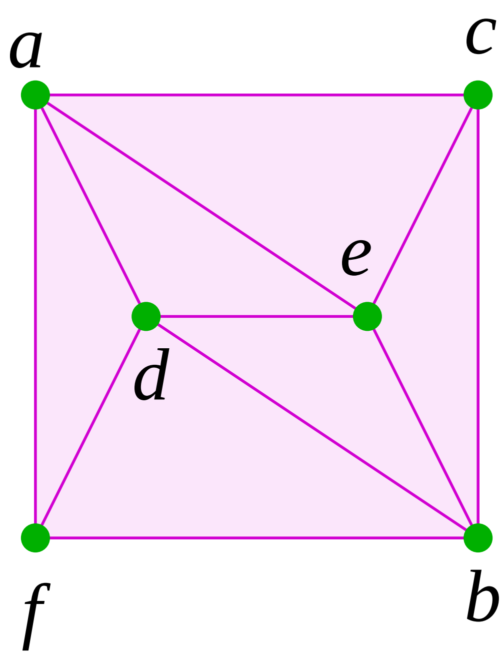
an abstract simplicial complex.

Write $\boldsymbol{\Delta(x,y)}$ for the order complex of the **open** interval $(x,y) = \{z \in P \ : \ x < z < y\}$.

Example of an order complex



P



$\Delta(P)$

Euler characteristic

Δ: finite simplicial complex

$f_i = \#\ i$-dimensional faces of Δ

Note: $f_{-1} = 1$ unless $\Delta = \emptyset$.

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reduced Euler characteristic:
$\boldsymbol{\tilde{\chi}(\Delta)} = -f_{-1} + f_0 - f_1 + f_2 - \cdots$

Note: $\tilde{\chi}(\Delta) = \chi(\Delta) - 1$ unless $\Delta = \emptyset$.

Philip Hall's theorem restated

Theorem. *For* $x < y$ *in a finite poset,*

$$\mu(x, y) = \tilde{\chi}(\Delta(x, y)).$$

Philip Hall's theorem restated

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Recall for any finite simplicial complex Δ,

$$\tilde{\chi}(\Delta) = \sum_j (-1)^j \dim \widetilde{H}_j(\Delta; K),$$

where $\boldsymbol{\widetilde{H}_j(\Delta; K)}$ denotes reduced simplicial homology over the field K.

A topological question

For $x < y$ in $L(\mathcal{A})$, with $i = \mathrm{rank}(x, y)$, we have

$$\boldsymbol{d} := \dim \Delta(x, y) = i - 2.$$

In particular, $(-1)^d = (-1)^i$.

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Is there a topological reason for this?

Folkman's theorem

Previous slide: $\sum_{j=0}^{d}(-1)^{d-j}\dim \widetilde{H}_j(\Delta; K) > 0.$

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Theorem (Folkman, 1966).

$$\widetilde{H}_j(\Delta;K)\begin{cases} = & 0,\ j\neq d \\ \neq & 0,\ j=d. \end{cases}$$

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Early result in **topological combinatorics**.

Cohen-Macaulay posets

A finite poset P is **Cohen-Macaulay** (over K) if after adjoining a top and bottom element to P, every interval $[x, y]$ satisfies:

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Folkman's theorem, restated. *If $\mathcal{A}$ is central then $L(\mathcal{A})$ is Cohen-Macaulay.*

Modular elements

Let $\mathcal{A}$ be central. An element $x \in L(\mathcal{A})$ is **modular** if for all $y \in L$ we have

