Some Combinatorial Aspects of Cyclotomic Polynomials

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March 26, 2025

Partitions

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partition of n \ge 0: an integer sequence \lambda = (\lambda_1, \lambda_2, \dots) satisfying \lambda_1 \ge \lambda_2 \ge \dots \ge 0 and \sum \lambda_i = n
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partitions of 5: 5, 41, 32, 311, 221, 2111, 11111

Terminology example. The partition (6, 4, 4, 3, 2, 2, 2, 1) has two parts equal to 4. Equivalently, 4 has **multiplicity** two as a part.

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f(n): number of partitions of n for which no part appears exactly once

Example. f(8) = 6: 44, 3311, 2222, 22211, 221111, 11111111



A theorem of MacMahon

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Proof.
$$\sum_{n\geq 0} f(n)x^n = \prod_{i\geq 1} (1+x^{2i}+x^{3i}+x^{4i}+\cdots)$$

$$= \prod_{i\geq 1} \left(\frac{1}{1-x^i}-x^i\right)$$

$$= \prod_{i\geq 1} \frac{1-x^i+x^{2i}}{1-x^i}$$

$$= \prod_{i\geq 1} \frac{1-x^{6i}}{(1-x^{2i})(1-x^{3i})}$$

$$= \prod_{j\not\equiv \pm 1 \bmod 6} (1-x^j)^{-1}. \quad \Box$$

Why does this work?

 $\Phi_n(x)$: the *n*th cyclotomic polynomial

$$\Phi_1(x) = 1 - x \quad (x - 1 \text{ is standard})$$

$$\Phi_n(x) = \prod_{\substack{1 \le j \le n \\ \gcd(j,n)=1}} \left(x - e^{2\pi i j/n} \right) = \prod_{d \mid n} (1 - x^d)^{\mu(n/d)}, \ n \ge 2$$

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$$\begin{split} \Phi_1(x) &= 1 - x \quad (x - 1 \text{ is standard}) \\ \Phi_n(x) &= \prod_{\substack{1 \le j \le n \\ \gcd(j,n) = 1}} \left(x - e^{2\pi i j/n} \right) = \prod_{d \mid n} (1 - x^d)^{\mu(n/d)}, \ n \ge 2 \\ &= \prod_{i=1}^k (1 - x^i)^{a_i}, \ a_i \in \mathbb{Z} \end{split}$$

Two points

1. (the main point)

$$F(x) := \frac{1}{1-x} - x = \frac{\Phi_6(x)}{1-x} = \frac{1-x^6}{(1-x^2)(1-x^3)}$$

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$$\sum_{n\geq 0} f(n)x^n = F(x)F(x^2)F(x^3)\cdots$$

$$= \frac{(1-x^6)(1-x^{12})(1-x^{18})\cdots}{(1-x^2)(1-x^4)(1-x^6)\cdots(1-x^3)(1-x^6)(1-x^9)\cdots}$$

$$= \frac{1}{(1-x^2)(1-x^3)(1-x^4)(1-x^6)(1-x^8)(1-x^9)\cdots}$$

Cyclotomic sets

Definition. A **cyclotomic set** is a subset S of $\mathbb{P} = \{1, 2, ...\}$ such that

$$F_S(x) := \frac{1}{1-x} - \sum_{j \in S} x^j = \frac{N_S(x)}{D_S(x)},$$

where $N_S(x)$ and $D_S(x)$ are finite products of cyclotomic polynomials.

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Think of S as the set of "forbidden part multiplicities."

An example: $S = \{1, 2, 3, 5, 7, 11\}$

$$F_S(x) := \frac{1}{1-x} - (x + x^2 + x^3 + x^5 + x^7 + x^{11})$$

$$= \frac{\Phi_6(x)\Phi_{12}(x)\Phi_{18}(x)}{1-x}$$

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$$F_{S}(x)F_{S}(x^{2})F_{S}(x^{3})\cdots = \prod_{i} (1-x^{i})^{-1},$$

$$i \equiv 0, 4, 6, 8, 9, 12, 16, 18, 20, 24, 27, 28, 30, 32 \pmod{36}$$
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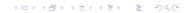
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Theorem. For all $n \ge 0$, the number of partitions of n such that no part occurs exactly 1, 2, 3, 5, 7 or 11 times equals the number of partitions of n into parts i satisfying (*).



