Some Combinatorial Aspects of Cyclotomic Polynomials

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Partitions

partition of $n \geq 0$: an integer sequence $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots)$ satisfying $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$ and $\sum \lambda_i = n$

partitions of 5: 5, 41, 32, 311, 221, 2111, 11111

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$\boldsymbol{f(n)}$: number of partitions of n for which no part appears exactly once

Example. $f(8) = 6$: 44, 3311, 2222, 22211, 221111, 11111111

A theorem of MacMahon

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Proof.

$$\begin{aligned}
\sum_{n\geq 0} f(n)x^n &= \prod_{i\geq 1}(1 + x^{2i} + x^{3i} + x^{4i} + \cdots) \\
&= \prod_{i\geq 1}\left(\frac{1}{1-x^i} - x^i\right) \\
&= \prod_{i\geq 1}\frac{1-x^i+x^{2i}}{1-x^i} \\
&= \prod_{i\geq 1}\frac{1-x^{6i}}{(1-x^{2i})(1-x^{3i})} \\
&= \prod_{j\not\equiv\pm 1 \bmod 6)} (1-x^j)^{-1}. \qquad \square
\end{aligned}$$

Why does this work?

$\Phi_n(x)$: the nth **cyclotomic polynomial**

$$\Phi_1(x) = 1 - x \quad (x - 1 \text{ is standard})$$

$$\Phi_n(x) = \prod_{\substack{1 \le j \le n \\ \gcd(j,n)=1}} \left(x - e^{2\pi i j/n}\right) = \prod_{d|n} (1 - x^d)^{\mu(n/d)}, \quad n \ge 2$$

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$$= \prod_{i=1}^{k} (1 - x^i)^{a_i}, \quad a_i \in \mathbb{Z}$$

Two points

1. (the main point)

$$F(x) := \frac{1}{1-x} - x = \frac{\Phi_6(x)}{1-x} = \frac{1-x^6}{(1-x^2)(1-x^3)}$$

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2.

$$\sum_{n\geq 0} f(n)x^n = F(x)F(x^2)F(x^3)\cdots$$

$$= \frac{(1-x^6)(1-x^{12})(1-x^{18})\cdots}{(1-x^2)(1-x^4)(1-x^6)\cdots(1-x^3)(1-x^6)(1-x^9)\cdots}$$

$$= \frac{1}{(1-x^2)(1-x^3)(1-x^4)(1-x^6)(1-x^8)(1-x^9)\cdots}$$

Cyclotomic sets

Definition. A **cyclotomic set** is a subset S of $\mathbb{P} = \{1, 2, \dots\}$ such that

$$F_S(x) := \frac{1}{1-x} - \sum_{j \in S} x^j = \frac{N_S(x)}{D_S(x)},$$

where $N_S(x)$ and $D_S(x)$ are finite products of cyclotomic polynomials.

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Think of S as the set of “forbidden part multiplicities.”

An example: $S = \{1, 2, 3, 5, 7, 11\}$

$$
\begin{aligned}
F_S(x) &:= \frac{1}{1-x} - (x + x^2 + x^3 + x^5 + x^7 + x^{11}) \\
&= \frac{\Phi_6(x)\Phi_{12}(x)\Phi_{18}(x)}{1-x} \\
&= \frac{(1-x^{12})(1-x^{18})}{(1-x^4)(1-x^6)(1-x^9)}
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F_S(x)F_S(x^2)F_S(x^3)\cdots = \prod_i (1-x^i)^{-1},
$$

$$
i \equiv 0, 4, 6, 8, 9, 12, 16, 18, 20, 24, 27, 28, 30, 32 \pmod{36}. \quad (*)
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Theorem. *For all $n \geq 0$, the number of partitions of n such that no part occurs exactly $1, 2, 3, 5, 7$ or 11 times equals the number of partitions of n into parts i satisfying (*).*

A further example

$S = \{2, 3, 4, \dots\}$ is cyclotomic:

$$\frac{1}{1-x} - (x^2 + x^3 + \cdots) = 1 + x = \frac{1-x^2}{1-x}$$

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Theorem (**Euler**). *The number of partitions of n into distinct parts equals the number of partitions of n into odd parts.*