$$\text{rk}(x) + \text{rk}(y) = \text{rk}(x \wedge y) + \text{rk}(x \vee y).$$

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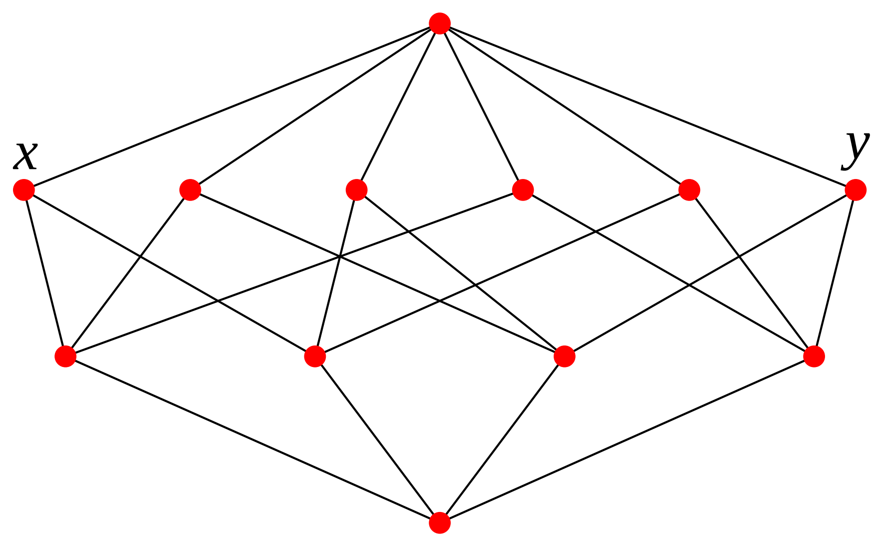

x is **not** modular: $\mathrm{rk}(x) + \mathrm{rk}(y) = 2 + 2 = 4$,
$\mathrm{rk}(x \wedge y) + \mathrm{rk}(x \vee y) = 0 + 3 = 3$

Simple properties

Easy: $\hat{0} = K^n$, $\hat{1} = \bigcap_{H \in \mathcal{A}} H$ (the top element), and each $H \in \mathcal{A}$ is modular.

More properties

$x, y \in L(\mathcal{A})$ are **complements** if $x \wedge y = \hat{0}$, $x \vee y = \hat{1}$.

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Theorem. *Let* $r = \mathrm{rk}(\mathcal{A})$. *Let* $x \in L$. *The following four conditions are equivalent.*

(i) x *is a modular element of* L.

(ii) *If* $x \wedge y = \hat{0}$, *then* $\mathrm{rk}(x) + \mathrm{rk}(y) = \mathrm{rk}(x \vee y)$.

(iii) *If* x *and* y *are complements, then* $\mathrm{rk}(x) + \mathrm{rk}(y) = n$.

(iv) *All complements of* x *are incomparable.*

Two additional results

Theorem.

(a) (transitivity of modularity) *If x is a modular element of L and y is modular in the interval $[\hat{0}, x]$, then y is a modular element of L.*

(b) *If x and y are modular elements of L, then $x \wedge y$ is also modular.*

Modular element factorization thm.

Theorem. *Let z be a modular element of $L(\mathcal{A})$, $\mathcal{A}$ central of rank r. Write* $\boldsymbol{\chi_z(t)} = \chi_{[\hat{0},z]}(t)$. *Then*

$$\chi_L(t) = \chi_z(t) \left[\sum_{y\,:\,y \wedge z = \hat{0}} \mu_L(y) t^{n-\mathrm{rk}(y)-\mathrm{rk}(z)} \right].$$

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Since each $H \in \mathcal{A}$ is modular in $L(\mathcal{A})$, we get:

Corollary. For all $H \in \mathcal{A}$,

$$\chi_L(t) = (t-1) \sum_{y \wedge H = \hat{0}} \mu(y) t^{n-1-\mathrm{rk}(y)}.$$

Supersolvability

A central arrangement $\mathcal{A}$ (or $L(\mathcal{A})$) is **supersolvable** if $L(\mathcal{A})$ has a maximal chain $\hat{0} = x_0 < x_1 < \cdots < x_r = \hat{1}$ of modular elements x_i.

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In this case, let
$\boldsymbol{a_i} = \#\{H \in \mathcal{A} : H \leq x_i,\ H \not\leq x_{i-1}\}$.

Corollary. *If $\mathcal{A}$ is supersolvable, then*

$$\chi_{\mathcal{A}}(t) = t^{n-r}(t - a_1)(t - a_2) \cdots (t - a_r).$$

Chordal graphs, revisited

For what graphs G is $\mathcal{A}_G$ supersolvable?

