

S_n : symmetric group on $\{1, 2, \dots, n\}$

$\ell(w) = \#\{(i, j) : i < j, w(i) > w(j)\}$

$s_i = (i, i + 1)$ (**adjacent transposition**)

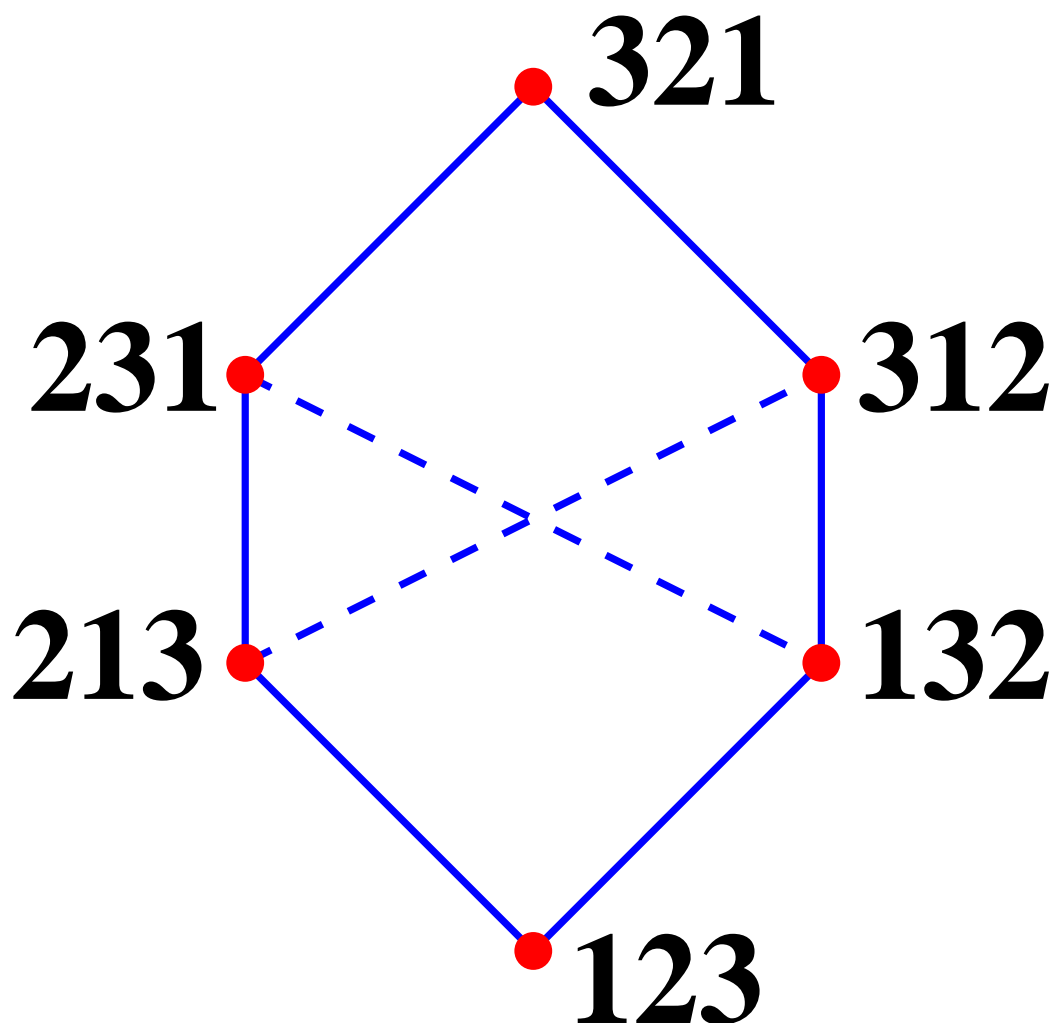
$W(S_n)$: **weak (Bruhat) order** on S_n ,
with cover relations:

