Let $\boldsymbol{\lambda}, \boldsymbol{\nu} \vdash n$. Let $\boldsymbol{\chi^{\lambda}(\nu)}$ denote the irreducible character χ^{λ} of $\mathfrak{S}_n$ evaluated at a permutation $w \in \mathfrak{S}_n$ of cycle type ν.

If $\mu \vdash k \leq n$ let

$$\boldsymbol{(\mu, 1^{n-k})} = (\mu, \underbrace{1, \ldots, 1}_{n-k\ 1'\mathrm{s}}) \vdash n.$$

Normalized character:

$$\widehat{\boldsymbol{\chi}}^{\lambda}(\mu, 1^{n-k}) = \frac{(n)_k\, \chi^{\lambda}(\mu, 1^{n-k})}{\chi^{\lambda}(1^n)},$$

where

$$\boldsymbol{\chi^{\lambda}(1^n)} = \dim \chi^{\lambda} = f^{\lambda}$$

$$\boldsymbol{(n)_k} = n(n-1)\cdots(n-k+1).$$

Let $\boldsymbol{p \times q} = (\underbrace{q, \ldots, q}_{p\ q'\mathrm{s}})$, and let $\boldsymbol{\kappa(w)}$ denote the number of cycles of $w \in \mathfrak{S}_k$.

Theorem. *Let $\mu \vdash k$ and fix a permutation $w_\mu \in \mathfrak{S}_k$ of cycle type μ. Then*

$$\widehat{\chi}^{p\times q}(\mu, 1^{pq-k}) = (-1)^k \sum_{u \in \mathfrak{S}_k} p^{\kappa(u)}(-q)^{\kappa(uw_\mu)}.$$

Proof based on Murnaghan-Nakayama rule. Another proof by Rattan based on shift Schur functions of Okounkov and Olshanski.

Example.

$$\mu = (1): \quad pq$$

$$\mu = (2): \quad -p^2q + pq^2$$

$$\mu = (1,1): \quad pq(pq-1)$$

$$\mu = (3): \quad p^3q - 3p^2q^2 + pq^3 + pq$$

$$\mu = (2,1): \quad (-p^2q + pq^2)(pq-2)$$

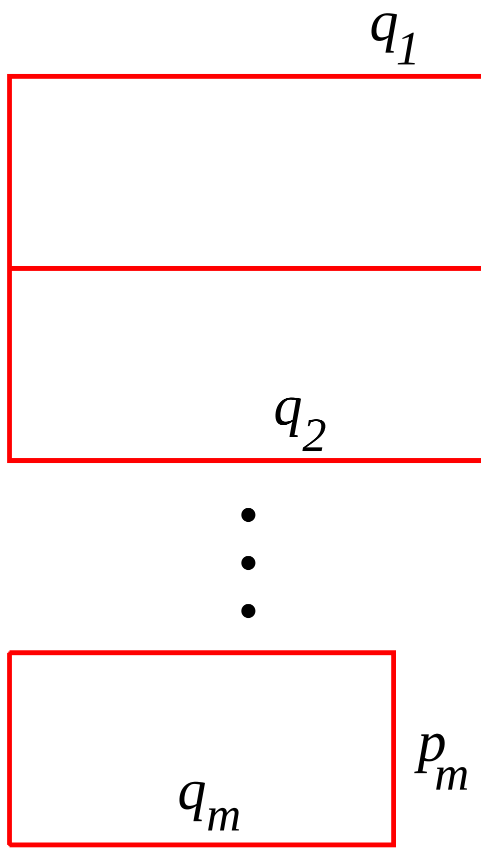
$$\mu = (1,1,1): \quad pq(pq-1)(pq-2)$$

$$\mu = (4): \quad -p^4q + 6p^3q^2 - 6p^2q^3 + pq^4 \\ -5p^2q + 5pq^2$$

$$\mu = (2,2): \quad p^4q^2 - 2p^3q^3 + p^2q^4 - 4p^3q \\ +10p^2q^2 - 4pq^3 - 2pq$$

Generalization of rectangular shape.

Define the shape σ by



Theorem (Katriel & RS). *Fix* $\mu \vdash k$. *Set* $n = |\sigma|$ *and*

$$\boldsymbol{F_\mu(p_1,\ldots,p_m;q_1,\ldots,q_m)} = \widehat{\chi}^\sigma(\mu, 1^{n-k}).$$

Then $F_\mu(p_1,\ldots,p_m;q_1,\ldots,q_m)$ *is a polynomial function of the* p_i*'s and* q_i*'s with integer coefficients, satisfying*

$$(-1)^k F_\mu(1,\ldots,1;-1,\ldots,-1) = (k+m-1)_k.$$

Let

$$\boldsymbol{c(k,i)} = \#w \in \mathfrak{S}_k,\ i \text{ cycles},$$

a **signless Stirling number of the first kind**. Thus

$$\sum_i c(k,i)x^i = (x+k-1)_k,$$

so

$$(-1)^k F_\mu(1,\ldots,1;-1,\ldots,-1) = \sum_i c(k,i)m^i.$$

$$(-1)^k F_\mu(1,\ldots,1;-1,\ldots,-1) = \sum_i c(k,i)m^i,$$

the number of $w \in \mathfrak{S}_k$ with each cycle colored from $1,2,\ldots,m$. Let $\boldsymbol{\mathfrak{S}_k^{(m)}}$ denote the set of such "colored permutations."