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If v_i is connected to a_i vertices of G_{i-1}, then

$$\chi_G(q) = (q - a_1)(q - a_2) \cdots (q - a_n).$$

Supersolvable graphs

Suggests that

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In fact:

Theorem. *G is chordal if and only if $\mathcal{A}_G$ is supersolvable.*

Free arrangements

Saito defined **free arrangements** $\mathcal{A}$. **Terao** (1980) proved

$$\chi_{\mathcal{A}}(t) = (t - a_1) \cdots (t - a_n),$$

where $a_i \in \{0, 1, 2, \dots\}$. (Definition not given here.)

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Supersolvable arrangements are free.

Open: is freeness of $\mathcal{A}$ a combinatorial property? That is, does it just depend on $\chi_{\mathcal{A}}(t)$?

Finite fields and good reduction

$\mathcal{A}$: arrangement over $\mathbb{Q}$

By multiplying hyperplane equations by a suitable integer, can assume $\mathcal{A}$ is defined over $\mathbb{Z}$.

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Consider coefficients modulo a prime p to get an arrangment $\mathcal{A}_q$ defined over the finite field $\mathbb{F}_q$, $q = p^k$.

$\mathcal{A}_q$ has **good reduction** if $L_{\mathcal{A}} \cong L_{\mathcal{A}_q}$.

Almost always good reduction

Example. $\mathcal{A} = \{2, 10\}$: affine arrangement in $\mathbb{Q}^1 = \mathbb{Q}$. Good reduction $\Leftrightarrow p \neq 2, 5$.

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Proof idea. Consider minors of the coefficient matrix, etc. $\square$

The finite field method

Theorem. *Let $\mathcal{A}$ be an arrangement in $\mathbb{Q}^n$, and suppose that $L(\mathcal{A}) \cong L(\mathcal{A}_q)$ for some prime power q. Then*

$$\begin{aligned}\chi_{\mathcal{A}}(q) &= \#\left(\mathbb{F}_q^n - \bigcup_{H\in\mathcal{A}_q} H\right)\\ &= q^n - \#\bigcup_{H\in\mathcal{A}_q} H.\end{aligned}$$

Proof

Let $x \in L(\mathcal{A}_q)$ so $\#x = q^{\dim(x)}$ (computed either over $\mathbb{Q}$ or F_q). Define $f, g : L(\mathcal{A}_q) \to \mathbb{Z}$ by

$$\begin{aligned} f(x) &= \#x \\ g(x) &= \#\left(x - \bigcup_{y>x} y\right) \\ \Rightarrow g(\hat{0}) &= g(\mathbb{F}_q^n) = \#\left(\mathbb{F}_q^n - \bigcup_{H\in\mathcal{A}_q} H\right). \end{aligned}$$

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$$x = \hat{0} \Rightarrow g(\hat{0}) = \sum_{y} \mu(y) q^{\dim(y)} = \chi_{\mathcal{A}}(q) \quad \square$$

Graphical arrangements

G: graph on vertex set $1, 2, \ldots, n$

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The braid arrangement $\mathcal{B}(B_n)$

$$x_i - x_j = 0,\ \ 1 \leq i < j \leq n$$
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Choose α_1 in $q-1$ ways, then α_2 in $q-3$ ways, etc.

Characteristic polynomial of $\mathcal{B}(B_n)$

$$\Rightarrow \chi_{\mathcal{B}(B_n)}(q) = (q-1)(q-3)\cdots(q-2n+1)$$

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In fact, $\mathcal{B}(B_n)$ is supersolvable.

$\mathcal{B}(D_n)$

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Exercise: If $n \ge 3$ then

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Not supersolvable ($n \ge 4$), but it is free.