A further example

$$S = \{2, 3, 4, \dots\}$$
 is cyclotomic:

$$\frac{1}{1-x} - (x^2 + x^3 + \cdots) = 1 + x = \frac{1-x^2}{1-x}$$

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Theorem (Euler). The number of partitions of n into distinct parts equals the number of partitions of n into odd parts.

A property of finite cyclotomic sets

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Theorem. Assume that S is finite. For $0 \le j \le d = \max(S)$, exactly one of j and d-j belongs to S. Hence #S = (d+1)/2.

Proof sketch. Symmetry or antisymmetry of $\Phi_n(x)$ implies

$$P_S(x) + x^d P_S(1/x) = 1 + x + \dots + x^d$$
, where $P_S(x) = \sum_{i \in S} x^i$. \square

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Corollary. *S* finite \Rightarrow max(*S*) is odd.

Some data

Let d be odd. There are $2^{(d-1)/2}$ sets $S \subset \mathbb{P}$ with $\max(S) = d$ such that $N_S(x)$ is symmetric. Let f(d) be the number of these that are cyclotomic. Then

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Can show: $f(d) \le a \exp b\sqrt{d}$.

Two infinite families

Write e.g.
$$125 = \{1, 2, 5\}.$$

$$1, 23, 345, 4567, 56789, \dots$$

$$1, 13, 135, 1357, \dots$$

Cleanness

Note. Any $f(x) \in \mathbb{Z}[[x]]$ with f(0) = 1 can be uniquely written (formally) as

$$f(x) = \prod_{n>1} (1-x^n)^{-a_n}, \quad a_n \in \mathbb{Z}.$$

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Let S be a subset of \mathbb{P} and

$$F(x) = \frac{1}{1-x} - \sum_{i \in S} x^{i}.$$

S is clean if

$$F(x)F(x^2)F(x^3)\cdots = \prod_{n>1} (1-x^n)^{-a_n},$$

where each $a_n = 0, 1$. (Get a "clean" partition identity—no weighted or colored parts.)



An example

Not every cyclotomic set S is clean, e.g., $S = \{1, 5, 7, 8, 9, 11\}$, for which

$$F(x)F(x^{2})F(x^{3})\cdots = \frac{(1-x^{5})(1-x^{25})(1-x^{35})(1-x^{55})\cdots}{(1-x^{2})(1-x^{3})(1-x^{6})(1-x^{6})(1-x^{8})(1-x^{9})(1-x^{10})(1-x^{12})\cdots}$$

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No nice theory of clean sets.

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$$A_M(x) = \frac{1}{1-x} - \sum_{i \in \mathbb{N}-M} x^i$$
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Cyclotomic numerical semigroups

Definition (E.-A. Ciolan, et al.) A numerical semigroup M is cyclotomic if $(1-x)A_M(x)$ is a product of cyclotomic polynomials. Equivalently, $\mathbb{N}-M$ is a cyclotomic set.

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Example. $M = \langle a, b \rangle$, where $a, b \geq 2$, gcd(a, b) = 1. Then

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Example. (a) $M = \langle 4, 6, 7 \rangle = \mathbb{N} - \{1, 2, 3, 5, 9\}$ is cyclotomic.

(b) $M = \langle 5, 6, 7 \rangle = \mathbb{N} - \{1, 2, 3, 4, 8, 9\}$ is not cyclotomic.



Consequence of $\langle a, b \rangle$ being cyclotomic and clean

Theorem. Let $a, b \ge 2$, gcd(a, b) = 1. Let $M = \langle a, b \rangle$. Then for all $n \ge 0$, the following numbers are equal:

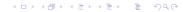
- ► the number of partitions of n all of whose part multiplicities belong to M
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MacMahon identity: a = 2, b = 3



Semigroup algebra

The **semigroup algebra** K[M] (over K) of a numerical semigroup M is

$$K[M] = K[z^i : i \in M].$$

Definition. Let $M = \langle a_1, \ldots, a_r \rangle$. K[M] is a **complete intersection** if all the relations among the generators z^{a_1}, \ldots, z^{a_r} are consequences of r-1 of them (the minimum possible).