A property of finite cyclotomic sets

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Theorem. *Assume that S is finite. For $0 \leq j \leq d = \max(S)$, exactly one of j and $d - j$ belongs to S. Hence $\#S = (d+1)/2$.*

Proof sketch. Symmetry or antisymmetry of $\Phi_n(x)$ implies

$$P_S(x) + x^d P_S(1/x) = 1 + x + \cdots + x^d, \text{ where } P_S(x) = \sum_{i \in S} x^i. \quad \square$$

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Corollary. *S finite $\Rightarrow \max(S)$ is odd.*

Some data

Let d be odd. There are $2^{(d-1)/2}$ sets $S \subset \mathbb{P}$ with $\max(S) = d$ such that $N_S(x)$ is symmetric. Let $f(d)$ be the number of these that are cyclotomic. Then

d	1	3	5	7	9	11	13	15	17	19	21	23	25	27	29
$f(d)$	1	2	3	5	5	9	10	12	18	22	22	37	39	41	54

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Can show: $f(d) \leq a \exp b\sqrt{d}$.

Two infinite families

Write e.g. $125 = \{1, 2, 5\}$.

$$1, 23, 345, 4567, 56789, \ldots$$

$$1, 13, 135, 1357, \ldots$$

Cleanness

Note. Any $f(x) \in \mathbb{Z}[[x]]$ with $f(0) = 1$ can be uniquely written (formally) as

$$f(x) = \prod_{n \geq 1} (1 - x^n)^{-a_n}, \quad a_n \in \mathbb{Z}.$$

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Let ***S*** be a subset of $\mathbb{P}$ and

$$\boldsymbol{F(x)} = \frac{1}{1-x} - \sum_{j \in S} x^j.$$

S is **clean** if

$$F(x)F(x^2)F(x^3)\cdots = \prod_{n \geq 1} (1 - x^n)^{-a_n},$$

where each $a_n = 0, 1$. (Get a "clean" partition identity—no weighted or colored parts.)

An example

Not every cyclotomic set S is clean, e.g., $S = \{1, 5, 7, 8, 9, 11\}$, for which

$$F(x)F(x^2)F(x^3)\cdots = \frac{(1-x^5)(1-x^{25})(1-x^{35})(1-x^{55})\cdots}{(1-x^2)(1-x^3)(1-x^4)(1-x^6)(1-x^8)(1-x^9)(1-x^{10})(1-x^{12})\cdots}.$$

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No nice theory of clean sets.

Numerical semigroups

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Note $A_M(x) = \frac{1}{1-x} - \sum_{i \in \mathbb{N} - M} x^i$,

Cyclotomic numerical semigroups

Definition (**E.-A. Ciolan**, et al.) A numerical semigroup M is **cyclotomic** if $(1-x)A_M(x)$ is a product of cyclotomic polynomials. Equivalently, $\mathbb{N} - M$ is a cyclotomic set.

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Example. $M = \langle a, b \rangle$, where $a, b \geq 2$, $\gcd(a, b) = 1$. Then

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Example. (a) $M = \langle 4, 6, 7 \rangle = \mathbb{N} - \{1, 2, 3, 5, 9\}$ is cyclotomic.

(b) $M = \langle 5, 6, 7 \rangle = \mathbb{N} - \{1, 2, 3, 4, 8, 9\}$ is not cyclotomic.

Consequence of $\langle a, b \rangle$ being cyclotomic and clean

Theorem. *Let $a, b \geq 2$, $gcd(a, b) = 1$. Let $M = \langle a, b \rangle$. Then for all $n \geq 0$, the following numbers are equal:*

- *the number of partitions of n all of whose part multiplicities belong to M*
- *the number of partitions of n into parts divisible by a or b (or both)*

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MacMahon identity: $a = 2, b = 3$

Semigroup algebra

The **semigroup algebra** $K[M]$ (over K) of a numerical semigroup M is
$$K[M] = K[z^i : i \in M].$$

Definition. Let $M = \langle a_1, \ldots, a_r \rangle$. $K[M]$ is a **complete intersection** if all the relations among the generators $z^{a_1}, \ldots, z^{a_r}$ are consequences of $r - 1$ of them (the minimum possible).