The Shi arrangement

$$\mathcal{S}_n: \ x_i - x_j = 0, 1, \quad 1 \le i < j \le n$$

$$\dim \mathcal{S}_n = n, \ \ \mathrm{rk}\, \mathcal{S}_n = n-1, \ \ \#\mathcal{S}_n = n(n-1)$$

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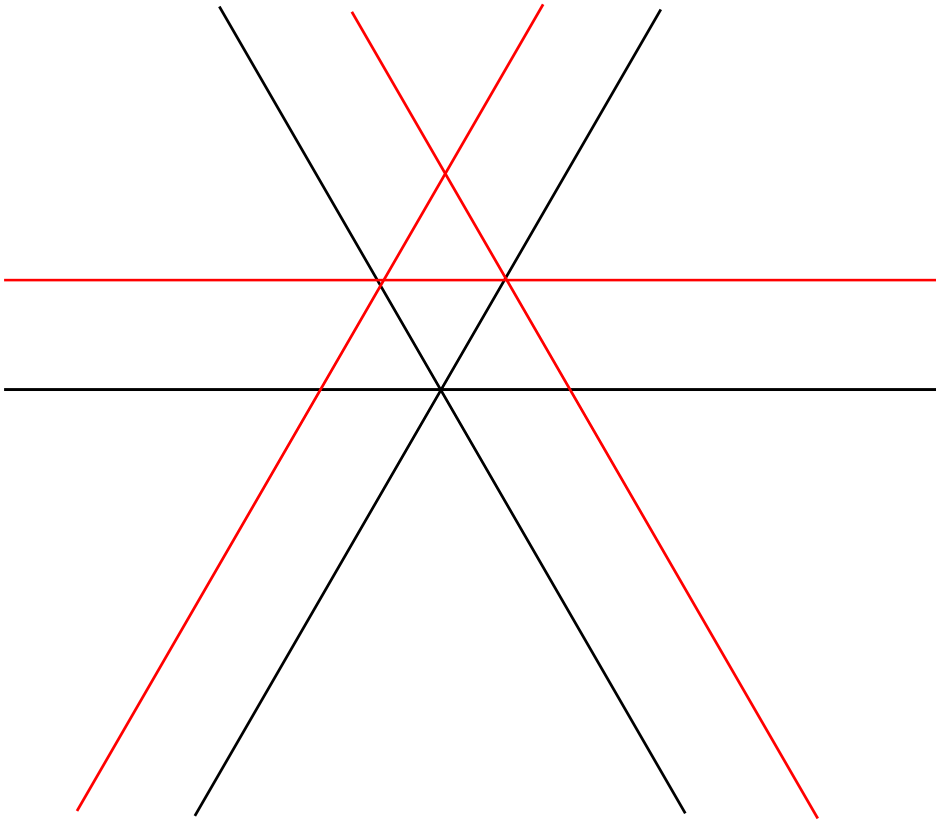

Characteristic polynomial of $\mathcal{S}_n$

Theorem. $\chi_{\mathcal{S}_n}(t) = t(t-n)^{n-1}$, so

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Proof. Finite field method $\Rightarrow$

$$\chi_{\mathcal{S}_n}(p) = \#\{(\alpha_1, \ldots, \alpha_n) \in \mathbb{F}_p^n :$$

$$i < j \Rightarrow \alpha_i \neq \alpha_j \text{ and } \alpha_i \neq \alpha_j + 1\},$$

for $p >> 0$ (actually, all p).

Proof continued

Choose $\boldsymbol{\pi} = (B_1, \ldots, B_{p-n})$ such that

$$\bigcup B_i = [n], \quad B_i \cap B_j = \emptyset \text{ if } i \neq j, \quad 1 \in B_1.$$

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For $2 \leq k \leq n$ there are $p - n$ choices for i such that $k \in B_i$, so $\boldsymbol{(p-n)^{n-1}}$ choices in all.

Circular placement of $\mathbb{F}_p$

Arrange the elements of $\mathbb{F}_p$ clockwise on a circle.

Place $1, 2, \ldots, n$ on some n of these points as follows.