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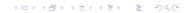
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Converse is **open** (main open problem on cyclotomic numerical semigroups).



Example. $M = \langle 4, 6, 7 \rangle = \mathbb{N} - \{1, 2, 3, 5, 9\}$. Generators of K[M] are $a = z^4, b = z^6, c = z^7$. Some relations: $a^3 = b^2, \ a^2b = c^2, \ a^7 = c^4, \ b^7 = c^6, \dots$

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All are consequences of the first two, so K[M] is a complete intersection. E.g.,

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$$c^4 = (a^2b)^2 = a^4b^2 = a^4a^3 = a^7.$$

The relation $a^3 = b^2$ has degree $3 \cdot 4 = 6 \cdot 2 = 12$. The relation $a^2b = c^2$ has degree $2 \cdot 4 + 6 = 2 \cdot 7 = 14$

$$\Rightarrow A_M(x) = \frac{(1-x^{12})(1-x^{14})}{(1-x^4)(1-x^6)(1-x^7)}.$$

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Note. Multiply $c^2 = a^5b^2$ by b: $c^2b = a^5b^3$. Substitute a^4c for b^3 : $c^2b = a^9c$. Divide by c: $bc = a^9$ (first relation). So why not just two relations?

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Answer: not allowed to divide (not a ring operation).



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Thus the main open problem on cyclotomic numerical semigroups is true for semigroups with at most three generators.

A more general framework

Key fact: a partition is determined by the multiplicity of each part, and these multiplicities are independent. Equivalently, define $\lambda \cup \mu$ by

$$m_i(\lambda \cup \mu) = m_i(\lambda) + m_i(\mu),$$

where m_i denotes the multiplicity of the part i. E.g.,

$$(6,6,4,1,1)+(6,5,4,1)=(6,6,6,5,4,4,1,1,1).$$

Then \cup makes the set of all partitions of all $n \ge 0$ into a free commutative monoid with unique basis $\{(1), (2), (3), \dots\}$.

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Further details omitted.

Where else can we find such monoids?



Polynomials over finite fields

Fix a prime power q.

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 $\beta(n)$: number of monic irreducible polynomials of degree n over \mathbb{F}_q .

$$\beta(n) = \frac{1}{n} \sum_{d|n} \mu(d) q^{n/d} \quad \text{(irrelevant)}$$

There are q^n monic polynomials of degree n over \mathbb{F}_q . Every such polynomial is uniquely (up to order of factors) a product of monic irreducible polynomials. Hence

$$\sum_{n\geq 0} q^n x^n = \frac{1}{1-qx} = \prod_{m\geq 1} (1-x^m)^{-\beta(m)}.$$

Powerful polynomials

Example. Let f(n) be the number of monic polynomials of degree n over \mathbb{F}_q such that every irreducible factor has multiplicity at least two (powerful polynomials). Thus

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$$\sum_{n\geq 0} f(n)x^{n} = \prod_{m\geq 1} (1+x^{2m}+x^{3m}+\cdots)^{\beta(m)}$$

$$= \prod_{m\geq 1} \left(\frac{1-x^{6m}}{(1-x^{2m})(1-x^{3m})}\right)^{\beta(m)}$$

$$= \frac{1-qx^{6}}{(1-qx^{2})(1-qx^{3})}$$

$$= \frac{1+x+x^{2}+x^{3}}{1-qx^{2}} - \frac{x(1+x+x^{2})}{1-qx^{3}}$$

$$\Rightarrow f(n) = q^{\lfloor n/2 \rfloor} + q^{\lfloor n/2 \rfloor - 1} - q^{\lfloor (n-1)/3 \rfloor}.$$

Generalization.