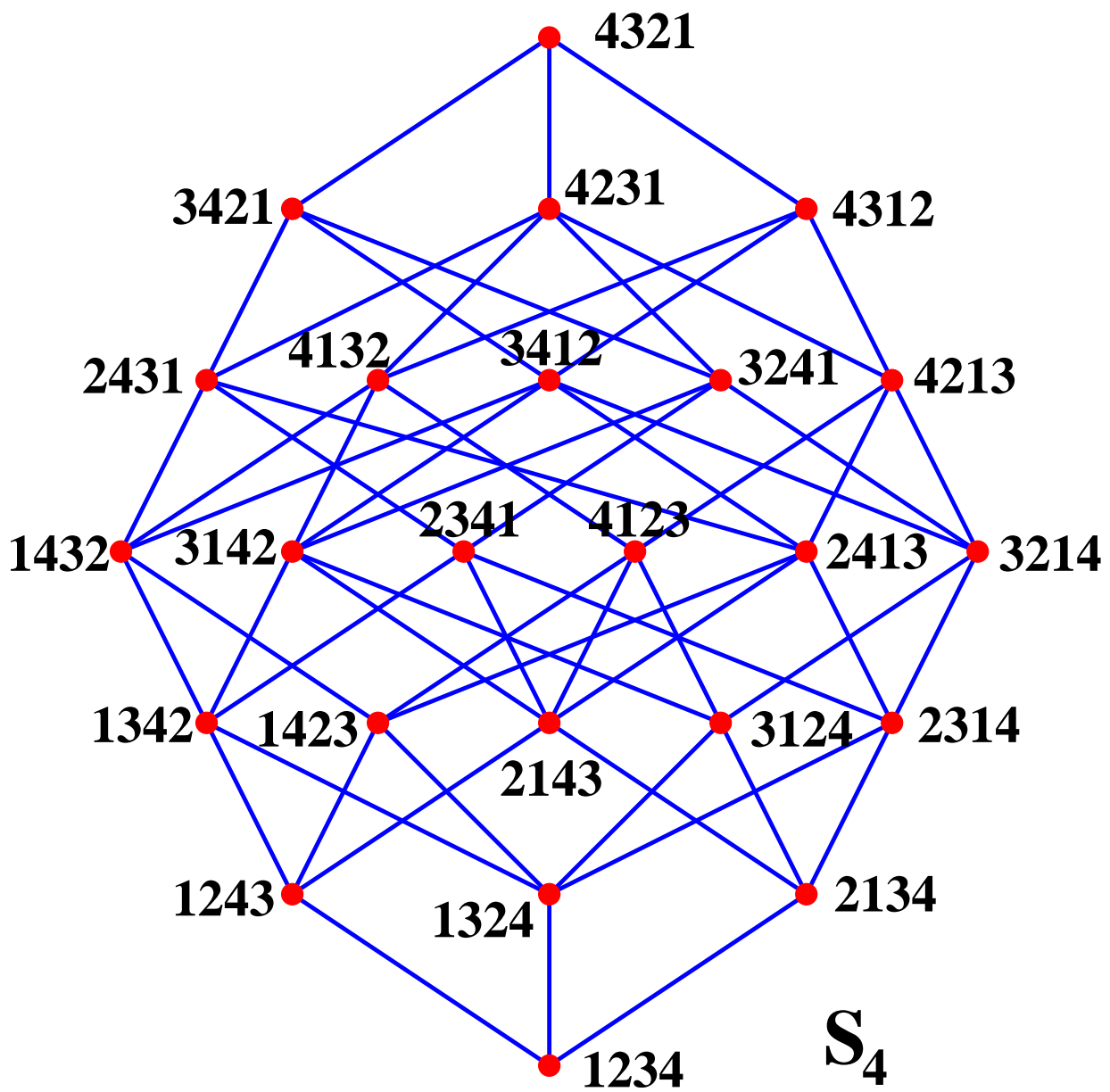
$u \prec^* v$ if $v = us_i$, $\ell(v) = 1 + \ell(u)$

S_n : **(strong) Bruhat order** on S_n ,
with cover relations:

$u \prec v$ if $v = u(i, j)$, $\ell(v) = 1 + \ell(u)$

$u = 6\underbrace{2718}_{\text{all } < 2 \text{ or } > 4}453 \prec 64718253 = u(2, 6)$





S_n is a graded poset, where $\text{rank}(w) = \ell(w)$. Thus the **rank-generating function** of S_n is given by

$$\begin{aligned} \mathbf{F}(S_n, q) &:= \sum_{w \in S_n} q^{\text{rank}(w)} \\ &= (1+q)(1+q+q^2) \cdots (1+q+\cdots+q^{n-1}). \end{aligned}$$

Motivation. Let K be a field and

$$\mathcal{F}(K^n) = \text{GL}(n, \mathbb{C})/B$$

the set of all (complete) **flags**

$$0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = K^n$$

of subspaces of K^n (so $\dim V_i = i$).

For every such flag F , there are unique vectors $v_1, \dots, v_n \in K^n$ such that:

- $\{v_1, \dots, v_i\}$ is a basis for V_i
- The $n \times n$ matrix with rows v_1, \dots, v_n has the form

$$\begin{array}{cccccc}
 * & * & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
 \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
 \mathbf{0} & * & \mathbf{0} & * & * & \mathbf{1} \\
 \mathbf{0} & * & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\
 \mathbf{0} & * & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\
 \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}
 \end{array}$$

The positions of the $\mathbf{1}$'s define a permutation $\mathbf{w}_F = 316452$. The number of $*$'s is $\ell(\mathbf{w}_F)$.

