Some Catalan Musings

Richard P. Stanley

An OEIS entry

A000108: $1, 1, 2, 5, 14, 42, 132, 429, \ldots$

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$C_n = \frac{1}{n+1}\binom{2n}{n}$, $n \geq 0$ (**Catalan number**)

Catalan monograph

R. Stanley, *Catalan Numbers*, Cambridge University Press, 2015, to appear.

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R. Stanley, *Catalan Numbers*, Cambridge University Press, 2015, to appear.

Includes 214 combinatorial interpretations of C_n and 68 additional problems.

An early version (1970's)

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①

CATALAN NUMBERS

$$C_n = \frac{1}{n+1}\binom{2n}{n} \qquad C_1 = 1,\ C_2 = 2,\ C_3 = 5,\ C_4 = 14$$

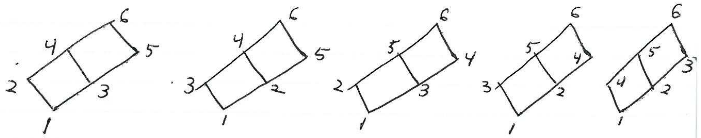
1. $e(\underline{2} \times \underline{n})$
2. no. of lattice paths in an $(n+1) \times (n+1)$ grid not going below diagonal
3. no. of order ideals of $\mathcal{S}(\underline{n-1})^{-\{\phi\}}$ (or $[0, \lambda]$ in $J(\underline{N}^2)$, where $\lambda = (n-1, n-2, \ldots, 1)$)
4. no. of ways of parenthesizing $n+1$ factors
5. no. of ways of dividing an $n+2$-gon into triangles by non-intersecting diagonals
6. no. of non-isomorphic ordered sets of card n with no sub-ordered sets   or .
7. no. of permutations of $1, 2, \ldots, n$ with longest increasing subsequence of length ≤ 2
8. no. of two-sided ideals in the algebra of $(n-1) \times (n-1)$ upper triangular matrices over $\mathbb{Q}$
9. no. of sequences $1 \leq a_1 \leq \cdots \leq a_n$ with $a_i \leq i$
10. no. of sequences $\epsilon_1, \epsilon_2, \ldots, \epsilon_{2n}$ of ± 1's with every partial sum $s_k \geq 0$ and $s_{2n} = 0$ (ballot problem)
11. no. of size sequences of principal ideals of posets

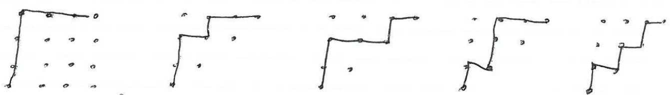
②

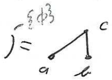
12. Berlekamp determinant with 1-1 boundary
13. no. of plane binary trees with n vertices (order ideal interpretation)
14. no. of plane planted trees with $n+1$ vertices
15. no. of partitions of $\{1, 2, \ldots, n\}$ ~~such~~ that if ~~a~b and c~d~~ $a < b < c < d$, then we never have $a \sim c$ and $b \sim d$ unless $a \sim b \sim c \sim d$
16. ~~$|\mu(0,1)|$ for the~~ $(-1)^{n-1}\mu(0,1)$ for the ordered set of partitions of $\{1, 2, \ldots, n+1\}$ satisfying (15)
17. no. of ways $2n$ points on the circumference of a circle can be joined in pairs by n non-intersecting chords
18. no. of planted (root has degree 1) trivalent plane trees on $2n+2$ vertices
19. no. of n-tuples $a_1, \ldots, a_n$, $a_i \in \mathbb{P}$, such that in the sequence $1\, a_1, a_2 \cdots a_n\, 1$, each a_i divides the sum of its two neighbors
20. no. of permutations $a_1, \ldots, a_n$ of $[n]$ with no subsequence a_i, a_j, a_k $(i<j<k)$ satisfying $a_j < a_k < a_i$
21. no. of permutations $a_1, \ldots, a_{2n}$ of the multiset $\{1^2, 2^2, \ldots, n^2\}$ such that: (i) first occurrences of $1, \ldots, n$ appear in increasing order, (ii) no subsequence of the form $\alpha\beta\alpha\beta$. (The second occurrences of $1, \ldots, n$ form a permutation as in 20.)

③

EXAMPLES $(n=3)$

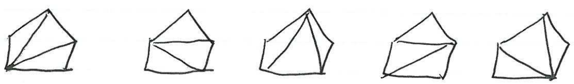
1. 

2. 

3. $\mathcal{S}(2) = $ 

ϕ, a, b, ab, abc

4. $x(x^2 \cdot x)$ $x(x \cdot x^2)$ $(x^2 \cdot x)x$ $(x \cdot x^2)x$ $x^2 \cdot x^2$

5. 

6. 

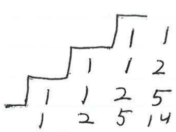
7. 132   213   231   312   321

8. (obtained from 3.)

9. 111   112   113   122   123

10. 111---   11-1--   11--1-   1-11--   1-1-1-

11. (same as 9.)

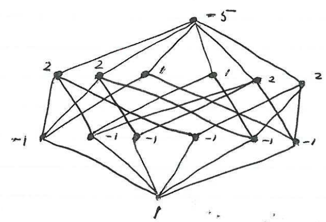
12. 

④

13. 

14. 

15. 123   12-3   13-2   1-23   1-2-3

16. 

17. 

18. 

19. 14321   13521   13231   12531   12341

20. 123   132   213   231   321

21. 112233   112332   122331   123321   122133

How to sample?

Compare **D. E. Knuth**, *3:16 Bible Texts Illuminated*.

Sample from Bible by choosing verse 3:16 from each chapter.

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I will be less random.

History

Sharabiin Myangat, also known as **Minggatu**, **Ming'antu** (明安图), and **Jing An** (c. 1692–c. 1763): a Mongolian astronomer, mathematician, and topographic scientist who worked at the Qing court in China.

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Typical result (1730's):

$$\sin(2\alpha) = 2\sin\alpha - \sum_{n=1}^{\infty} \frac{C_{n-1}}{4^{n-1}} \sin^{2n+1}\alpha$$

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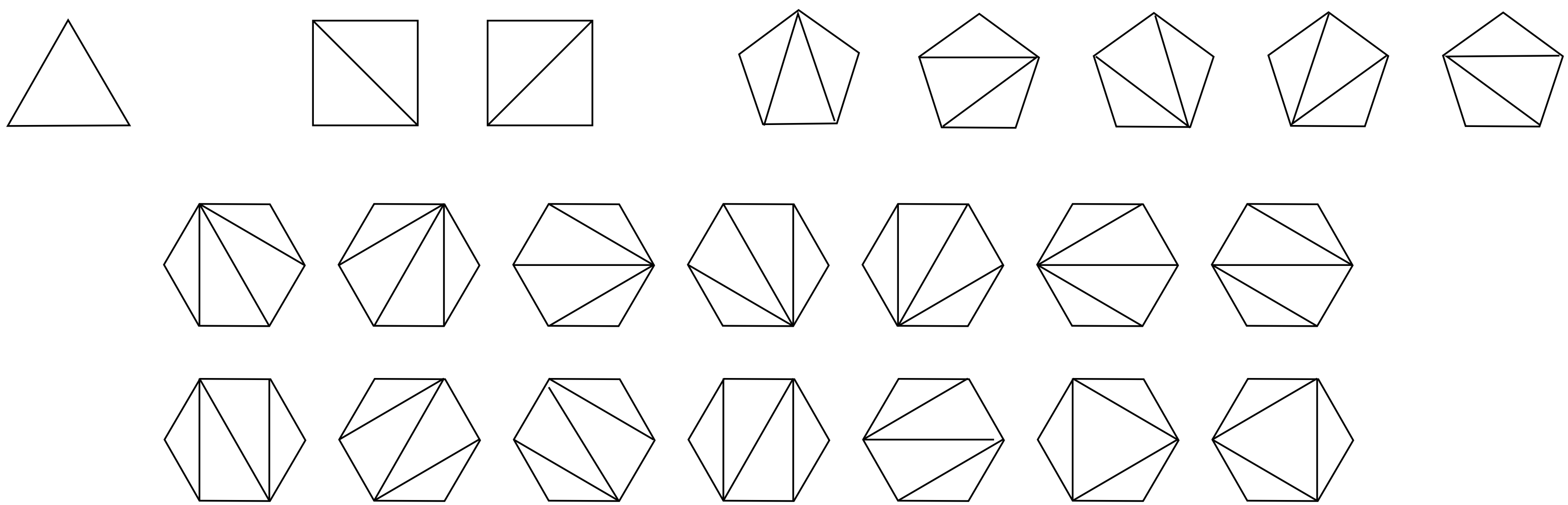
No combinatorics, no further work in China.

More history, via Igor Pak

- **Euler** (1751): conjectured formula for number C_n of triangulations of a convex $(n+2)$-gon



Completion of proof

- **Goldbach and Segner** (1758–1759): helped Euler complete the proof, in pieces.
- **Lamé** (1838): first self-contained, complete proof.

Catalan

- **Eugène Charles Catalan** (1838): wrote C_n in the form $\frac{(2n)!}{n!\,(n+1)!}$ and showed they counted (nonassociative) **bracketings** (or **parenthesizations**) of a string of $n+1$ letters.

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Born in 1814 in Bruges (now in Belgium, then under Dutch rule). Studied in France and worked in France and Liège, Belgium. Died in Liège in 1894.

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- **Gardner** (1976): used the term in his Mathematical Games column in *Scientific American*. Real popularity began.