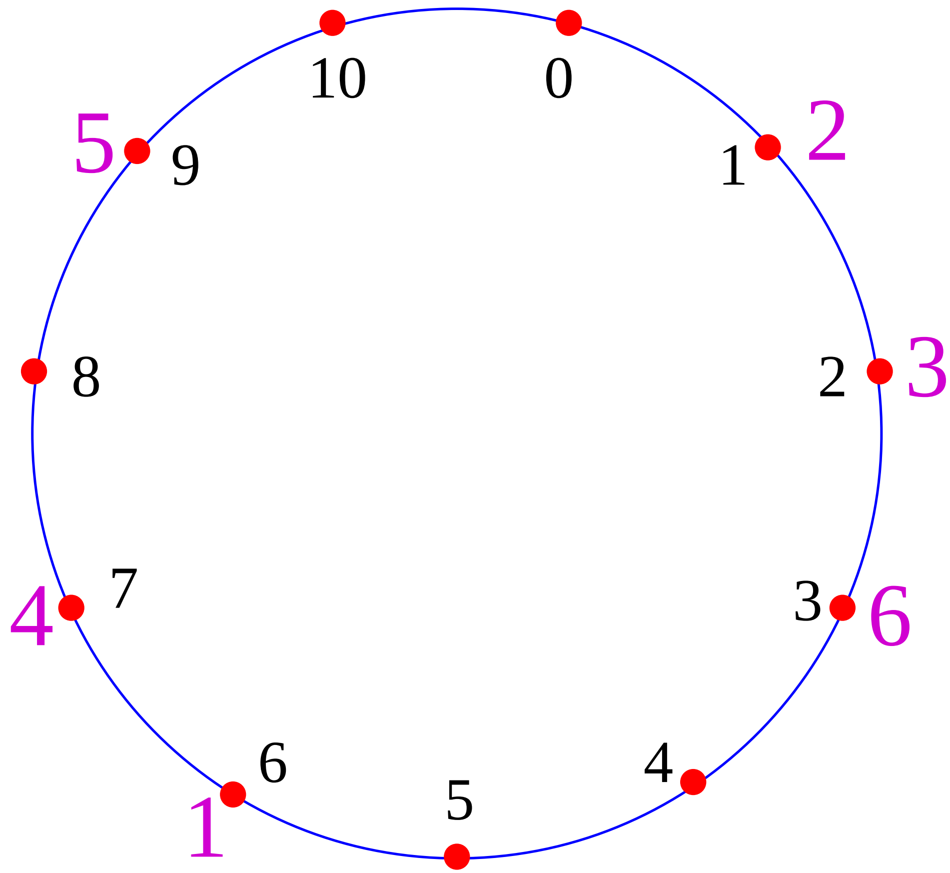
Place elements of B_1 consecutively (clockwise) in increasing order with 1 placed at some element $\alpha_1 \in \mathbb{F}_p$.

Skip a space and place the elements of B_2 consecutively in increasing order.

Skip another space and place the elements of B_3 consecutively in increasing order, etc.

Example for $p = 11, n = 6$

$$\pi = (\{1,4\}, \{5\}, \emptyset, \{2,3,6\}, \emptyset)$$



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Gives bijection

$$\{(\pi = (B_1, \ldots, B_{p-n}), \alpha_1)\} \to \mathbb{F}_p^n - \bigcup_{H \in (\mathcal{S}_n)_p} H.$$

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$(p-n)^{n-1}$ choices for π and p choices for α_1, so

$$\chi_{\mathcal{S}_n}(p) = p(p-n)^{n-1}.$$

The Catalan arrangement

$$\boldsymbol{\mathcal{C}_n} : \; x_i - x_j = 0, -1, 1, \quad 1 \leq i < j \leq n$$

$$\dim \mathcal{C}_n = n, \;\; \operatorname{rk} \mathcal{C}_n = n - 1, \;\; \#\mathcal{S}_n = 3\binom{n}{2}$$