For $w \in S_n$ define the **Bruhat cell**

$$\Omega_w = \{F \in \mathcal{F}(K^n) : w = w_F\}.$$

Thus

$$\mathcal{F}(K^n) = \bigsqcup_{w \in S_n} \Omega_w,$$

the **Bruhat decomposition** of $\mathcal{F}(K^n)$.

$$\begin{array}{ccc} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} * & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & * & 1 \\ 0 & 1 & 0 \end{bmatrix} \\ \begin{bmatrix} * & 1 & 0 \\ * & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} & \begin{bmatrix} * & * & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} & \begin{bmatrix} * & * & 1 \\ * & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

$\overline{\Omega}_w$: **closed** Bruhat cell

Theorem (Ehresmann, 1934)

$$\overline{\Omega}_v \subseteq \overline{\Omega}_w \Leftrightarrow v \leq w$$

(Bruhat order).

Example. $213 < 312$

$$\begin{bmatrix} * & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} < \begin{bmatrix} * & * & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$
$$\begin{bmatrix} ax & x & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{x \rightarrow \infty} \begin{bmatrix} a & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Topology of the Bruhat order

P : finite poset

Define the **Möbius function**

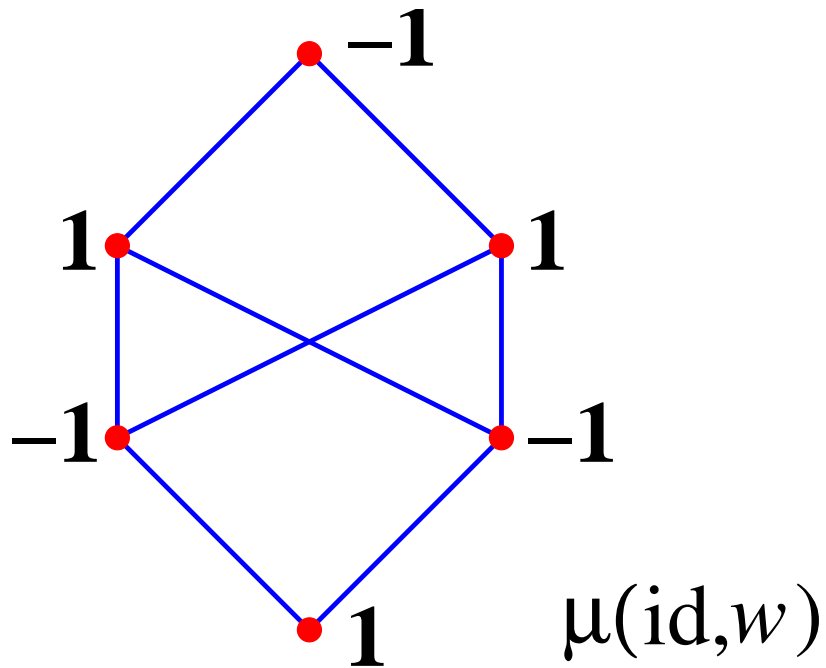
$$\mu : P \times P \rightarrow \mathbb{Z}$$

recursively by:

$$\mu(x, y) = \begin{cases} 0, & \text{unless } x \leq y \\ 1, & x = y \\ - \sum_{x \leq z < y} \mu(x, z), & x < y \end{cases}$$

Thus

$$x < y \Rightarrow \sum_{x \leq z \leq y} \mu(x, z) = 0.$$



Theorem (Verma, 1971) *For $v \leq w$ in S_n we have*

$$\mu(v, w) = (-1)^{\ell(w) - \ell(v)}.$$

For $x \leq y$ in any finite poset P , let c_i be the number of chains

$$x < x_0 < x_1 < \cdots < x_i < y,$$

with $c_{-1} = 1$.

Theorem (P. Hall, 1936)