The primary recurrence

$$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}, \quad C_0 = 1$$

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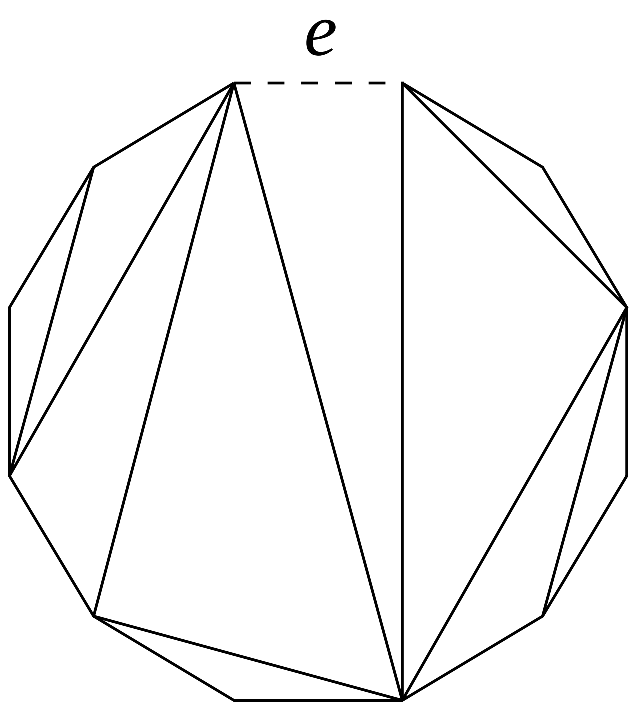

“Transparent” interpretations

3. Binary **parenthesizations** or **bracketings** of a string of $n+1$ letters

$$(xx \cdot x)x \quad x(xx \cdot x) \quad (x \cdot xx)x \quad x(x \cdot xx) \quad xx \cdot xx$$

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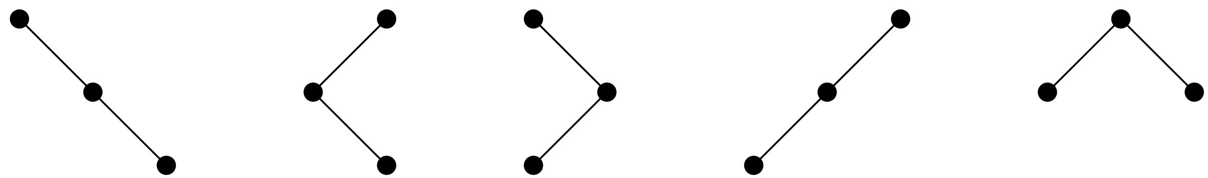
Binary trees

4. Binary trees with n vertices



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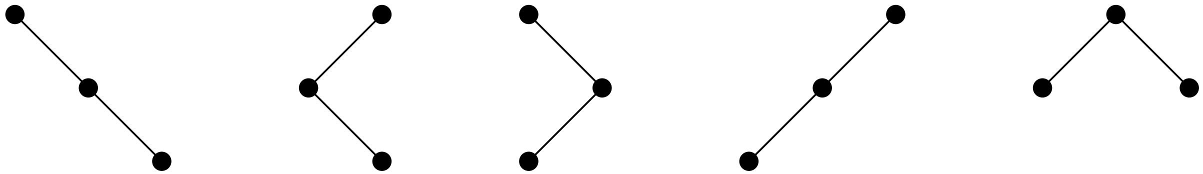

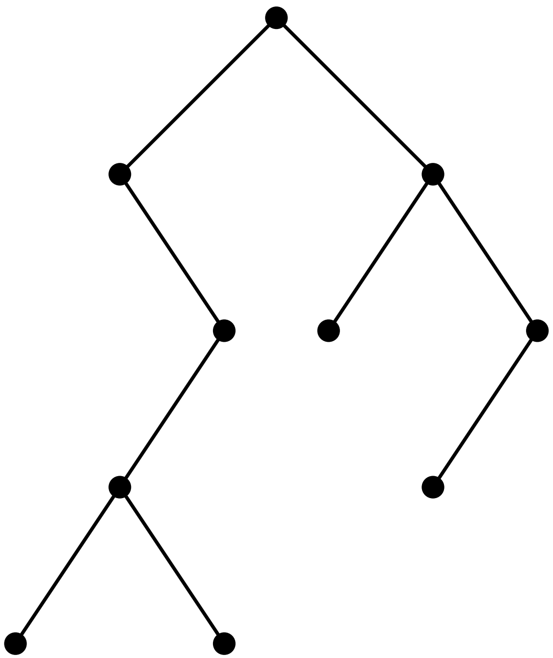

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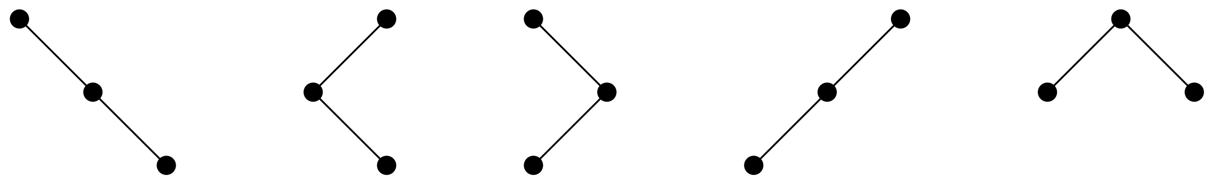
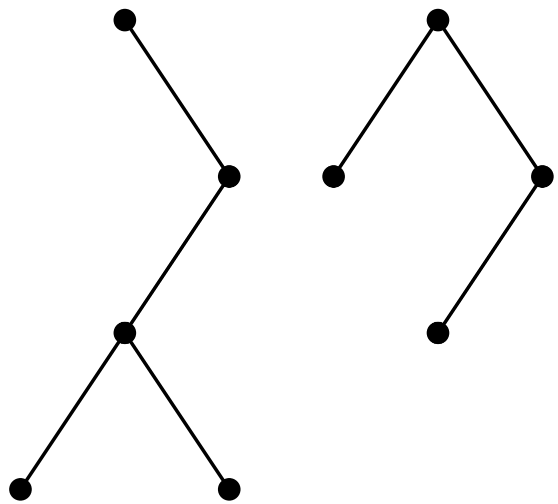
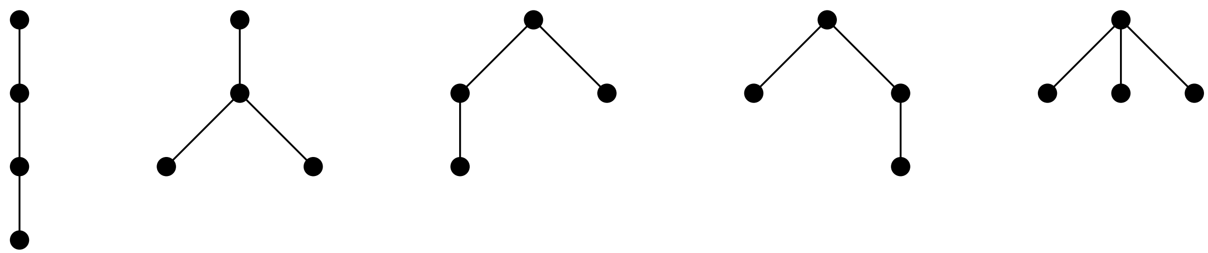