Recall:

$$\widehat{\chi}^{p\times q}(\mu, 1^{pq-k}) = (-1)^k \sum_{u\in\mathfrak{S}_k} p^{\kappa(u)}(-q)^{\kappa(uw_\mu)}.$$

Suggests: let $\boldsymbol{\kappa_i(u)}$ be the number of cycles of u colored i. Let

$$\boldsymbol{p}^{\boldsymbol{\kappa}(u)} = p_1^{\kappa_1(u)} p_2^{\kappa_2(u)} \cdots,$$

and similarly $\boldsymbol{q}^{\boldsymbol{\kappa}(u)}$. Then

$$(-1)^k F_\mu(\boldsymbol{p}; \boldsymbol{q})$$

$$= (-1)^k \sum_{u\in\mathfrak{S}_k^{(m)}} \boldsymbol{p}^{\boldsymbol{\kappa}(u)}(-\boldsymbol{q})^{\boldsymbol{\kappa}(u\diamondsuit w_\mu)},$$

for some "product" $u\diamondsuit w_\mu \in \mathfrak{S}_k^{(m)}$.

How to define

$$\diamondsuit : \mathfrak{S}_k^{(m)} \times \mathfrak{S}_k \to \mathfrak{S}_k^{(m)}?$$

For $u \in \mathfrak{S}_k$, let

$$\boldsymbol{C(u)} = \{\text{cycles of } u\}.$$

Formally define

$$\boldsymbol{\mathfrak{S}_k^{(m)}} = \{(u, \varphi) \mid u \in \mathfrak{S}_k,\ \varphi : C(u) \to [m]\}.$$

Suppose $(u, \varphi)\diamondsuit w = (v, \psi)$.

- $uw = v$ in $\mathfrak{S}_k$
- Let $\boldsymbol{\tau} = (a_1, \ldots, a_j) \in C(v)$. Let $\boldsymbol{\rho_i}$ be the cycle of u containing a_i. Set

$$\boldsymbol{\psi(\tau)} = \max\{\varphi(\rho_1), \ldots, \varphi(\rho_j)\}.$$

Example. Multiplying left-to-right:

$$(\overbrace{1,2,3}^{1})(\overbrace{4,5}^{2})(\overbrace{6,7}^{3})(\overbrace{\ 8\ }^{2})\diamondsuit(1,7)(2,4,8,5)(3,5)$$

$$=(\overbrace{1,4,2,6}^{3})(\overbrace{3,7}^{3})(\overbrace{5,8}^{2}).$$

Note. The product $\diamondsuit$ does **not** define an action of $\mathfrak{S}_k$ on $\mathfrak{S}_k^{(m)}$, i.e., it is **false** in general that

$$(w,\psi)\diamondsuit uv=((w,\psi)\diamondsuit u)\diamondsuit v.$$

Recall: let $\kappa_i(u,\varphi)$ be the number of cycles of (u,φ) colored i. Let

$$\boldsymbol{p}^{\boldsymbol{\kappa}(u,\varphi)} = p_1^{\kappa_1(u,\varphi)} p_2^{\kappa_2(u,\varphi)} \cdots,$$

and similarly $\boldsymbol{q}^{\boldsymbol{\kappa}(u,\psi)}$. Let $\boldsymbol{w_\mu} \in \mathfrak{S}_k$ have cycle type μ.

Conjecture.

$$(-1)^k F_\mu(\boldsymbol{p};\boldsymbol{q})$$

$$= (-1)^k \sum_{u \in \mathfrak{S}_k^{(m)}} \boldsymbol{p}^{\boldsymbol{\kappa}(u,\varphi)} (-\boldsymbol{q})^{\boldsymbol{\kappa}(u \diamondsuit w_\mu)}.$$

Example.