Theorem. Let S be a cyclotomic subset of \mathbb{P} , so

$$\frac{1}{1-x} - \sum_{i \in S} x^{i} = \frac{\prod (1-x^{i})^{a_{i}}}{\prod (1-x^{j})^{b_{j}}},$$

where the products are **finite**. Let f(n) be the number of monic polynomials of degree n over \mathbb{F}_q such that no irreducible factor has multiplicity $m \in S$. Then

$$\sum f(n)x^{n} = \frac{\prod_{i}(1 - qx^{i})^{a_{i}}}{\prod_{i}(1 - qx^{j})^{b_{j}}}.$$

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Can convert to a partial fraction in q and find an explicit (though in general very lengthy) formula for f(n).

$$S = \{1, 2, 3, 5, 7, 11\}$$

$$\sum_{n\geq 0} f(n)x^n = \frac{(1-qx^{12})(1-qx^{18})}{(1-qx^4)(1-qx^6)(1-qx^9)}$$

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$$= \frac{\Phi_2 \Phi_4 \Phi_8 \Phi_7 \Phi_{14}}{\Phi_5 (1 - qx^4)} + \frac{\Phi_3 \Phi_9 x^8}{\Phi_5 (1 - qx^9)}$$

$$-\frac{\Phi_2 \Phi_3 \Phi_4 \Phi_6^2 \Phi_{12} x^2}{1 - qx^6},$$

where $\Phi_j = \Phi_j(x)$.

Yet another example

Let $S = \{2, 3, 4, \dots\}$. Recall

$$\frac{1}{1-x} - \sum_{i \in S} x^i = 1 + x = \frac{1-x^2}{1-x}.$$

f(n): number of squarefree monic polynomials of degree n over \mathbb{F}_a . Then

$$\sum_{n\geq 0} f(n)x^n = \frac{1-qx^2}{1-qx}$$

$$= 1+qx+\sum_{n\geq 2} (q-1)q^{n-1}x^n$$

$$\Rightarrow f(n)=(q-1)q^{n-1}, n\geq 2 \text{ (well-known)},$$

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$$= 1 + qx + \sum_{n\geq 2} (q - 1)q^{n-1}x^{n}$$

$$\Rightarrow f(n) = (q - 1)q^{n-1}, \ n \geq 2 \text{ (well-known)},$$

a kind of analogue (though not a q-analogue in the usual sense) of Euler's result on partitions of n into distinct parts and into odd parts.

Factorization of integers

Most familiar free commutative monoid: $\mathbb{P} = \{1, 2, 3, \dots\}$ under multiplication.

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For functions f(n) involving factorization of integers into primes, often convenient to use **Dirichlet series** $\sum_{n\geq 1} f(n) n^{-s}$. In particular,

$$\zeta(s) = \sum_{n\geq 1} n^{-s}
= \prod_{p} (1 + p^{-s} + p^{-2s} + p^{-3s} + \cdots)
= \prod_{p} \frac{1}{1 - p^{-s}}.$$

Powerful numbers

A positive integer is **powerful** if $p|n \Rightarrow p^2|n$ when p is prime.

$$F(s) := \sum_{\substack{n \ge 1 \\ n \text{ powerful}}} n^{-s}$$

$$= \prod_{p} (1 + p^{-2s} + p^{-3s} + p^{-4s} + \cdots)$$

$$= \prod_{p} \left(\frac{1}{1 - p^{-s}} - p^{-s} \right)$$

$$= \prod_{p} \frac{1 - p^{-6s}}{(1 - p^{-2s})(1 - p^{-3s})}$$

$$= \frac{\zeta(2s)\zeta(3s)}{\zeta(6s)} \quad \text{(Golomb, 1970)}$$



Insignificant corollary

$$\zeta(4) = \frac{\pi^4}{90}, \quad \zeta(6) = \frac{\pi^6}{945}, \quad \zeta(12) = \frac{691\pi^{12}}{638512875}$$

$$\Rightarrow \sum_{\substack{n \ge 1 \\ n \text{ powerful}}} \frac{1}{n^2} = \frac{\zeta(4)\zeta(6)}{\zeta(12)}$$

$$= \frac{15015}{1382\pi^2}$$

$$\approx 1.100823 \cdots$$

A general result

Theorem. Let S be a finite cyclotomic subset of \mathbb{P} , so

$$\frac{1}{1-x} - \sum_{i \in S} x^i = \frac{\prod (1-x)^{a_i}}{\prod (1-x)^{b_j}}$$
 (finite products).

Then

$$\sum_{n} n^{-s} = \frac{\prod \zeta(b_{i}s)}{\prod \zeta(a_{j}s)},$$

where n ranges over all positive integers such that if $m \in S$, then no prime p divides n with multiplicity m.

The final slide

The final slide