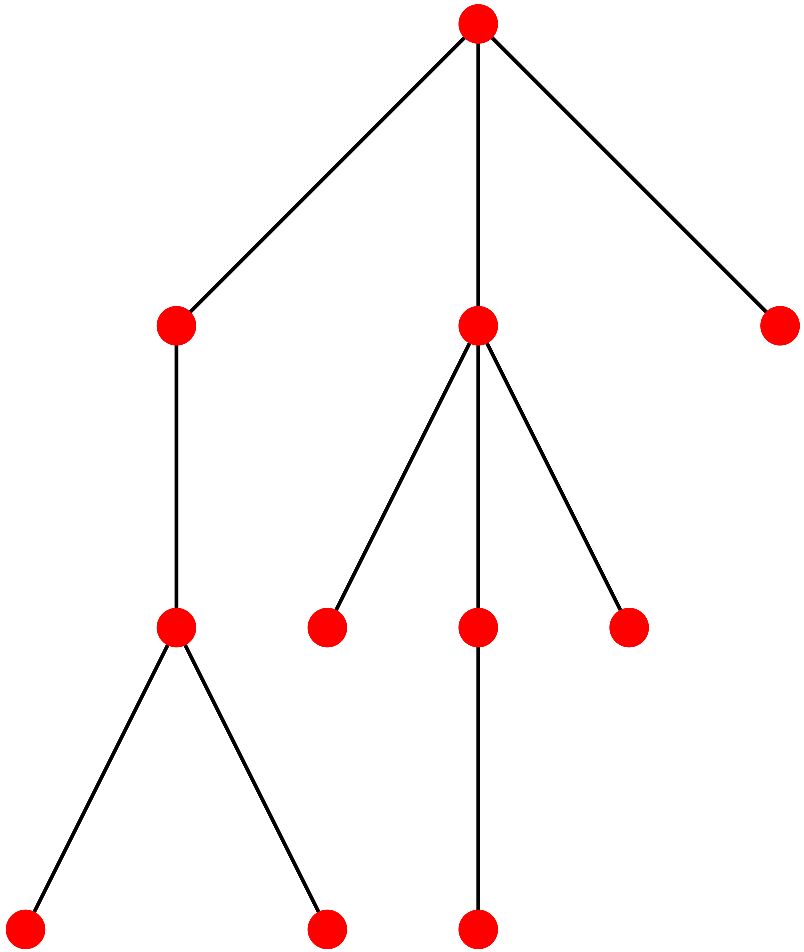
Plane trees

Plane tree: subtrees of a vertex are linearly ordered

6. Plane trees with $n+1$ vertices



Plane tree recurrence



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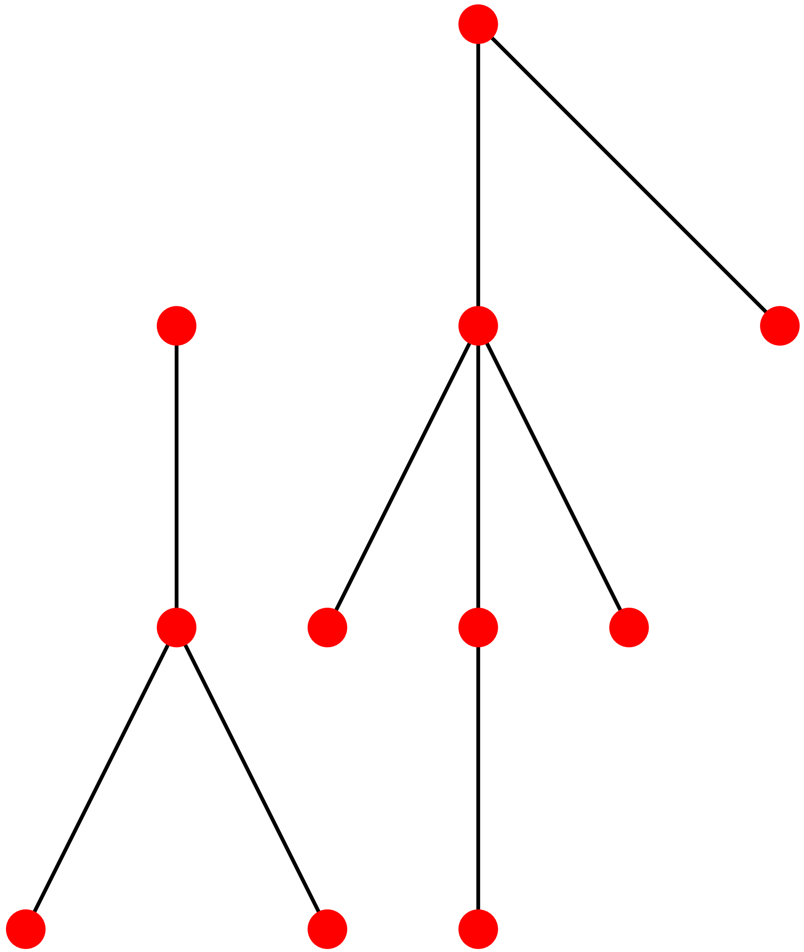
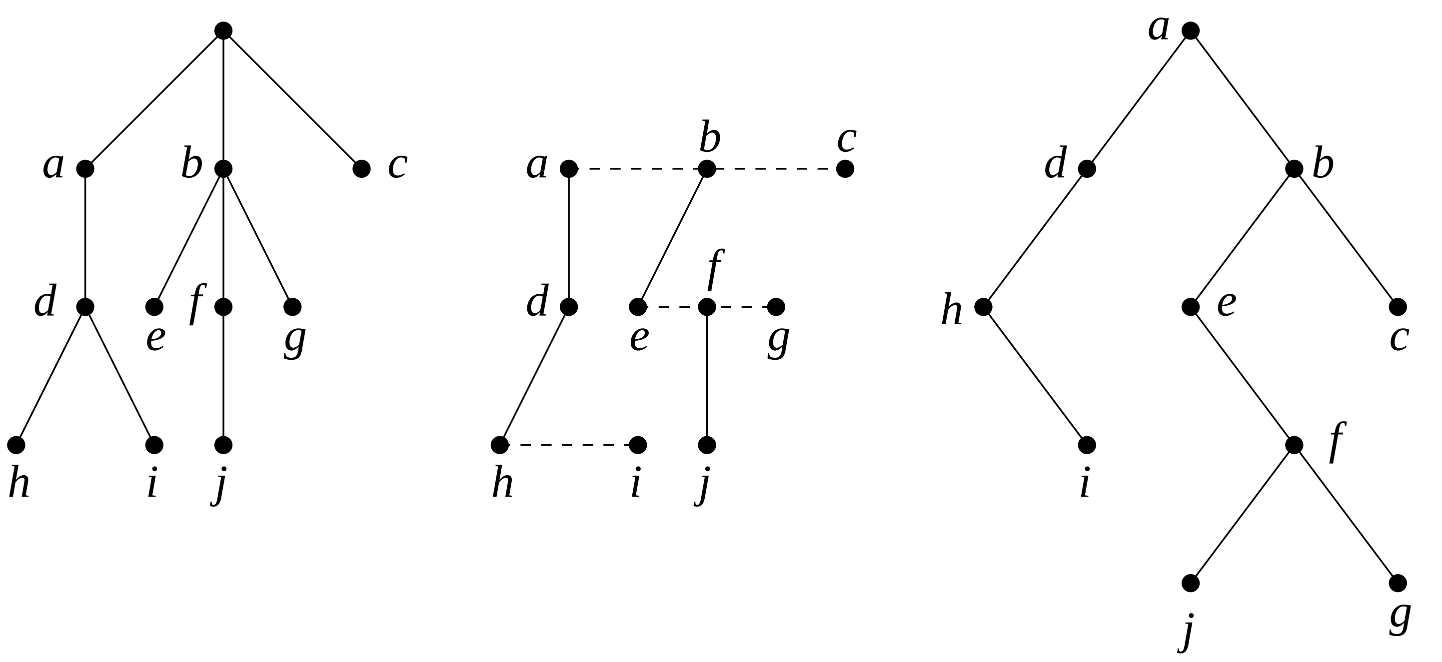
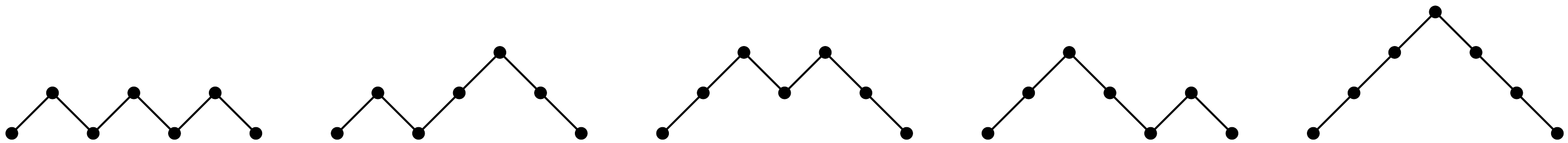


The “natural bijection”



Dyck paths

25. Dyck paths of length $2n$, i.e., lattice paths from $(0,0)$ to $(2n,0)$ with steps $(1,1)$ and $(1,-1)$, never falling below the x-axis



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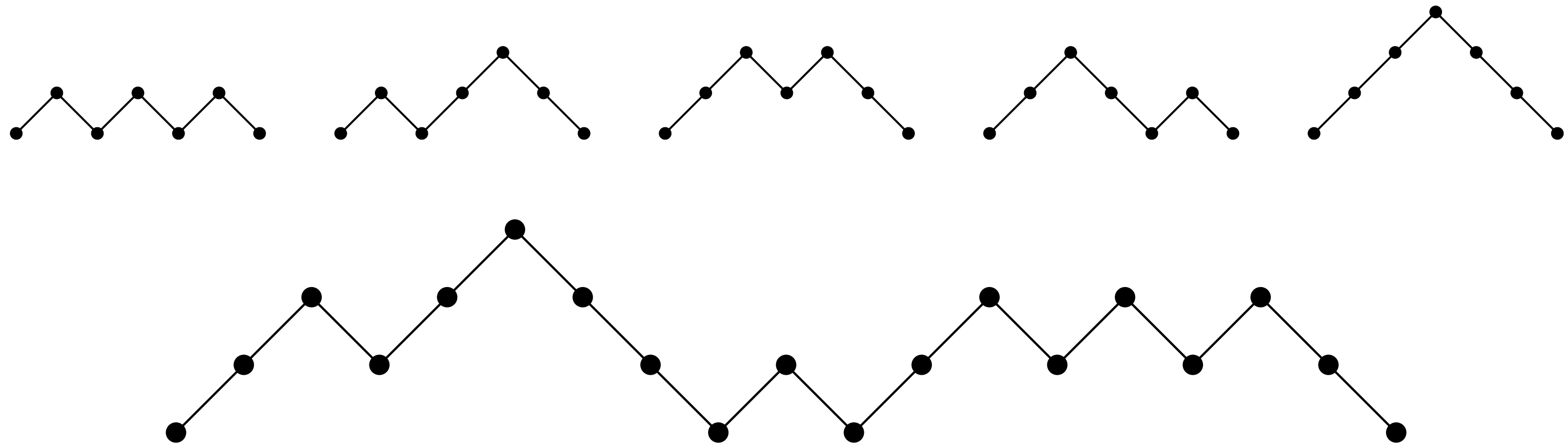

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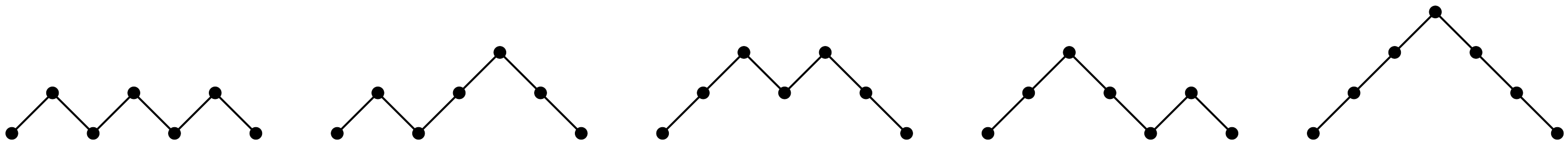

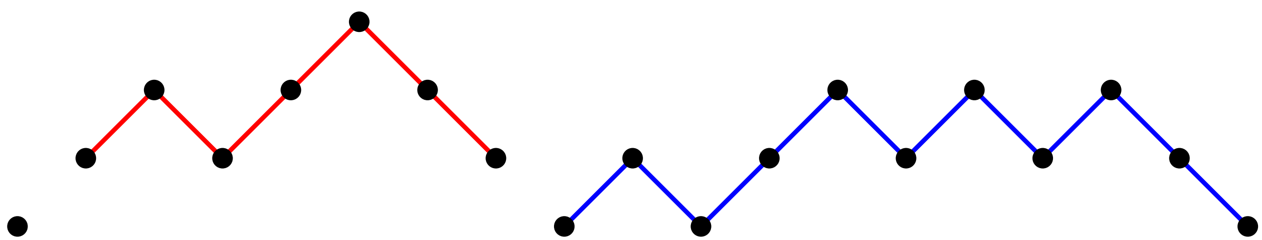

312-avoiding permutations

116. Permutations $a_1a_2\cdots a_n$ of $1,2,\ldots,n$ for which there does not exist $i<j<k$ and $a_j<a_k<a_i$ (called **312-avoiding**) permutations)

$$123 \quad 132 \quad 213 \quad 231 \quad 321$$

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$$34251768$$

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$$123 \quad 132 \quad 213 \quad 231 \quad 321$$

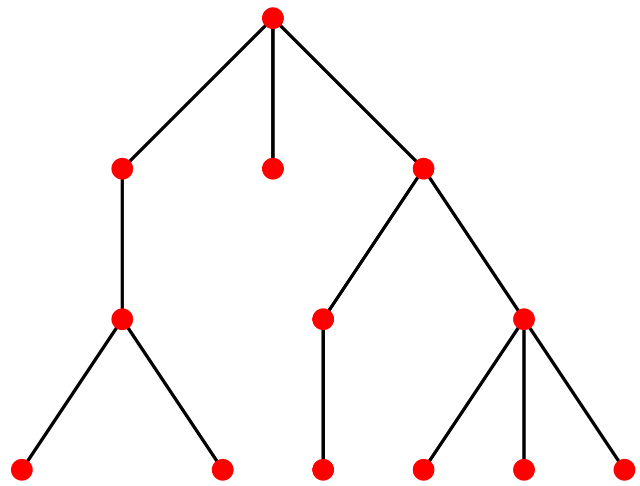
$$\mathbf{3425\ 768}$$

Less transparent interpretations

159. Noncrossing partitions of $1, 2, \ldots, n$, i.e., partitions $\pi = \{B_1, \ldots, B_k\} \in \Pi_n$ such that if $a < b < c < d$ and $a, c \in B_i$ and $b, d \in B_j$, then $i = j$

$$123 \quad 12\text{–}3 \quad 13\text{–}2 \quad 23\text{–}1 \quad 1\text{–}2\text{–}3$$