$$\mu(x, y) = -c_{-1} + c_0 - c_1 + c_2 - \cdots$$

order complex $\Delta(x, y)$: the abstract simplicial complex on the set

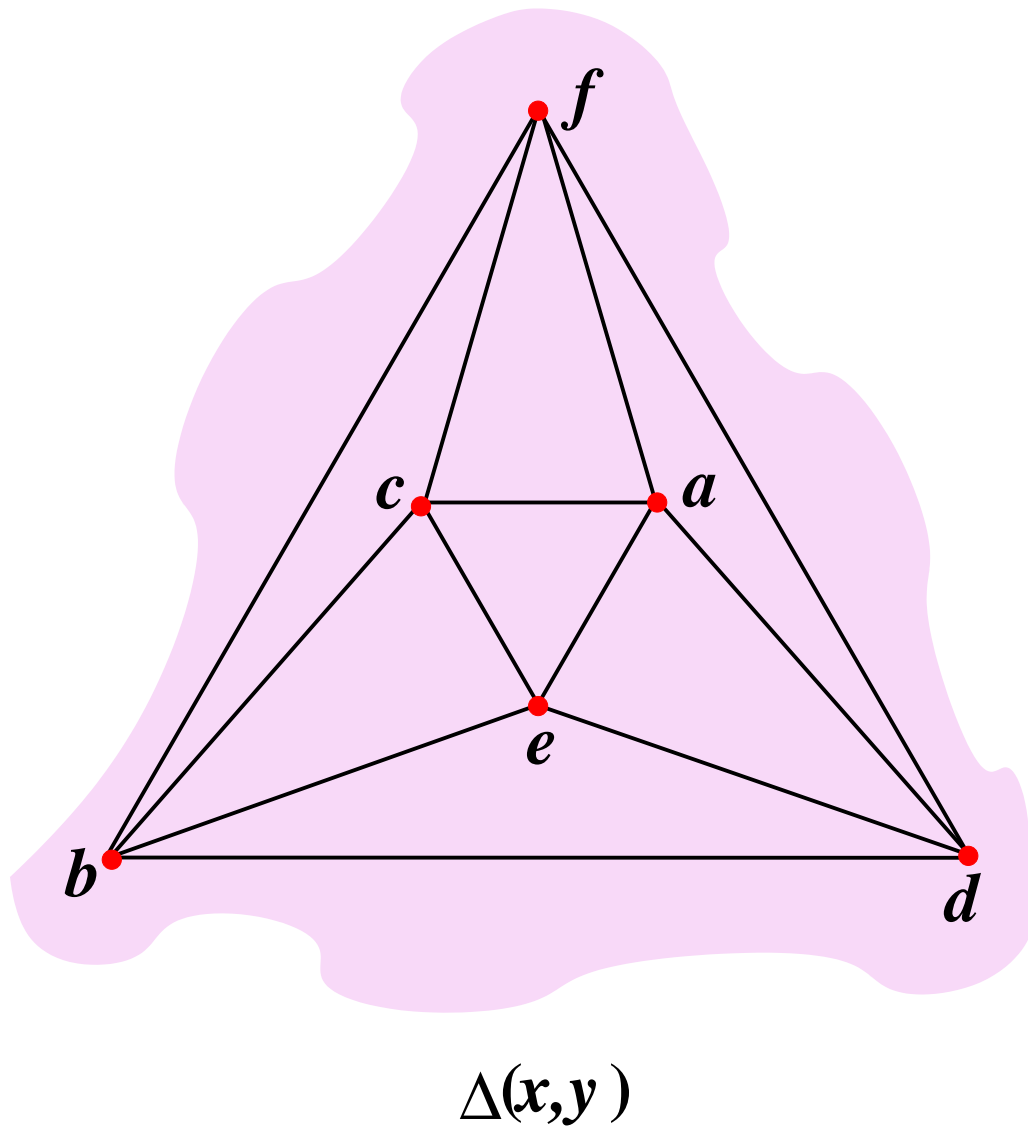
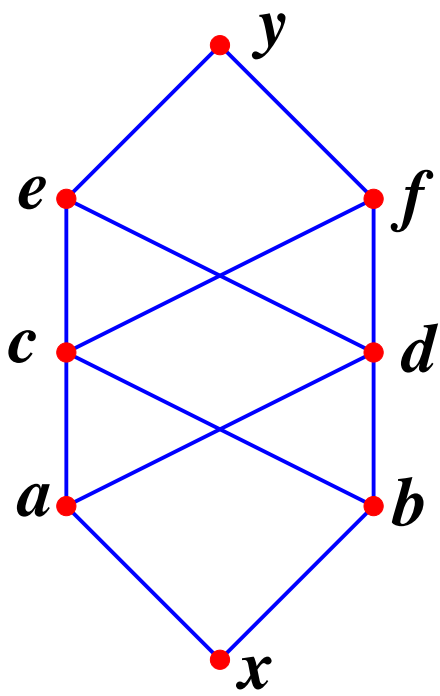
$$(x, y) = \{z \in P : x < z < y\}$$

whose faces (simplices) are the chains in (x, y) .

P. Hall's theorem restated:

$$\mu(x, y) = \tilde{\chi}(\Delta(x, y)),$$

the **reduced Euler characteristic** of $\Delta(x, y)$.



Verma's theorem on μ for S_n suggests:

Conjecture. For all $v \leq w$ in S_n , $\Delta(x, y)$ is a triangulation of a sphere.

Note. Given an abstract simplicial complex Δ , it is **undecidable** whether Δ triangulates a sphere.

Basic tool: lexicographic shellability (Björner, Wachs). Let P be a finite graded poset with $\hat{0}$ and $\hat{1}$, with

$$\mu(x, y) = (-1)^{\text{rank}(y) - \text{rank}(x)} \quad \forall x \leq y$$

(i.e., P is **Eulerian**). Let

$$\lambda : \mathcal{E}_P \rightarrow \{1, 2, \dots\}$$

be a labeling of the edges of the (Hasse) diagram of P satisfying

- For all $x < y$, \exists a unique saturated **increasing** chain

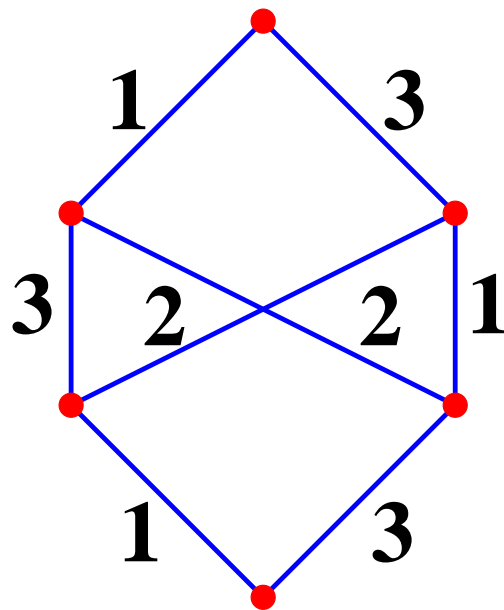
$$C : x_0 \prec x_1 \prec \cdots \prec x_r = y, \text{ i.e.,}$$