$$m = 2, \mu = (2), w_\mu = (1,2)$$

unbarred cycle: color 1
barred cycle: color 2

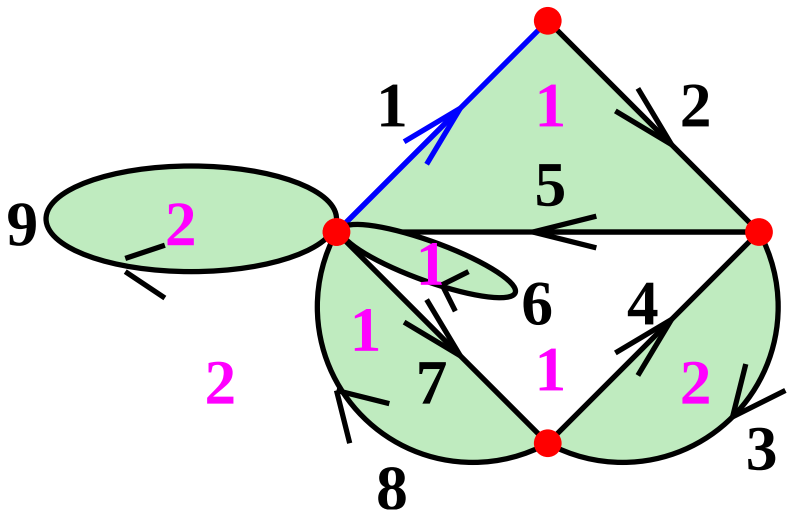
α	$\alpha(1,2)$	$\boldsymbol{p}^{\boldsymbol{\kappa}(\alpha)}\boldsymbol{q}^{\boldsymbol{\kappa}(\alpha(1,2))}$
$(1)(2)$	$(1,2)$	$p_1^2\, q_1$
$(\overline{1})(2)$	$(\overline{1,2})$	$p_1\, p_2\, q_2$
$(1)(\overline{2})$	$(\overline{1,2})$	$p_1\, p_2\, q_2$
$(\overline{1})(\overline{2})$	$(\overline{1,2})$	$p_2^2\, q_2$
$(1,2)$	$(1)(2)$	$p_1\, q_1^2$
$(\overline{1,2})$	$(\overline{1})(\overline{2})$	$p_2\, q_2^2.$

$$F_2(p_1, p_2; q_1, q_2) =$$
$$-p_1^2 q_1 - 2p_1 p_2 q_2 - p_2^2 q_2 + p_1 q_1^2 + p_2 q_2^2.$$

Evidence:

- True for small cases.
- True for $m = 1$ (rectangular shapes).
- If true for each $q_i = 1$, i.e., for $\lambda = (p_1, p_2, \dots, p_m)$, then true in general.
- True for terms of top degree (A. Rattan, math.CO/0610557).

Relation to maps.



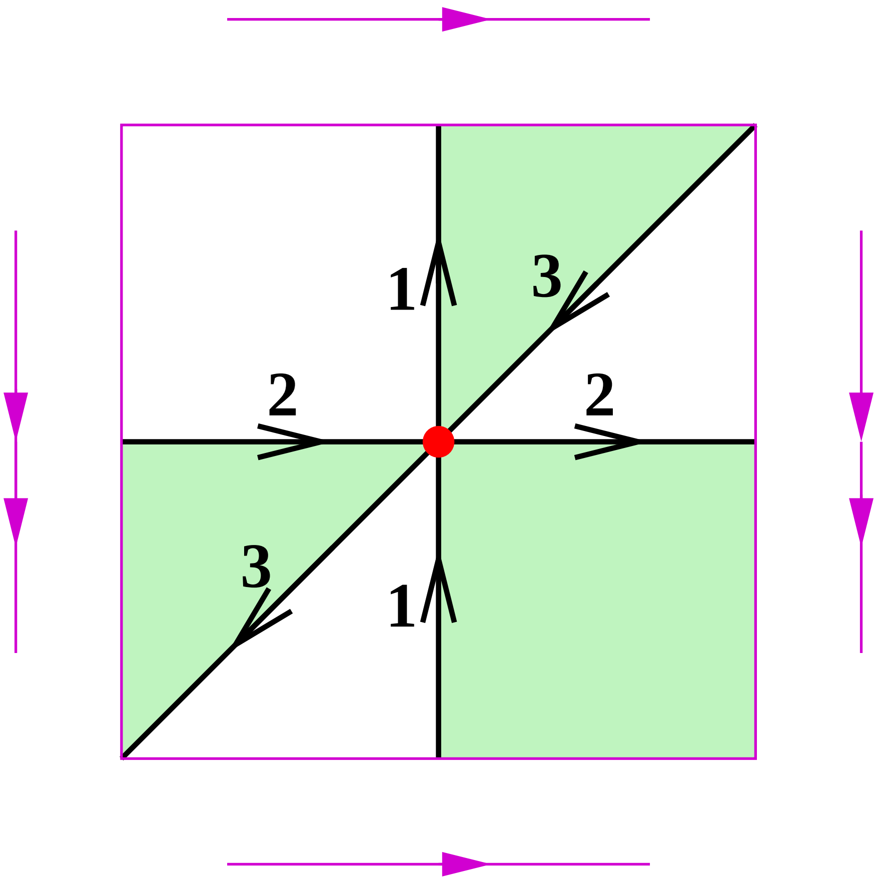
$\boldsymbol{w_\mu}$: edges **out** of a vertex

$\boldsymbol{u}$: white faces

$\boldsymbol{uw_\mu}$: green faces

label of white face = max label of bordering green face

$$\begin{aligned} w_\mu &= (2)(53)(7619)(48) \\ u &= (12389)(4567) \\ uw_\mu &= (125)(34)(67)(78)(9) \end{aligned}$$



$$(123) \cdot (123) = (132)$$