Bijection with plane trees



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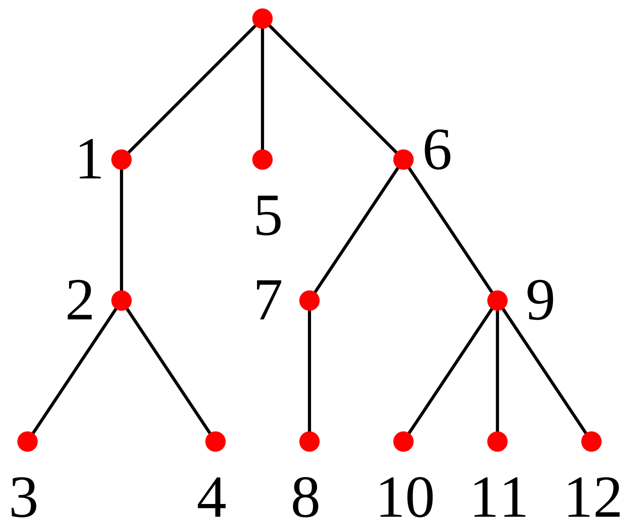

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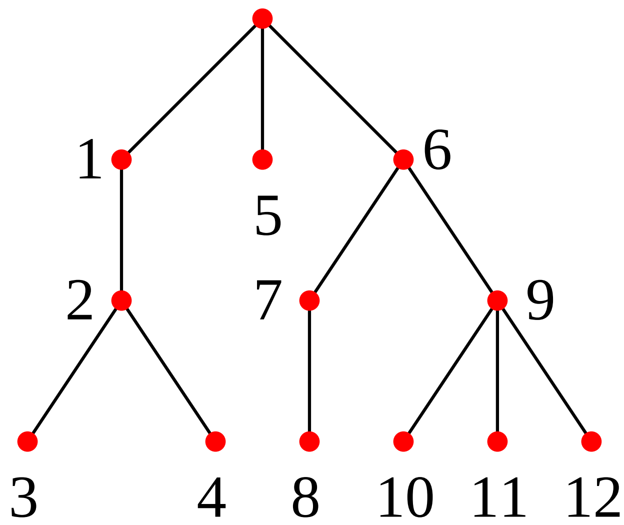
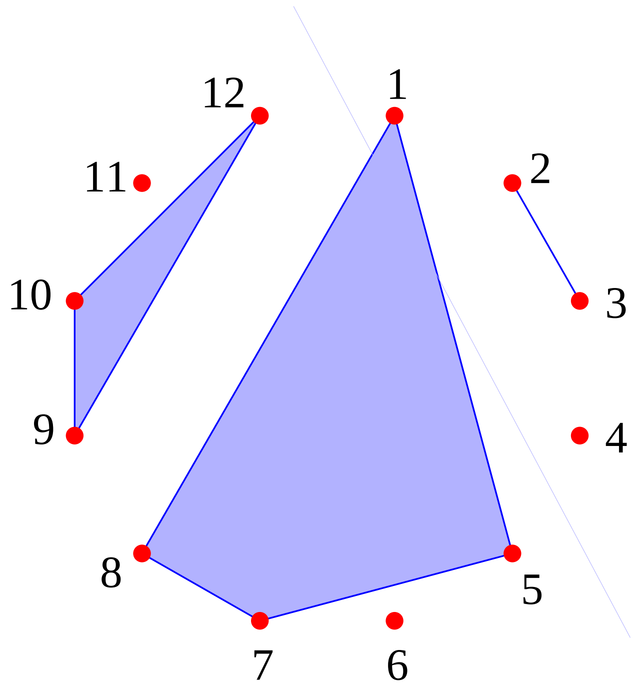


Children of nonleaf vertices:

$$\{1,5,6\},\ \{2\},\{3,4\},\{7,9\},\{8\},\{10,11,12\}$$

Noncrossing partition recurrence



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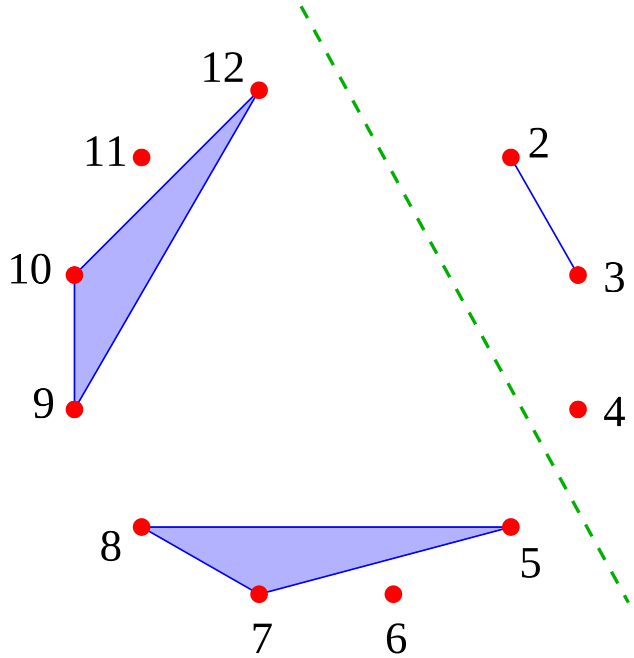

321-avoiding permutations

115. Permutations $a_1 a_2 \cdots a_n$ of $1, 2, \ldots, n$ with longest decreasing subsequence of length at most two (i.e., there does not exist $i < j < k$, $a_i > a_j > a_k$), called **321-avoiding** permutations

$$123 \quad 213 \quad 132 \quad 312 \quad 231$$

Bijection with Dyck paths

$$w = 412573968$$

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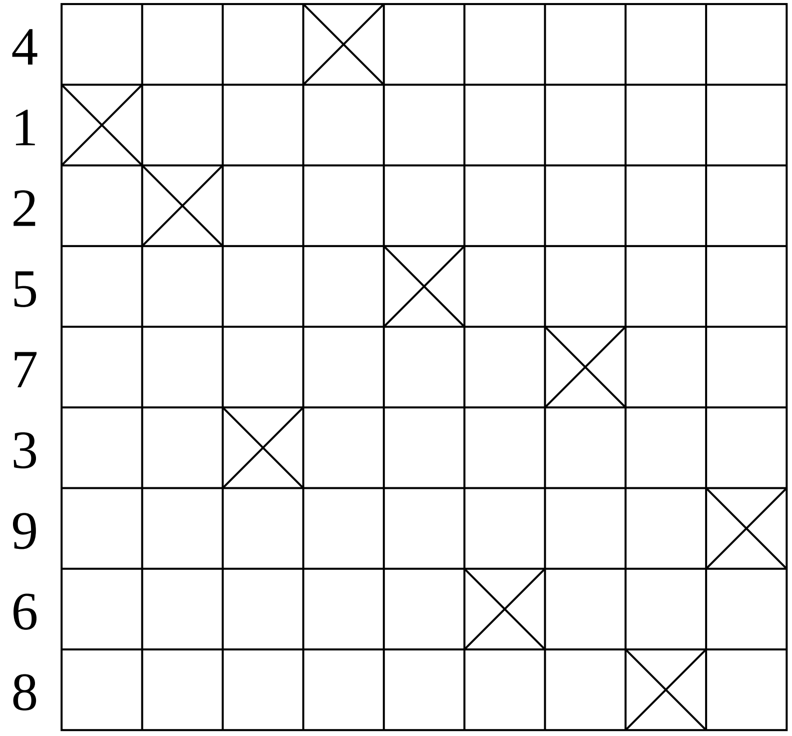

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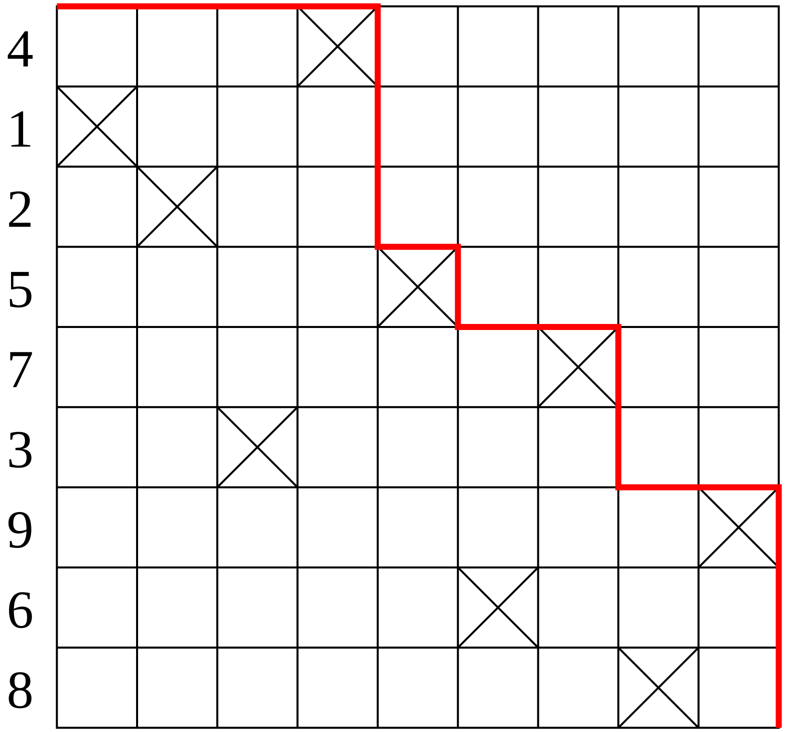
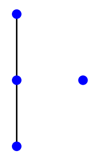
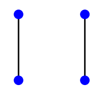