$$\lambda(x_0, x_1) \leq \lambda(x_1, x_2) \leq \cdots \leq \lambda(x_{r-1}, x_r).$$

- The label sequence of C lexicographically precedes that of all other saturated chains from x to y .

Call λ an **EL-labeling**.

Theorem (Björner) *Let P be a finite Eulerian poset with an EL-labeling. Then for all $x < y$ in P , $\Delta(x, y)$ triangulates a sphere.*



First EL-labeling of S_n due to Edelman (1981):

Let $\tau_1, \tau_2, \dots, \tau_{\binom{n}{2}}$ be the transpositions in S_n in lexicographic order. E.g., $n = 4$:

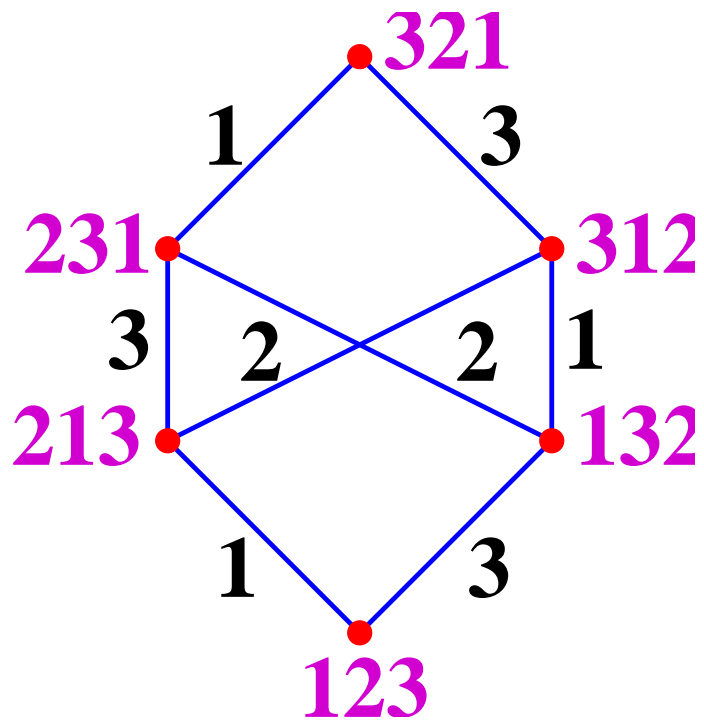
$$\tau_1 = (1, 2), \tau_2 = (1, 3), \tau_3 = (1, 4)$$

$$\tau_4 = (2, 3), \tau_5 = (2, 4), \tau_6 = (3, 4)$$

Let $w \succ v$ in S_n . Define

$$\lambda(v, w) = j \quad \text{if} \quad \tau_j v = w.$$

Theorem (Edelman). λ is an EL-labeling of S_n , so $\forall v < w$ in S_n , $\Delta(v, w)$ triangulates a sphere.



Counting maximal chains in S_n

(with A. Postnikov)

Let $u \triangleleft^* us_i$ in $W(S_n)$. Define

$$m^*(u, us_i) = i.$$

If

$$C : u_0 \triangleleft^* u_1 \triangleleft^* u_2 \triangleleft^* \cdots \triangleleft^* u_k$$