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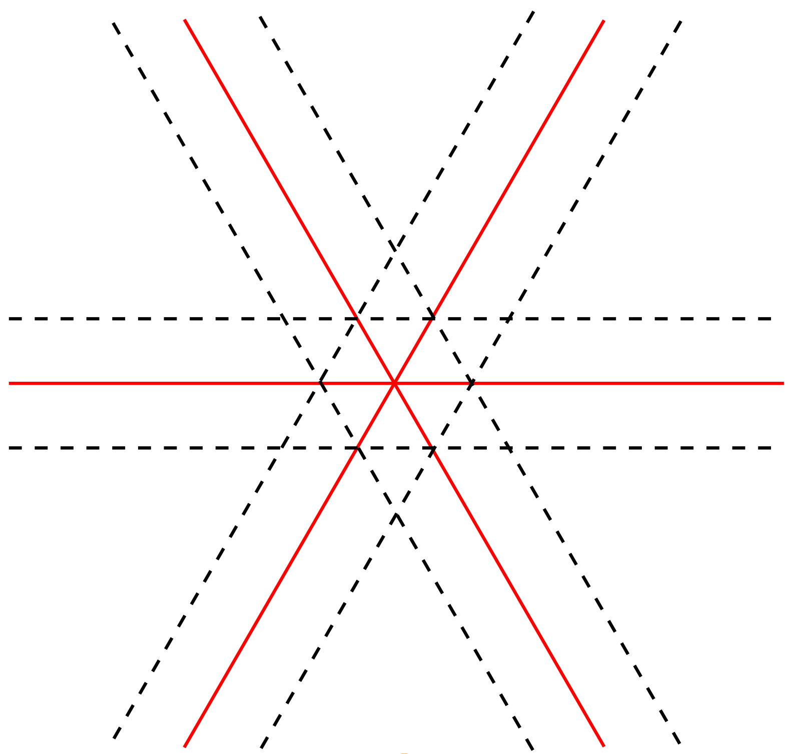

Char. poly. of Catalan arrangment

Theorem.

$\chi_{\mathcal{C}_n}(t) = t(t-n-1)(t-n-2)(t-n-3)\cdots(t-2n+1)$,

so

$$r(\mathcal{C}_n) = n!C_n, \quad b(\mathcal{C}_n) = n!C_{n-1},$$

where $\boldsymbol{C_m} = \frac{1}{m+1}\binom{2m}{m}$ (**Catalan number**).

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Easy to prove using finite field method.

Each region of the braid arrangement $\mathcal{B}_n$ contains C_n regions and C_{n-1} relatively bounded regions of the Catalan arrangment $\mathcal{C}_n$.

Catalan numbers

≥ 172 combinatorial interpretations of C_n at

`math.mit.edu/~rstan/ec`

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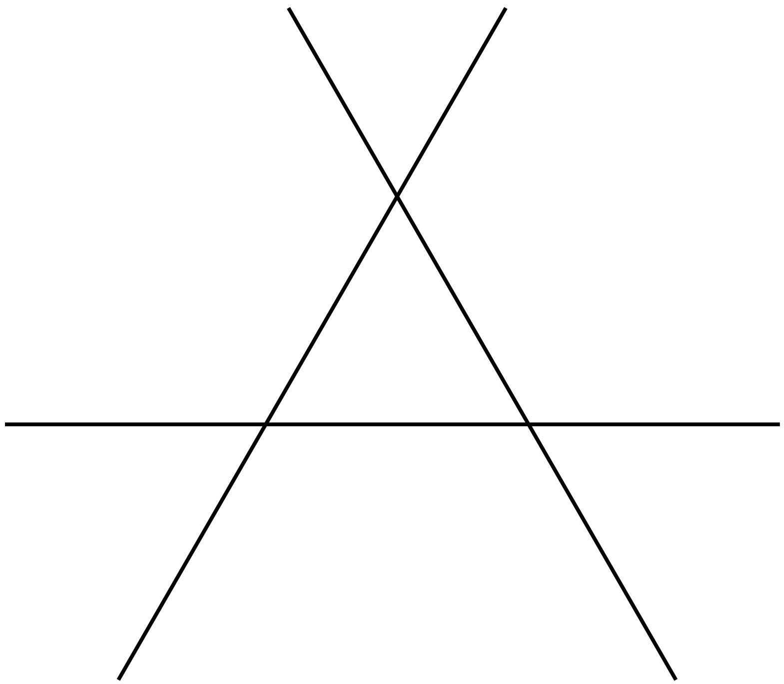

Char. poly. of Linial arrangment

Theorem. $\chi_{\mathcal{L}_n}(t) = \frac{t}{2^n} \sum_{k=1}^{n} \binom{n}{k} (t-k)^{n-1}$,

so

$$r(\mathcal{L}_n) = \frac{1}{2^n} \sum_{k=1}^{n} \binom{n}{k} (k+1)^{n-1}$$

$$b(\mathcal{L}_n) = \frac{1}{2^n} \sum_{k=1}^{n} \binom{n}{k} (k-1)^{n-1}$$

Two proofs

Postnikov: (difficult) proof using Whitney's theorem

Athanasiadis: (difficult) proof using finite field method

Alternating trees

An **alternating tree** on $[n]$ is a tree on the vertex set $[n]$ such that every vertex is either less than all its neighbors or greater than all its neighbors.