Semiorders

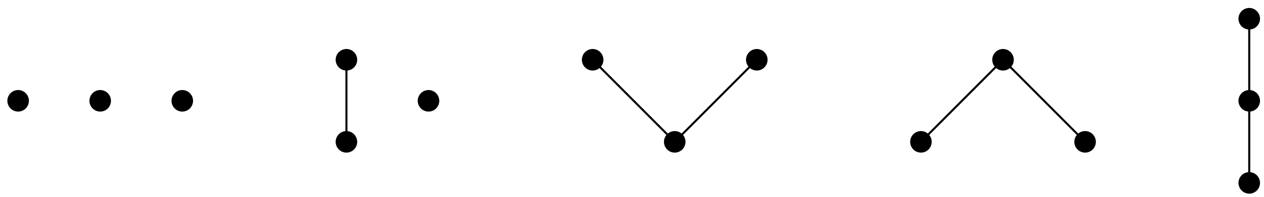
(finite) **semiorder** or unit interval order: a finite subset P of $\mathbb{R}$ with the partial order:

$$x <_P y \iff x <_{\mathbb{R}} y - 1$$

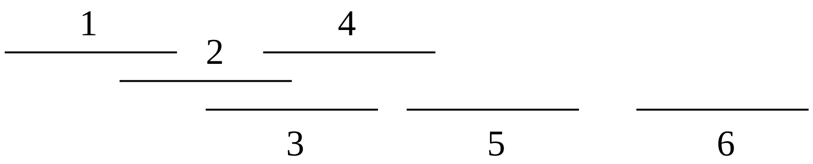
Equivalently, no induced  $(\mathbf{3}+\mathbf{1})$ or  $(\mathbf{2}+\mathbf{2})$

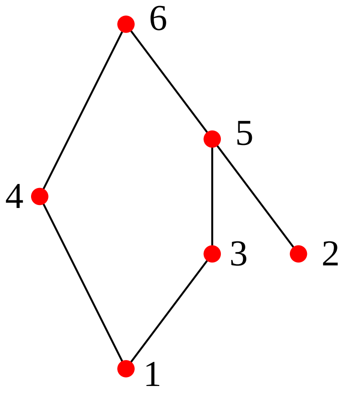
Semiorders (cont.)

180. Nonisomorphic n-element posets with no induced subposet isomorphic to $\mathbf{2}+\mathbf{2}$ or $\mathbf{3}+\mathbf{1}$



Semiorders and Dyck paths





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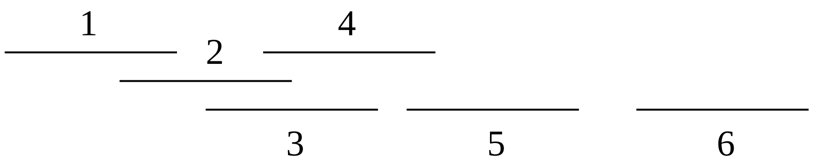
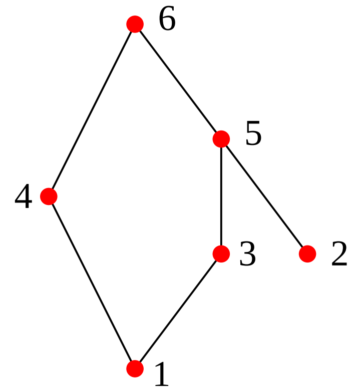
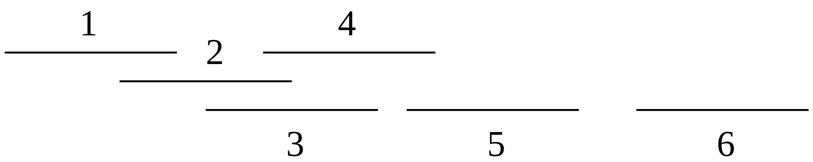
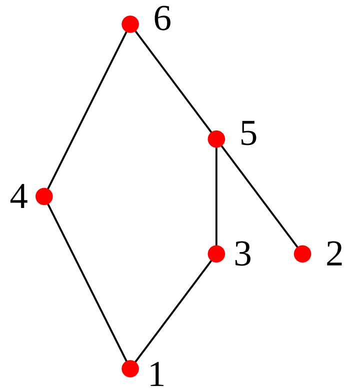


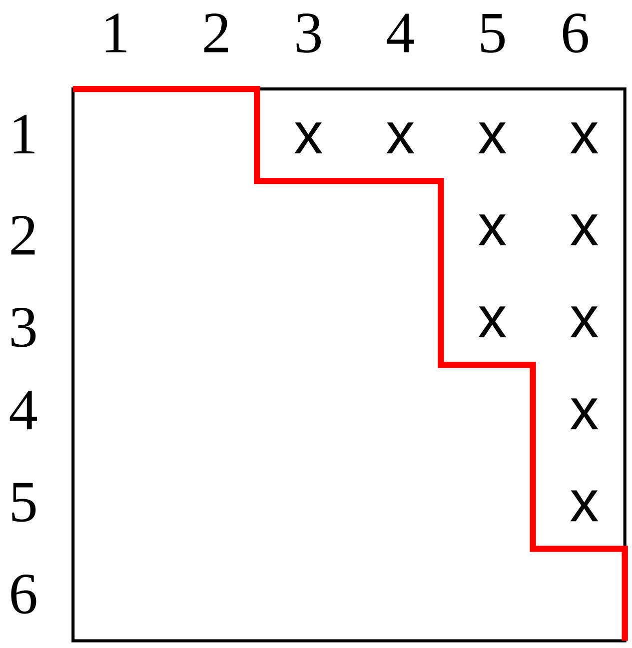


	1	2	3	4	5	6
1			x	x	x	x
2					x	x
3					x	x
4						x
5						x
6						

Semiorders and Dyck paths

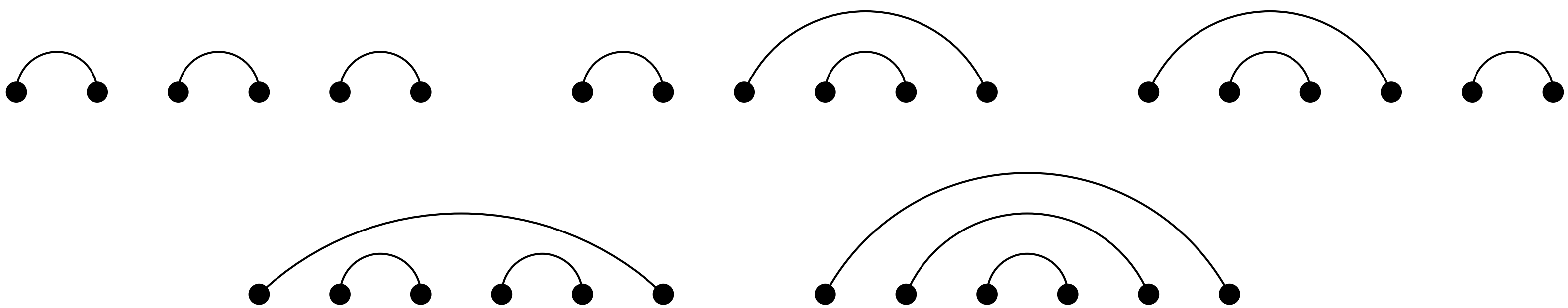




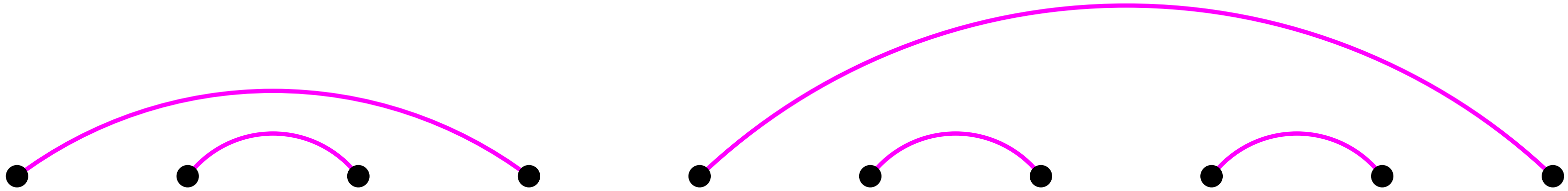


Noncrossing matchings

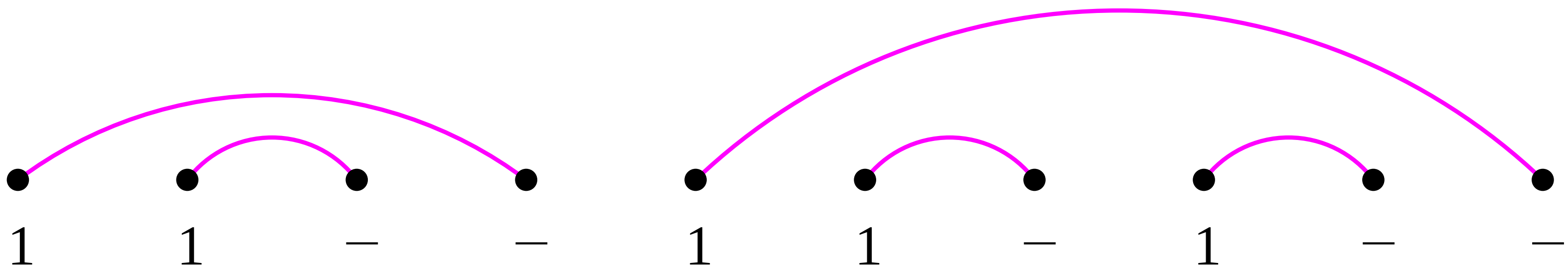
61. Noncrossing (complete) matchings on $2n$ vertices, i.e., ways of connecting $2n$ points in the plane lying on a horizontal line by n nonintersecting arcs, each arc connecting two of the points and lying above the points



Bijection to ballot sequences



Bijection to ballot sequences



left endpoint: 1

right endpoint: -1

Inverse bijection

•	•	•	•	•	•	•	•	•	•
1	1	–	–	1	1	–	1	–	–

Inverse bijection

•	•	•	•	•	•	•	•	•	•
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Scan ballot sequence from right-to-left. Connect each 1 with **leftmost** available -1.