in $W(S_n)$, then define

$$m_C^* = m^*(u_0, u_1)m^*(u_1, u_2) \cdots m^*(u_{k-1}, u_k).$$

Similarly let $u \triangleleft u(i, j)$ in S_n , and define

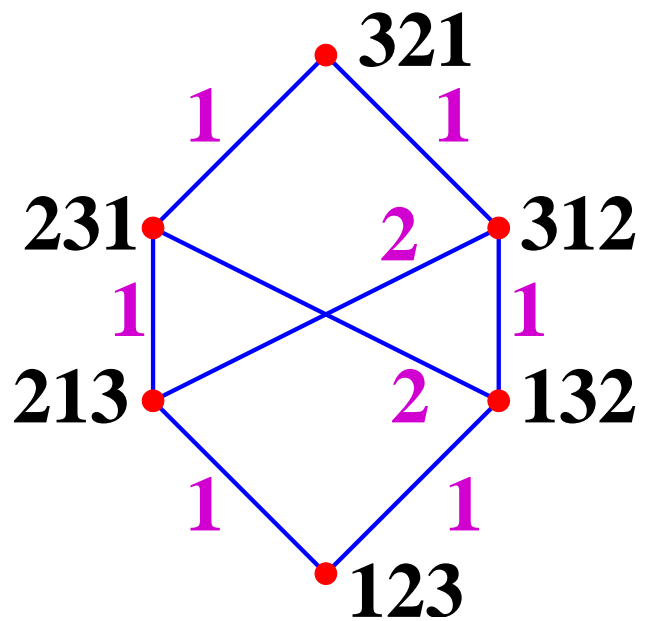
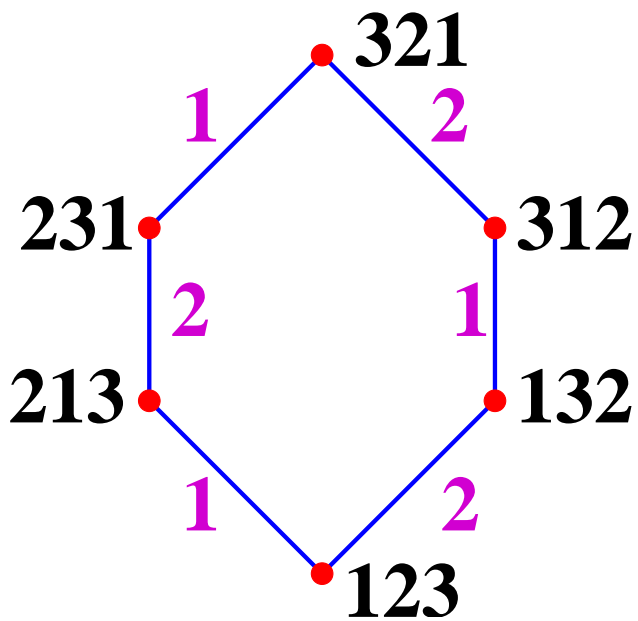
$$m(u, u(i, j)) = j - i.$$

If

$$C : u_0 \triangleleft u_1 \triangleleft u_2 \triangleleft \cdots \triangleleft u_k$$

in S_n , then define

$$m_C = m(u_0, u_1)m(u_1, u_2) \cdots m(u_{k-1}, u_k).$$



Let $\mathcal{M}(P)$ denote the set of maximal chains of the poset P . Thus

$$\begin{aligned} \sum_{C \in \mathcal{M}(W(S_3))} m_C^* &= 1 \cdot 2 \cdot 1 + 2 \cdot 1 \cdot 2 \\ &= 6 \end{aligned}$$

$$\begin{aligned} \sum_{C \in \mathcal{M}(S_3)} m_C &= 1 \cdot 1 \cdot 1 + 1 \cdot 2 \cdot 1 \\ &\quad + 1 \cdot 2 \cdot 1 + 1 \cdot 1 \cdot 1 \\ &= 6. \end{aligned}$$

Theorem. (a) (Macdonald; Fomin & RS)

$$\sum_{C \in \mathcal{M}(W(S_n))} m_C^* = \binom{n}{2}!$$

(b) (Stembridge (explicitly))

$$\sum_{C \in \mathcal{M}(S_n)} m_C = \binom{n}{2}!$$

Open. A bijective proof of (a) or (b), or a bijective proof that (a) = (b).

Generalize the definition $m(u, u(i, j))$ to

$$\mathbf{m}(u, u(i, j)) = \lambda_i - \lambda_j.$$

(Original definition corresponds to $\lambda_i = -i$.)

As before, if

$$C : u_0 \triangleleft u_1 \triangleleft u_2 \triangleleft \cdots \triangleleft u_k$$

in S_n , then define

$$\mathbf{m}_C(\lambda) = m(u_0, u_1)m(u_1, u_2) \cdots m(u_{k-1}, u_k).$$

If $u \leq v$ in S_n , define

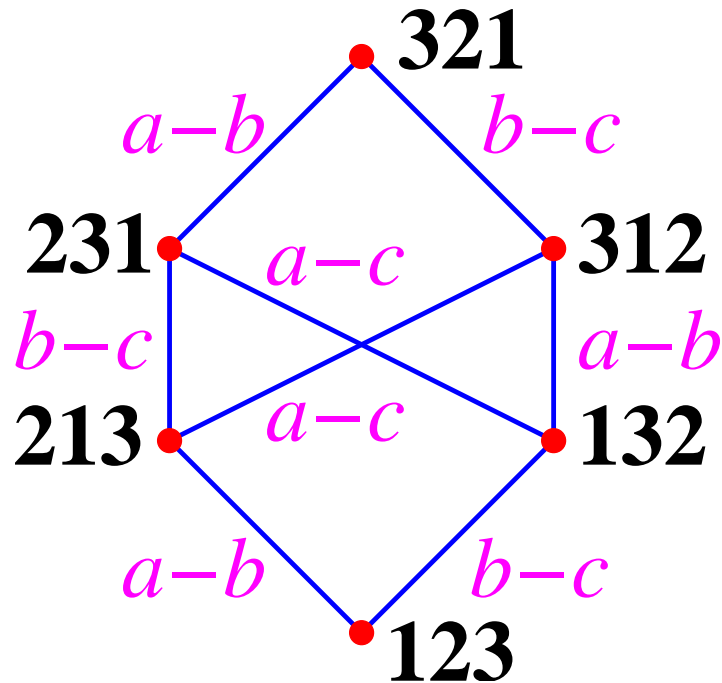
$$\mathbf{D}_{u,v}(\lambda) = \frac{1}{(\ell(v) - \ell(u))!} \sum_C m_C(\lambda),$$

where C ranges over all saturated chains from u to v in S_n .