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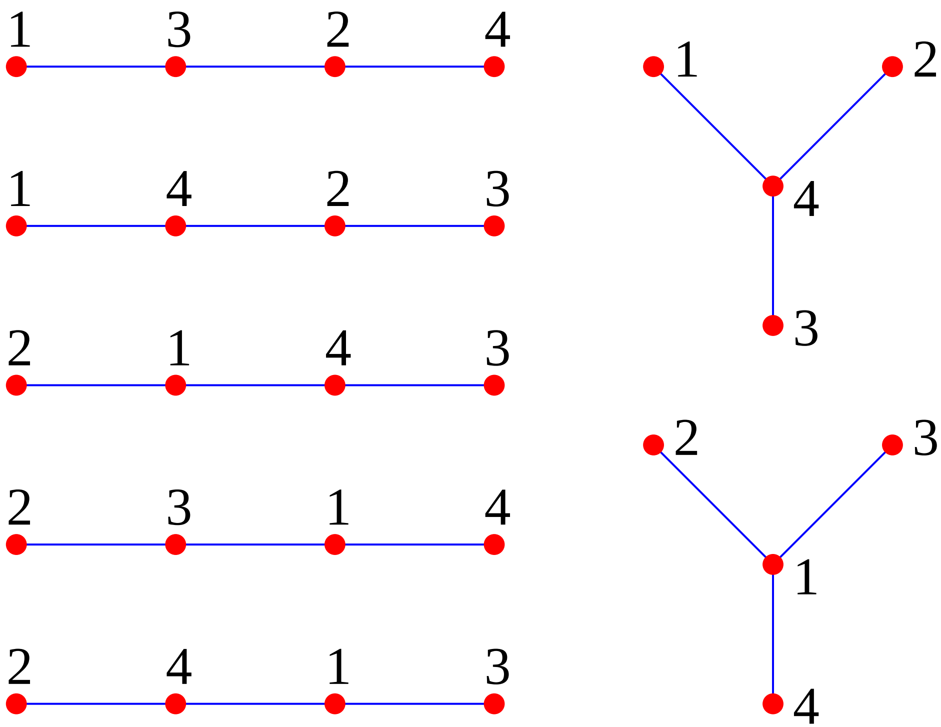

Alternating trees and $\mathcal{L}_n$

$f(n)$: number of alternating trees on $[n]$

Theorem (**Kuznetsov, Pak, Postnikov**, 1994).

$$f(n+1) = \frac{1}{2^n} \sum_{k=1}^{n} \binom{n}{k} (k+1)^{n-1}$$

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No combinatorial proof known!

The threshold arrangment

$$\mathcal{T}_n : \; x_i + x_j = 0, \quad 1 \le i < j \le n$$

$$\dim \mathcal{T}_n = n, \;\; \mathrm{rk}\, \mathcal{T}_n = n, \;\; \#\mathcal{T}_n = \binom{n}{2}$$

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threshold graph:

- $\emptyset$ is a threshold graph
- G threshhold $\Rightarrow G \cup \{\mathrm{vertex}\}$ threshold
- G threshold $\Rightarrow \mathrm{join}(G, v)$ threshold

Char. poly. of threshold arrangemen

Theorem. $r(\mathcal{T}_n) = \#$ threshold graphs on $[n]$.
Hence (by a known result on threshold graphs)

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Theorem. $\displaystyle\sum_{n\geq 0} \chi_{\mathcal{T}_n}(t)\frac{x^n}{n!} = (1+x)(2e^x-1)^{(t-1)/2}$

Small values of $\chi_{\mathcal{T}_n}(t)$

$$\begin{aligned}
\chi_{\mathcal{T}_3}(t) &= t^3 - 3t^2 + 3t - 1 \\
\chi_{\mathcal{T}_4}(t) &= t^4 - 6t^3 + 15t^2 - 17t + 7 \\
\chi_{\mathcal{T}_5}(t) &= t^5 - 10t^4 + 45t^3 - 105t^2 + 120t - 51.
\end{aligned}$$

Coefficients of $\chi_{\mathcal{T}_n}(t)$

Let

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Open: interpret a_i as the number of threshold graphs on $[n]$ with some property.