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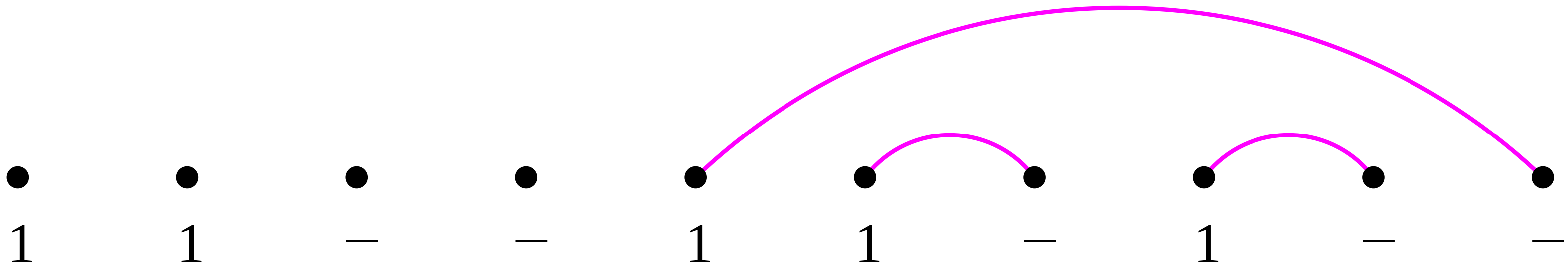

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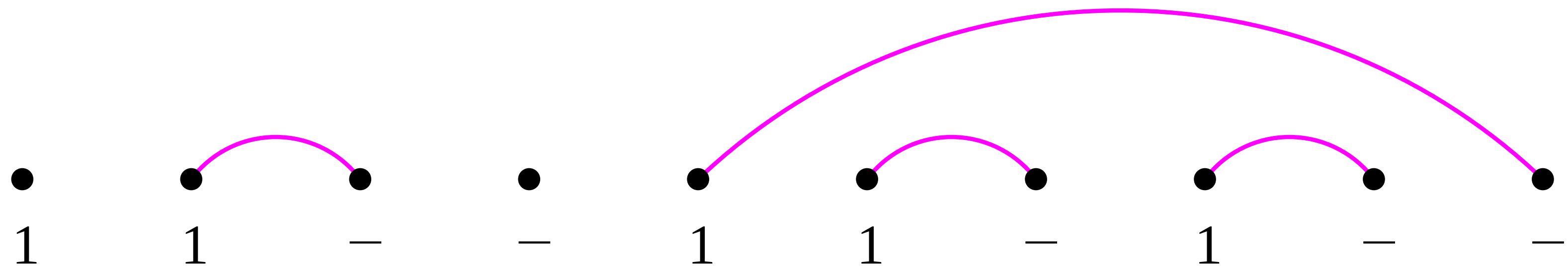

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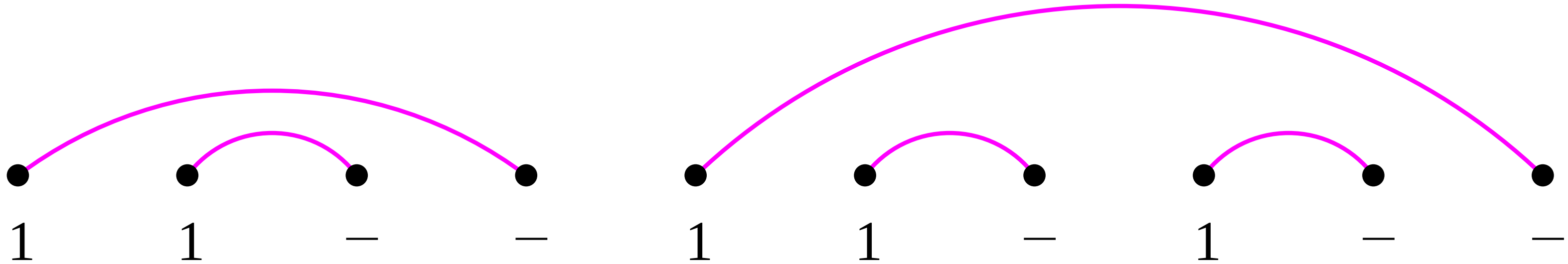

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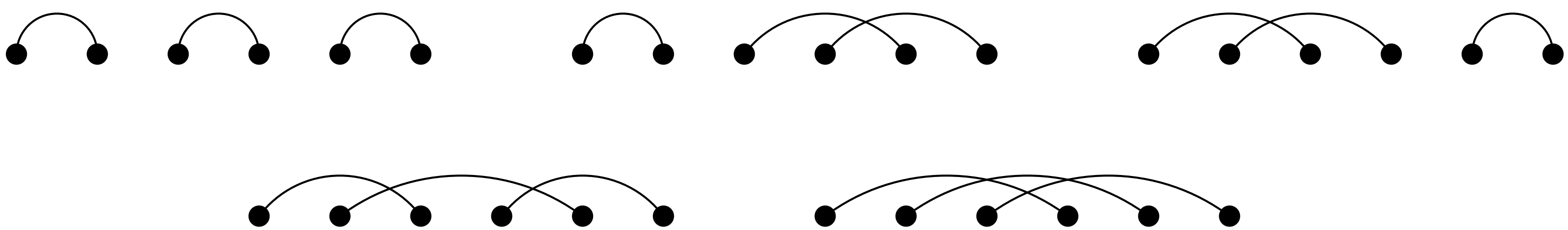
Nonnesting matchings

64. Nonnesting matchings on $[2n]$, i.e., ways of connecting $2n$ points in the plane lying on a horizontal line by n arcs, each arc connecting two of the points and lying above the points, such that no arc is contained entirely below another



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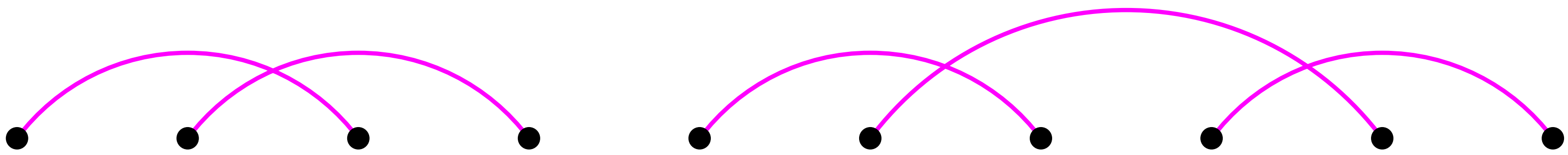

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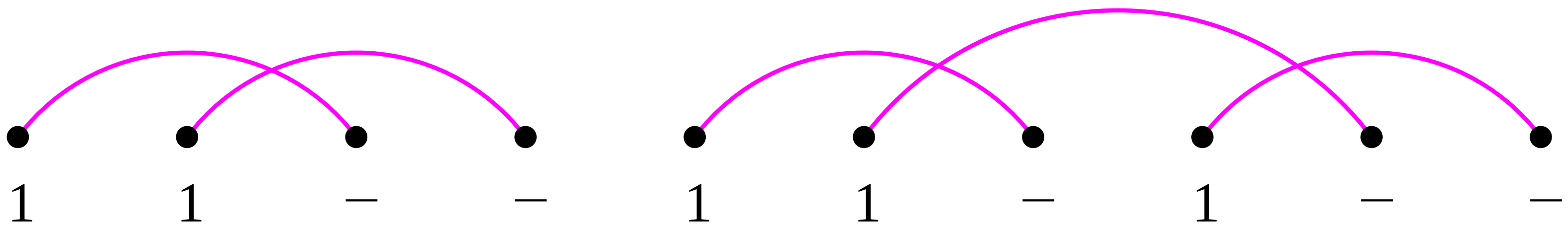
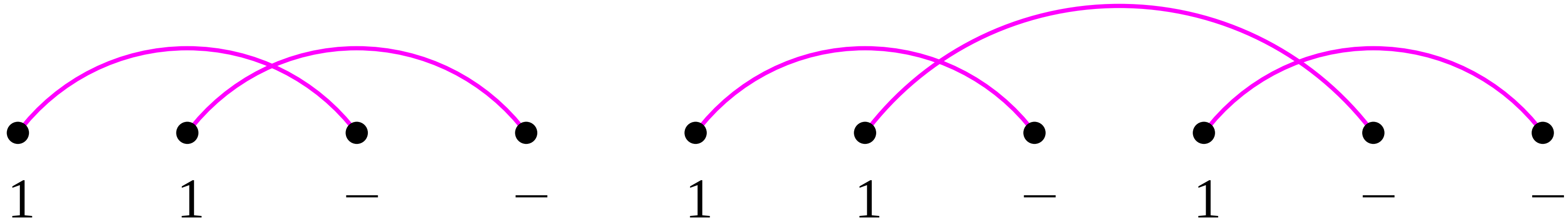


left endpoint: 1

right endpoint: -1

Bijection to ballot sequences



left endpoint: 1

right endpoint: -1

Same rule as for noncrossing matchings!

Inverse bijection

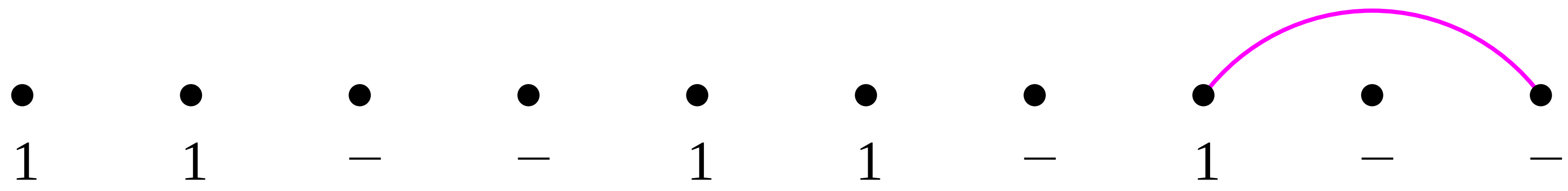
•	•	•	•	•	•	•	•	•	•
1	1	—	—	1	1	—	1	—	—

Inverse bijection

•	•	•	•	•	•	•	•	•	•
1	1	−	−	1	1	−	1	−	−

Scan ballot sequence from right-to-left. Connect each 1 with **rightmost** available -1.