Set

$$\mathbf{D}_w(\lambda) = \mathbf{D}_{\text{id},w}(\lambda).$$

Write $a = \lambda_1$, $b = \lambda_2$, $c = \lambda_3$.



$$u = 123 = \text{id}, \quad v = 321 = w_0$$

$$\begin{aligned} \mathcal{D}_{321}(\lambda) &= \frac{1}{3!}((a-b)(b-c)(a-b) \\ &\quad + (a-b)(a-c)(b-c) \\ &\quad + (b-c)(a-c)(a-b) \\ &\quad + (b-c)(a-b)(b-c)) \\ &= \frac{1}{2}(a-b)(a-c)(b-c) \end{aligned}$$

Schubert polynomials. Define the **divided difference operator** ∂_i by

$$\partial_i f(x_i, x_{i+1}) = \frac{f(x_i, x_{i+1}) - f(x_{i+1}, x_i)}{x_i - x_{i+1}}.$$

Let (a_1, a_2, \dots, a_p) be a **reduced decomposition** of $w^{-1}w_0 \in S_n$, i.e.,

$$w^{-1}w_0 = s_{a_1} \cdots s_{a_p}, \quad p = \ell(w^{-1})w_0.$$

Schubert polynomial:

$$\mathfrak{S}_w = \partial_{a_1} \cdots \partial_{a_p} x_1^{n-1} x_2^{n-2} \cdots x_{n-1}.$$

Example. (a) $\mathfrak{S}_{w_0} = x_1^{n-1} x_2^{n-2} \cdots x_{n-1}$
 (b) $(s_2)^{-1}w_0 = s_1 s_2$, so

$$\begin{aligned} \mathfrak{S}_{s_2} &= \partial_1 \partial_2 x_1^2 x_2 \\ &= x_1 + x_2. \end{aligned}$$

Theorem (B-J-S).

$$\mathfrak{S}_w = \sum_{(a_1, \dots, a_p)} \sum_{(i_1, \dots, i_p)} x_{i_1} \cdots x_{i_p},$$

where

- (a_1, \dots, a_p) ranges over all reduced decompositions of w
- $1 \leq i_1 \leq \dots \leq i_p$
- $i_j < i_{j+1}$ if $a_j < a_{j+1}$
- $i_j \leq a_j$

Example. $w = 2143 = s_1 s_3 = s_3 s_1$

$$(a_1, a_2) = (1, 3) \Rightarrow (i_1, i_2) = (1, 2), (1, 3)$$

$$(a_1, a_2) = (3, 1) \Rightarrow (i_1, i_2) = (1, 1),$$

$$\text{so } \mathfrak{S}_{2143} = x_1^2 + x_1 x_2 + x_1 x_3.$$

Regard $S_n \subset S_{n+1}$ via $w(n+1) = n+1$ for $w \in S_n$. Let

$$S_\infty = \bigcup S_n,$$

the permutations of $\{1, 2, \dots\}$ moving finitely many letters. Then $\{\mathfrak{S}_w : w \in S_\infty\}$ is a \mathbb{Z} -basis for $\mathbb{Z}[x_1, x_2, \dots]$.

Monk's rule for $(x_1 + x_2 + \dots + x_i)\mathfrak{S}_w$ gives:

$$(\lambda_1 x_1 + \lambda_2 x_2 + \dots)^k \mathfrak{S}_u = k! \sum_{\ell(v)=k+\ell(u)} \mathcal{D}_{u,v}(\lambda) \mathfrak{S}_v.$$

Geometric interpretation of \mathfrak{S}_w .

$H^*(\mathcal{F}(\mathbb{C}^n); \mathbb{R})$: cohomology ring

basis : $\{[\bar{\Omega}_w] : w \in S_n\}$

Let

$$e_j = \sum_{1 \leq i_1 < \dots < i_j \leq n} x_{i_1} \cdots x_{i_j}.$$

Theorem.

$$H^*(\mathcal{F}(\mathbb{C}^n); \mathbb{R}) \cong \mathbb{R}[x_1, \dots, x_n] / (e_1, \dots, e_n)$$
$$[\bar{\Omega}_w] \leftrightarrow \mathfrak{S}_w$$

Geometric interpretation of $\mathcal{D}_w(\lambda)$.