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Converse is **open** (main open problem on cyclotomic numerical semigroups).

An example

Example. $M = \langle 4, 6, 7 \rangle = \mathbb{N} - \{1, 2, 3, 5, 9\}$. Generators of $K[M]$ are $a = z^4, b = z^6, c = z^7$. Some relations:

$$a^3 = b^2,\ a^2 b = c^2,\ a^7 = c^4,\ b^7 = c^6, \ldots.$$

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All are consequences of the first two, so $K[M]$ is a complete intersection. E.g.,

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$$c^4 = (a^2b)^2 = a^4b^2 = a^4a^3 = a^7.$$

The relation $a^3 = b^2$ has degree $3 \cdot 4 = 6 \cdot 2 = 12$.
The relation $a^2b = c^2$ has degree $2 \cdot 4 + 6 = 2 \cdot 7 = 14$

$$\Rightarrow A_M(x) = \frac{(1-x^{12})(1-x^{14})}{(1-x^4)(1-x^6)(1-x^7)}.$$

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Note. Multiply $c^2 = a^5b^2$ by b: $c^2b = a^5b^3$. Substitute a^4c for b^3: $c^2b = a^9c$. Divide by c: $bc = a^9$ (first relation). So why not just two relations?

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Answer: not allowed to divide (not a ring operation).

A theorem of Herzog

Theorem (**H. Herzog**, 1969) Let $M = \langle a, b, c \rangle$. The following conditions are equivalent.

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Thus the main open problem on cyclotomic numerical semigroups is true for semigroups with at most three generators.

A more general framework

Key fact: a partition is determined by the multiplicity of each part, and these multiplicities are independent. Equivalently, define $\lambda \cup \mu$ by

$$m_i(\lambda \cup \mu) = m_i(\lambda) + m_i(\mu),$$

where $\boldsymbol{m_i}$ denotes the multiplicity of the part i. E.g.,

$$(6,6,4,1,1) + (6,5,4,1) = (6,6,6,5,4,4,1,1,1).$$

Then $\cup$ makes the set of all partitions of all $n \geq 0$ into a free commutative monoid with unique basis $\{(1),(2),(3),\dots\}$.

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Where else can we find such monoids?

Polynomials over finite fields

Fix a prime power $\boldsymbol{q}$.

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There are q^n monic polynomials of degree n over $\mathbb{F}_q$. Every such polynomial is uniquely (up to order of factors) a product of monic irreducible polynomials. Hence

$$\sum_{n\geq 0} q^n x^n = \frac{1}{1-qx} = \prod_{m\geq 1}(1-x^m)^{-\beta(m)}.$$

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$$\begin{aligned}
\sum_{n\geq 0} f(n)x^n &= \prod_{m\geq 1}(1+x^{2m}+x^{3m}+\cdots)^{\beta(m)} \\
&= \prod_{m\geq 1}\left(\frac{1-x^{6m}}{(1-x^{2m})(1-x^{3m})}\right)^{\beta(m)} \\
&= \frac{1-qx^6}{(1-qx^2)(1-qx^3)} \\
&= \frac{1+x+x^2+x^3}{1-qx^2}-\frac{x(1+x+x^2)}{1-qx^3} \\
\Rightarrow\quad & f(n)=q^{\lfloor n/2\rfloor}+q^{\lfloor n/2\rfloor-1}-q^{\lfloor (n-1)/3\rfloor}.
\end{aligned}$$

Generalization.

Theorem. *Let S be a cyclotomic subset of $\mathbb{P}$, so*

$$\frac{1}{1-x} - \sum_{i \in S} x^i = \frac{\prod \left(1 - x^i\right)^{a_i}}{\prod \left(1 - x^j\right)^{b_j}},$$

where the products are **finite**. *Let $f(n)$ be the number of monic polynomials of degree n over $\mathbb{F}_q$ such that no irreducible factor has multiplicity $m \in S$. Then*

$$\sum f(n) x^n = \frac{\prod_i (1 - qx^i)^{a_i}}{\prod_j (1 - qx^j)^{b_j}}.$$

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Can convert to a partial fraction in q and find an explicit (though in general very lengthy) formula for $f(n)$.