Inverse bijection



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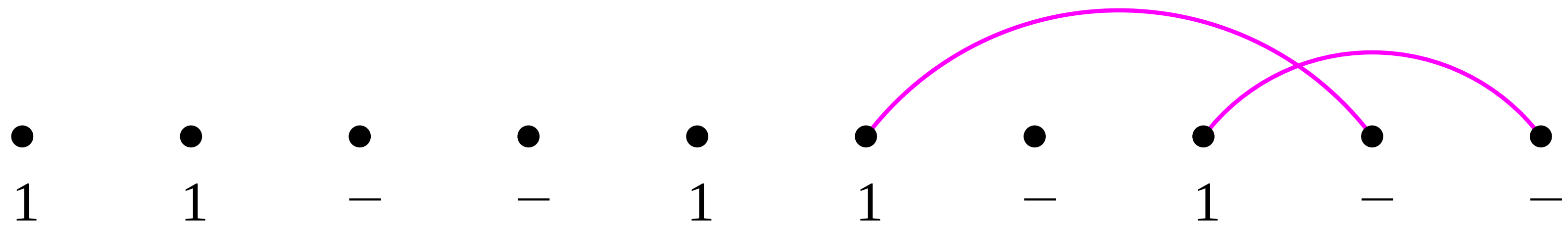

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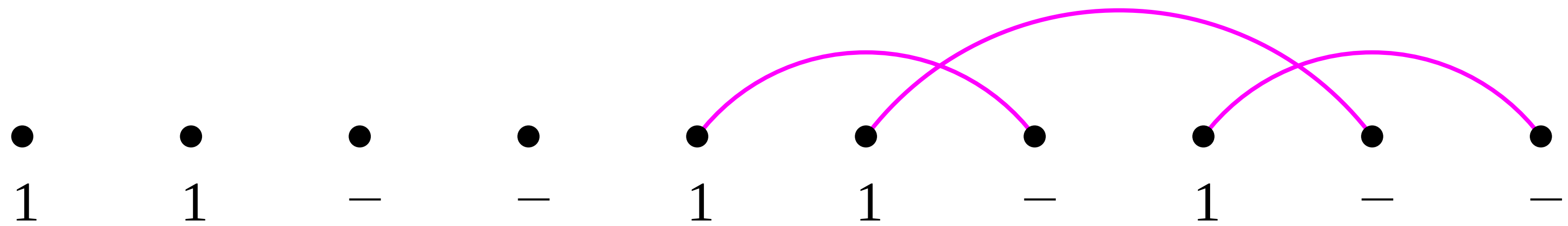

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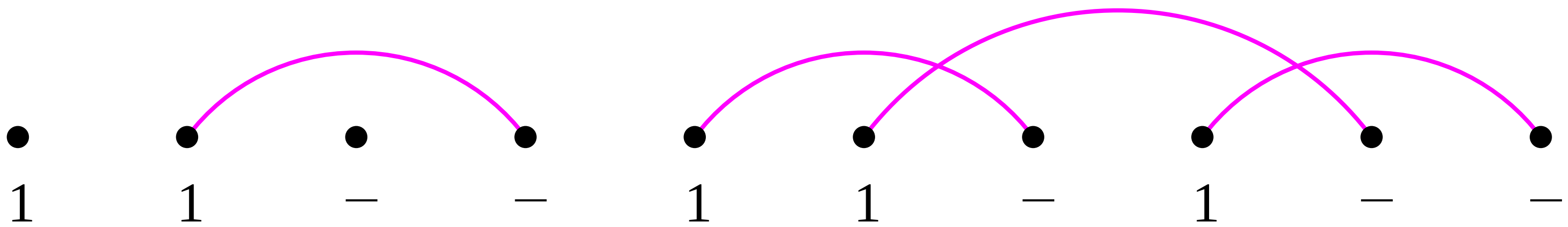

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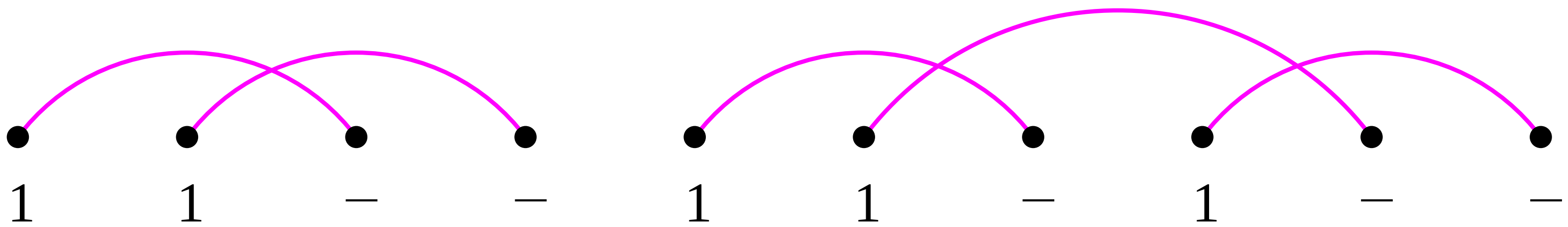

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Many interpretations

By changing the connection rule from the 1's to -1's, we get **infinitely** many combinatorial interpretations of Catalan numbers in terms of complete matchings. All have the same bijection rule from the matchings to the ballot sequences!

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Unexpected interpretations

92. n-tuples $(a_1, a_2, \ldots, a_n)$ of integers $a_i \geq 2$ such that in the sequence $1a_1a_2\cdots a_n1$, each a_i divides the sum of its two neighbors

$$14321 \quad 13521 \quad 13231 \quad 12531 \quad 12341$$

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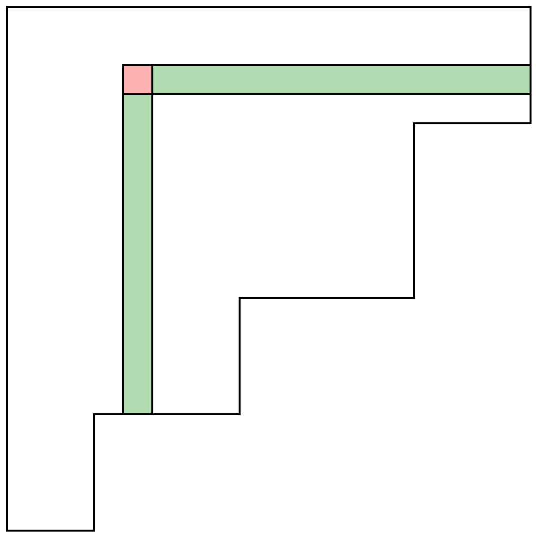
$$\to UDUUDDUD$$

Cores

hook lengths of a partition λ



8	5	4	1
6	3	2	
5	2	1	
2			
1			

p-core: a partition with no hook lengths equal to (equivalently, divisible by) p

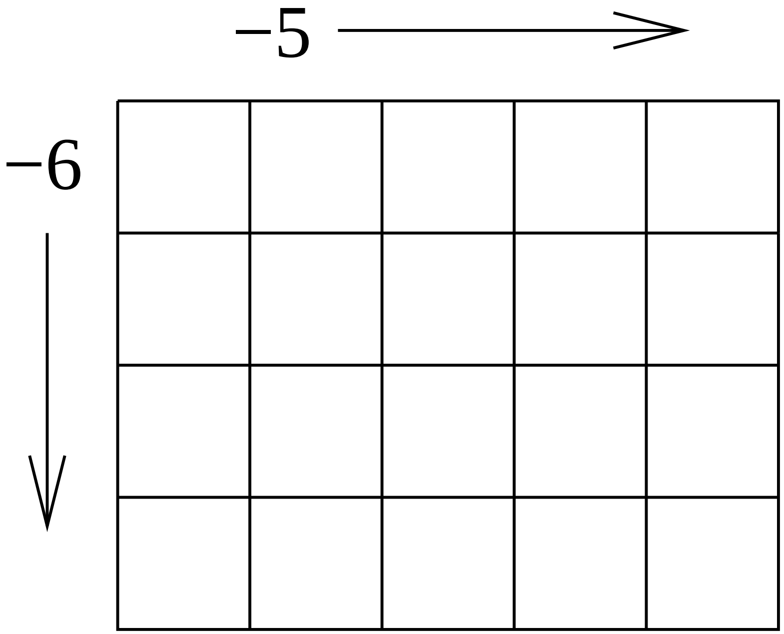
(p, q)-core: a partition that is simultaneously a p-core and q-core

$(n, n+1)$-cores

112. Integer partitions that are both n-cores and $(n+1)$-cores

$$\emptyset \quad 1 \quad 2 \quad 11 \quad 311$$

Constructing $(5, 6)$-cores



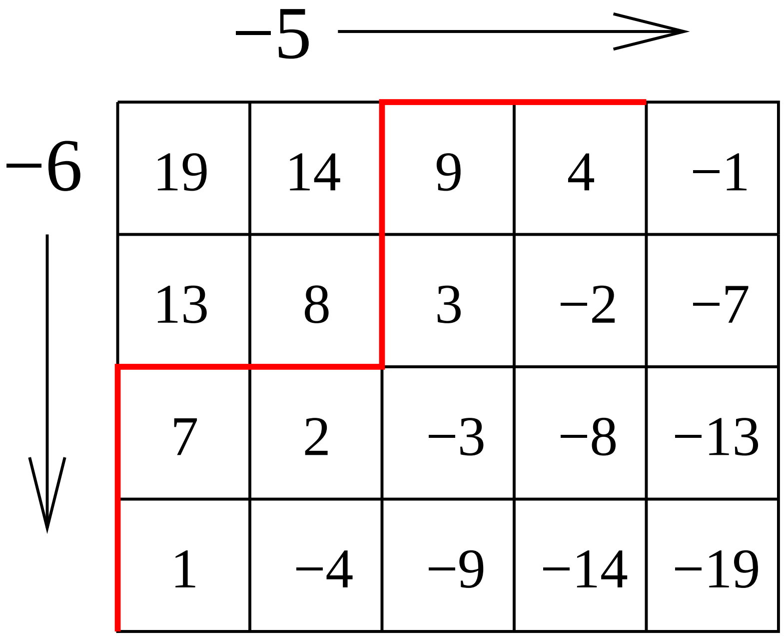
Constructing $(5, 6)$-cores

$-5 \longrightarrow$

$-6 \downarrow$

19	14	9	4	−1
13	8	3	−2	−7
7	2	−3	−8	−13
1	−4	−9	−14	−19

Constructing $(5,6)$-cores



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13	8	3	−2	−7
7	2	−3	−8	−13
1	−4	−9	−14	−19

$1 \quad 2 \quad 3 \quad 4 \quad 7 \quad 9$

Constructing $(5,6)$-cores

-5 (across) $\longrightarrow$

-6 (down) $\downarrow$

19	14	9	4	−1
13	8	3	−2	−7
7	2	−3	−8	−13
1	−4	−9	−14	−19

$$
\begin{array}{cccccccc}
 & 1 & 2 & 3 & 4 & 7 & 9 \\
- & 0 & 1 & 2 & 3 & 4 & 5 \\
 & 1 & 1 & 1 & 1 & 3 & 4
\end{array}
$$

$(4, 3, 1, 1, 1, 1)$ is a $(5, 6)$-core

9	4	3	1
7	2	1	
4			
3			
2			
1			

Inversions of permutations

inversion of $a_1a_2\cdots a_n \in \mathfrak{S}_n$: (a_i, a_j) such that $i < j$, $a_i > a_j$