Minkowski space $\mathbb{R}^{1,3}$

$\mathbb{R}^{1,3}$: Minkowski spacetime with one time and three space dimensions

$$\boldsymbol{p} = (t, \boldsymbol{x}) \in \mathbb{R}^{1,3}, \quad \boldsymbol{x} = (x, y, z) \in \mathbb{R}^3$$

$$|\boldsymbol{p}|^2 = t^2 - |\boldsymbol{x}|^2 = t^2 - (x^2 + y^2 + z^2)$$

Ordering events in $\mathbb{R}^{1,3}$

Let $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_k \in \mathbb{R}^{1,3}$. In different reference frames (at constant velocities with respect to each other) these events can occur in different orders (but never violating causality).

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Let $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_k \in \mathbb{R}^{1,3}$. In different reference frames (at constant velocities with respect to each other) these events can occur in different orders (but never violating causality).

Main question: what is the maximum number of different orders in which these events can occur?

The hyperplane of simultaneity

Let $\boldsymbol{p_1} = (t_1, \boldsymbol{x_1}),\ \boldsymbol{p_2} = (t_2, \boldsymbol{x_2}) \in \mathbb{R}^{1,3}$.

For a reference frame at velocity $\boldsymbol{v}$, the Lorentz transformation $\Rightarrow \boldsymbol{p_1}, \boldsymbol{p_2}$ occur at the same time if and only if

$$t_1 - t_2 = (\boldsymbol{x_1} - \boldsymbol{x_2}) \cdot \boldsymbol{v}.$$

The set of all such $\boldsymbol{v} \in \mathbb{R}^3$ forms a hyperplane.

The Einstein arrangement

Thus the number of different orders in which the events can occur is the number of regions R of the **Einstein arrangement**

$$\boldsymbol{\mathcal{E}} = \mathcal{E}(\boldsymbol{p_1}, \ldots, \boldsymbol{p_k})$$

defined by

$$t_i - t_j = (\boldsymbol{x_1} - \boldsymbol{x_2}) \cdot \boldsymbol{v}, \;\; 1 \leq i < j \leq k,$$

such that $|\boldsymbol{v}| < 1$ (the speed of light) for some $\boldsymbol{v} \in R$.

Intersection poset of $\mathcal{E}$

Can insure that $\boldsymbol{v} \in R$ for all R by taking $\boldsymbol{p_1}, \ldots, \boldsymbol{p_k}$ sufficiently "far apart".

Can maximize $r(\mathcal{E})$ for fixed k by choosing $\boldsymbol{p_1}, \ldots, \boldsymbol{p_k}$ generic.

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In this case, $L(\mathcal{E})$ is isomorphic to the **rank 3 truncation** of $L(\mathcal{B}_k) \cong \Pi_k$.

Coefficients of $\chi_{\mathcal{B}_k}(t)$

Recall

$$\begin{aligned}\chi_{\mathcal{B}_k}(t) &= t(t-1)\cdots(t-k+1)\\ &= c(k,k)t^k - c(k,k-1)t^{k-1} + \cdots\ ,\end{aligned}$$

where $\boldsymbol{c(k,i)}$ is the number of permutations of $1, 2, \ldots, k$ with i cycles (signless Stirling number of the first kind).

Computation of $r(\mathcal{E})$

Corollary.

$$\begin{aligned}\chi_{\mathcal{E}}(t) &= c(k,k)t^3 - c(k,k-1)t^2 + c(k,k-2)t \\ &\quad -c(k,k-3) \\ \Rightarrow r(\mathcal{E}) &= c(k,k) + c(k,k-1) + c(k,k-2) \\ &\quad +c(k,k-3) \\ &= \frac{1}{48}\big(k^6 - 7k^5 + 23k^4 - 37k^3 + 48k^2 \\ &\quad -28k + 48\big)\end{aligned}$$


THE END