An example

$$
\begin{aligned}
S &= \{1, 2, 3, 5, 7, 11\} \\
\sum_{n \geq 0} f(n) x^n &= \frac{(1 - qx^{12})(1 - qx^{18})}{(1 - qx^4)(1 - qx^6)(1 - qx^9)}
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&= \frac{\Phi_2\Phi_4\Phi_8\Phi_7\Phi_{14}}{\Phi_5(1-qx^4)} + \frac{\Phi_3\Phi_9\, x^8}{\Phi_5(1-qx^9)} \\
&\quad -\frac{\Phi_2\Phi_3\Phi_4\Phi_6^2\Phi_{12}\, x^2}{1-qx^6},
\end{aligned}
$$

where $\Phi_j = \Phi_j(x)$.

Yet another example

Let $S = \{2, 3, 4, \dots\}$. Recall

$$\frac{1}{1-x} - \sum_{i \in S} x^i = 1 + x = \frac{1-x^2}{1-x}.$$

***f*(*n*)**: number of squarefree monic polynomials of degree n over $\mathbb{F}_q$. Then

$$\begin{aligned} \sum_{n \geq 0} f(n) x^n &= \frac{1-qx^2}{1-qx} \\ &= 1 + qx + \sum_{n \geq 2} (q-1) q^{n-1} x^n \\ \Rightarrow\ & f(n) = (q-1)q^{n-1},\ n \geq 2 \ \ \text{(well-known)}, \end{aligned}$$

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\Rightarrow & \quad f(n) = (q-1)q^{n-1}, \ n \geq 2 \ \text{(well-known)},
\end{aligned}$$

a kind of analogue (though not a q-analogue in the usual sense) of Euler's result on partitions of n into distinct parts and into odd parts.

Factorization of integers

Most familiar free commutative monoid: $\mathbb{P} = \{1, 2, 3, \ldots\}$ under multiplication.

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For functions $f(n)$ involving factorization of integers into primes, often convenient to use **Dirichlet series** $\sum_{n\geq 1} f(n) n^{-s}$. In particular,

$$\begin{aligned} \zeta(s) &= \sum_{n\geq 1} n^{-s} \\ &= \prod_p (1 + p^{-s} + p^{-2s} + p^{-3s} + \cdots) \\ &= \prod_p \frac{1}{1 - p^{-s}}. \end{aligned}$$

Powerful numbers

A positive integer is **powerful** if $p|n \Rightarrow p^2|n$ when p is prime.

$$F(s) := \sum_{\substack{n\geq 1\\ n \text{ powerful}}} n^{-s}$$

$$= \prod_p (1 + p^{-2s} + p^{-3s} + p^{-4s} + \cdots)$$

$$= \prod_p \left(\frac{1}{1-p^{-s}} - p^{-s}\right)$$

$$= \prod_p \frac{1-p^{-6s}}{(1-p^{-2s})(1-p^{-3s})}$$

$$= \frac{\zeta(2s)\zeta(3s)}{\zeta(6s)} \quad \text{(Golomb, 1970)}$$

Insignificant corollary

$$\zeta(4) = \frac{\pi^4}{90}, \quad \zeta(6) = \frac{\pi^6}{945}, \quad \zeta(12) = \frac{691\pi^{12}}{638512875}$$

$$\Rightarrow \sum_{\substack{n \geq 1 \\ n \text{ powerful}}} \frac{1}{n^2} = \frac{\zeta(4)\zeta(6)}{\zeta(12)}$$

$$= \frac{15015}{1382\pi^2}$$

$$\approx 1.100823\cdots$$

A general result

Theorem. *Let S be a finite cyclotomic subset of $\mathbb{P}$, so*

$$\frac{1}{1-x} - \sum_{i \in S} x^i = \frac{\prod (1-x)^{a_i}}{\prod (1-x)^{b_j}} \quad \text{(finite products)}.$$

Then

$$\sum_n n^{-s} = \frac{\prod \zeta(b_i s)}{\prod \zeta(a_j s)},$$

where n ranges over all positive integers such that if $m \in S$, then no prime p divides n with multiplicity m.

The final slide

The final slide