Let $\Phi \subset \mathfrak{h}^*$ denote the root lattice for the Lie algebra $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ of $G = \mathrm{SL}(n, \mathbb{C})$.

Let

$$\Lambda = \{\lambda \in \mathfrak{h}^* : \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \text{ for any } \alpha \in \Phi\},$$

the weight lattice of \mathfrak{g} . Let $\lambda \in \Lambda^+$ be a dominant weight. Let

$$V_\lambda = \lambda\text{-weight space}$$

$$v_\lambda \in V_\lambda : \text{highest weight vector}$$

$\mathbb{P}(V_\lambda)$ = projectivization of V_λ

$$e : G/B \rightarrow \mathbb{P}(V_\lambda)$$

$$e(gB) = g(v_\lambda)$$

$\bar{\Omega}_w \subset G/B$ (Schubert variety)

Thus e is a projective embedding $G/B \hookrightarrow \mathbb{P}(V_\lambda)$. Define the **λ -degree** of $\bar{\Omega}_w$ by:

$$\mathbf{deg}_\lambda(\bar{\Omega}_w) = \#(e(\bar{\Omega}_w) \cap L),$$

where L is a generic linear subspace of $\mathbb{P}(V_\lambda)$ of complex codimension $\ell(w)$.

Theorem. $\mathbf{deg}_\lambda(\bar{\Omega}_w) = \ell(w)! \mathcal{D}_w(\lambda)$

An expression for $\mathcal{D}_{u,v}(\lambda)$.

Theorem. Let $w \in S_n$ and

$$V_n = \frac{1}{1! 2! \cdots (n-1)!} \prod_{1 \leq i < j \leq n} (\lambda_i - \lambda_j).$$

Then

$$\mathcal{D}_{u,v} = \mathfrak{S}_u \left(\frac{\partial}{\partial \lambda_1}, \frac{\partial}{\partial \lambda_2}, \dots \right) \mathfrak{S}_{w_0 v} \left(\frac{\partial}{\partial \lambda_1}, \frac{\partial}{\partial \lambda_2}, \dots \right) \cdot V_n.$$

In particular,

$$\mathcal{D}_w = \mathcal{D}_{\text{id},w} = \mathfrak{S}_{w_0 w} \left(\frac{\partial}{\partial \lambda_1}, \frac{\partial}{\partial \lambda_2}, \dots \right) \cdot V_n.$$

$$\mathcal{D}_{w_0} = V_n \quad (\dots, \text{Stembridge})$$

Corollary. $\{\mathcal{D}_w : w \in S_\infty\}$ is a \mathbb{Z} -basis for $\mathbb{Z}[\lambda_1, \lambda_2, \dots]$. Let

$$\mathfrak{S}_u \mathfrak{S}_v = \sum_w c_{u,v}^w \mathfrak{S}_w.$$

Then

$$\mathcal{D}_{u,w} = \sum_v c_{u,v}^w \mathcal{D}_v.$$

Note. (1) $\{\mathcal{D}_w : w \in S_n\}$ is a \mathbb{Z} -basis for Har_n , the harmonic polynomials in $\mathbb{Z}[\lambda_1, \dots, \lambda_n]$.

(2) $\langle \mathfrak{S}_u, \mathcal{D}_v \rangle = \delta_{uv}$ under the “ D -pairing”

$$\langle f, g \rangle = f \left(\frac{\partial}{\partial x_1}, \dots \right) g(x_1, x_2, \dots) \Big|_{x_i=0}.$$

Corollary. Let $w \in S_n$ be 312-avoiding, i.e.,

$$a < b < c \Rightarrow \mathbf{not} \ w(b) < w(c) < w(a).$$

Let $\mathbf{code}(w_0w) = (c_1, c_2, \dots)$, where

$$c_i = \#\{j : i < j, w(i) > w(j)\}.$$

Then

$$\mathcal{D}_w = \det \left(\frac{\lambda_i^{n-c_i-j}}{(n-c_i-j)!} \right)_{i,j=1}^n,$$

where $\alpha^k/k! = 0$ if $k < 0$.

Idea of proof.

w 312-avoiding $\Rightarrow w_0w$ 132-avoiding

(dominant)

$$\Rightarrow \mathfrak{S}_{w_0w} = x_1^{c_1} x_2^{c_2} \cdots, \text{ etc.}$$

Other special values are interesting, e.g.,

$$\mathcal{D}_{41532} = \frac{1}{12}(f(5, 4, 2) - f(5, 4, 1) - f(5, 3, 2) + f(5, 3, 1) + f(4, 3, 2) - f(4, 3, 1)),$$

where

$$f(i, j, k) = (x_i - x_j)(x_i - x_k)(x_j - x_k).$$

Note that $\text{code}(4, 1, 5, 3, 2) = (3, 0, 2, 1, 0)$.

Further connections:

- Demazure characters (key polynomials)
- Gelfand-Tsetlin polytopes
- inverse “Schubert Kostka” matrix