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186. Sets S of n non-identity permutations in $\mathfrak{S}_{n+1}$ such that every pair (i, j) with $1 \leq i < j \leq n$ is an inversion of exactly one permutation in S

$$\{1243, 2134, 3412\}, \{1324, 2314, 4123\}, \{2134, 3124, 4123\}$$

$$\{1324, 1423, 2341\}, \{1243, 1342, 2341\}$$

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due to **R. Dewji, I. Dimitrov, A. McCabe, M. Roth, D. Wehlau, J. Wilson**

A8. Algebraic interpretations

(a) Number of two-sided ideals of the algebra of all $(n-1) \times (n-1)$ upper triangular matrices over a field

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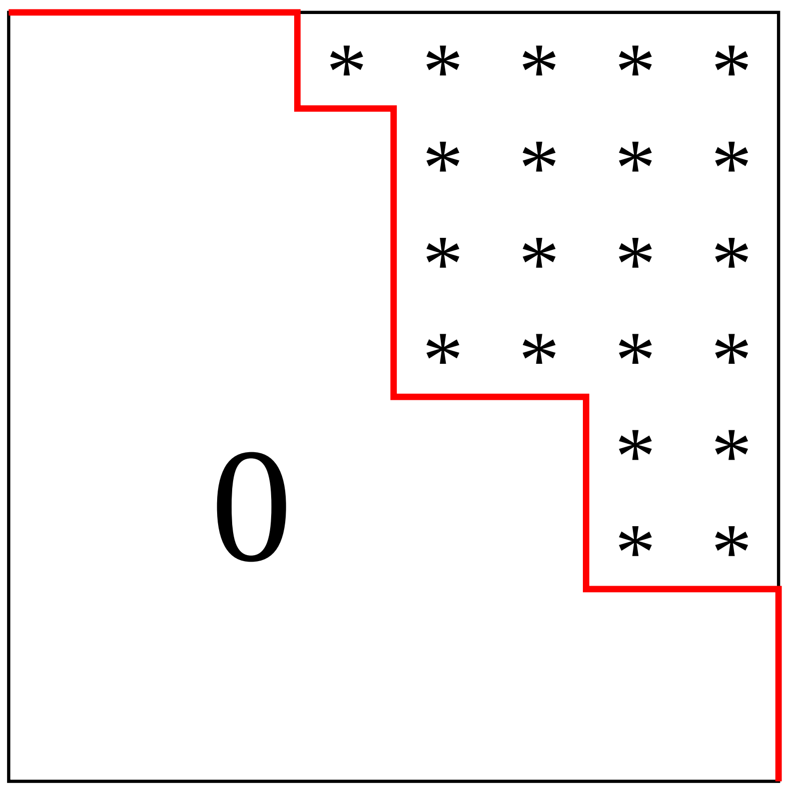

Quasisymmetric functions

Quasisymmetric function: a polynomial $f \in \mathbb{Q}[x_1, \ldots, x_n]$ such that if $i_1 < \cdots < i_n$ then

$$[x_{i_1}^{a_1} \cdots x_{i_n}^{a_n}]f = [x_1^{a_1} \cdots x_n^{a_n}]f.$$

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(k) Dimension (as a $\mathbb{Q}$-vector space) of the ring $\mathbb{Q}[x_1, \ldots, x_n]/Q_n$, where Q_n denotes the ideal of $\mathbb{Q}[x_1, \ldots, x_n]$ generated by all quasisymmetric functions in the variables $x_1, \ldots, x_n$ with 0 constant term

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Difficult proof by **J.-C. Aval**, **F. Bergeron** and **N. Bergeron**, 2004.

Diagonal harmonics

(i) Let the symmetric group $\mathfrak{S}_n$ act on the polynomial ring $A = \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ by $w \cdot f(x_1, \ldots, x_n, y_1, \ldots, y_n) = f(x_{w(1)}, \ldots, x_{w(n)}, y_{w(1)}, \ldots, y_{w(n)})$ for all $w \in \mathfrak{S}_n$. Let I be the ideal generated by all invariants of positive degree, i.e.,

$$I = \langle f \in A : w{\cdot}f = f \text{ for all } w \in \mathfrak{S}_n, \text{ and } f(0) = 0\rangle.$$

Diagonal harmonics (cont.)

Then C_n is the dimension of the subspace of A/I affording the sign representation, i.e.,

$$C_n = \dim\{f \in A/I : w\cdot f = (\text{sgn}\, w)f \text{ for all } w \in \mathfrak{S}_n\}.$$

Diagonal harmonics (cont.)

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Very deep proof by **M. Haiman**, 1994.

Generalizations & refinements

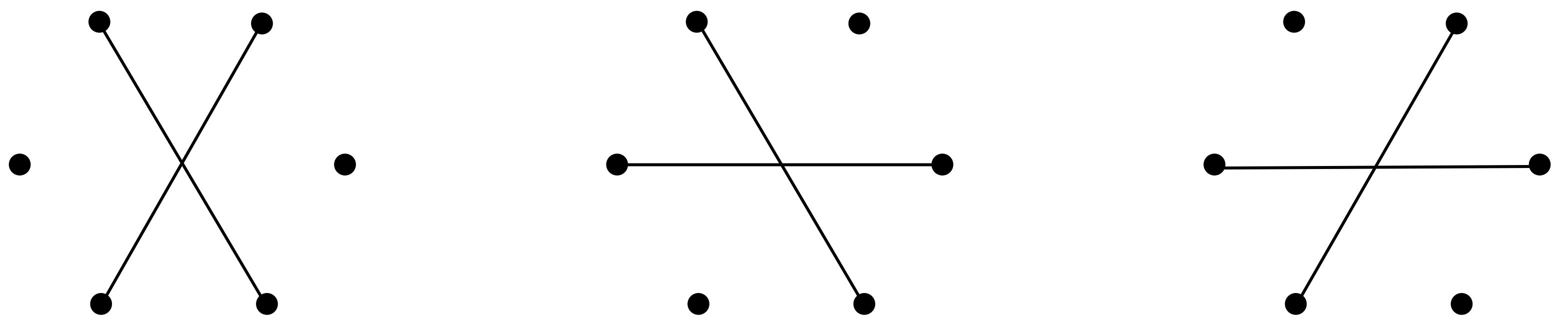
A12. ***k*-triangulation** of n-gon: maximal collections of diagonals such that no $k+1$ of them pairwise intersect in their interiors

$k=1$: an ordinary triangulation

superfluous edge: an edge between vertices at most k steps apart (along the boundary of the n-gon). They appear in all k-triangulations and are irrelevant.

An example

Example. 2-triangulations of a hexagon (superfluous edges omitted):



Some theorems

Theorem (**Nakamigawa**, **Dress-Koolen-Moulton**). *All* k*-triangulations of an* n*-gon have* $k(n-2k-1)$ *nonsuperfluous edges.*

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Theorem (**Nakamigawa, Dress-Koolen-Moulton**). *All k-triangulations of an n-gon have $k(n-2k-1)$ nonsuperfluous edges.*

Theorem (**Jonsson, Serrano-Stump**). *The number $T_k(n)$ of k-triangulations of an n-gon is given by*

$$T_k(n) = \det\left[C_{n-i-j}\right]_{i,j=1}^{k} = \prod_{1\leq i<j\leq n-2k} \frac{2k+i+j-1}{i+j-1}.$$

∎

Representation theory?

Note. The number $T_k(n)$ is the dimension of an irreducible representation of the symplectic group $\mathrm{Sp}(2n-4)$.

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Is there a direct connection?

Number theory

A61. Let $b(n)$ denote the number of 1's in the binary expansion of n. Using Kummer's theorem on binomial coefficients modulo a prime power, show that the exponent of the largest power of 2 dividing C_n is equal to $b(n+1)-1$.

Sums of three squares

Let $\boldsymbol{f(n)}$ denote the number of integers $1 \leq k \leq n$ such that k is the sum of three squares (of nonnegative integers). Well-known:

$$\lim_{n\to\infty} \frac{f(n)}{n} = \frac{5}{6}.$$

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Let $g(n)$ denote the number of integers $1 \le k \le n$ such that C_k is the sum of three squares. Then

$$\lim_{n\to\infty} \frac{g(n)}{n} = ??.$$

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A63. Let $g(n)$ denote the number of integers $1 \le k \le n$ such that C_k is the sum of three squares. Then

$$\lim_{n\to\infty} \frac{g(n)}{n} = \frac{7}{8}.$$

Analysis

A65.(b)

$$\sum_{n\geq 0} \frac{1}{C_n} = ??$$

Analysis

A65.(b)

$$\sum_{n\geq 0} \frac{1}{C_n} = 2 + \frac{4\sqrt{3}\pi}{27}.$$

Why?

A65.(a)

$$\sum_{n\geq 0}\frac{x^n}{C_n}=\frac{2(x+8)}{(4-x)^2}+\frac{24\sqrt{x}\sin^{-1}\left(\frac{1}{2}\sqrt{x}\right)}{(4-x)^{5/2}}.$$

Why?

A65.(a)

$$\sum_{n\geq 0}\frac{x^n}{C_n}=\frac{2(x+8)}{(4-x)^2}+\frac{24\sqrt{x}\sin^{-1}\left(\frac{1}{2}\sqrt{x}\right)}{(4-x)^{5/2}}.$$

Consequence of

$$2\left(\sin^{-1}\frac{x}{2}\right)^2=\sum_{n\geq 1}\frac{x^{2n}}{n^2\binom{2n}{n}}.$$

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Consequence of

$$2\left(\sin^{-1}\frac{x}{2}\right)^2=\sum_{n\geq 1}\frac{x^{2n}}{n^2\binom{2n}{n}}.$$

$$\sum_{n\geq 0}\frac{4-3n}{C_n}=2.$$

An outlier

Euler (1737):

$$e = 1 + \cfrac{2}{1 + \cfrac{1}{6 + \cfrac{1}{10 + \cfrac{1}{14 + \cfrac{1}{18 + \cfrac{1}{22 + \cdots}}}}}}.$$

Convergents: $1, 3, \frac{19}{7}, \frac{193}{71}, \ldots$.

A curious generating function

a_n: numerator of the nth convergent

$$a_1 = 1,\ a_2 = 3,\ a_3 = 19,\ a_4 = 193$$

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a_n: numerator of the nth convergent

$$a_1 = 1,\ a_2 = 3,\ a_3 = 19,\ a_4 = 193$$

$$1 + \sum_{n \geq 1} a_n \frac{x^n}{n!} = \exp \sum_{m \geq 0} C_m x^{m+1}$$

The last slide

The last slide

甲乙與丙庚爲第一率與

二位二率降爲四率四率

即如三率乘一率除

The last slide

The last slide

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