Two Analogues of Pascal's Triangle

Richard P. Stanley
U. Miami & M.I.T.

October 13, 2024

The diagrams P_{ib}

Let $i, b \geq 2$. Define the diagram (or poset) $\boldsymbol{P_{ib}}$ by

- There is a unique maximal element $\hat{1}$

The diagrams P_{ib}

Let $i, b \geq 2$. Define the diagram (or poset) $\boldsymbol{P_{ib}}$ by

- There is a unique maximal element $\hat{1}$
- Each element covers exactly i elements.

The diagrams P_{ib}

Let $i, b \geq 2$. Define the diagram (or poset) $\boldsymbol{P_{ib}}$ by

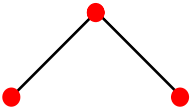
- There is a unique maximal element $\hat{1}$
- Each element covers exactly i elements.
- The diagram is planar.

The diagrams P_{ib}

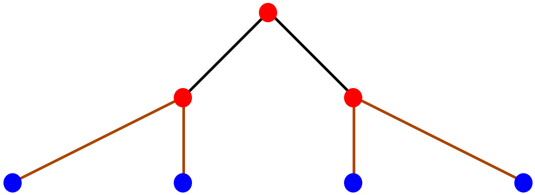
Let $i, b \geq 2$. Define the diagram (or poset) P_{ib} by

- There is a unique maximal element $\hat{1}$
- Each element covers exactly i elements.
- The diagram is planar.
- Every  extends to a $2b$-gon (b edges on each side)

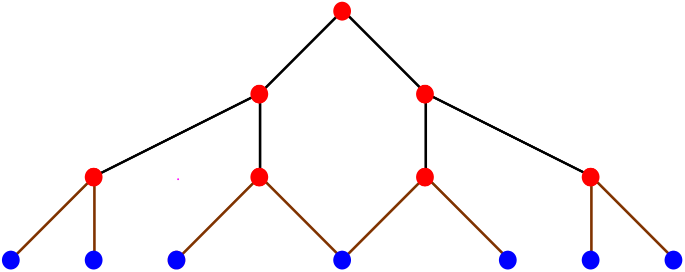
Construction of P_{23}



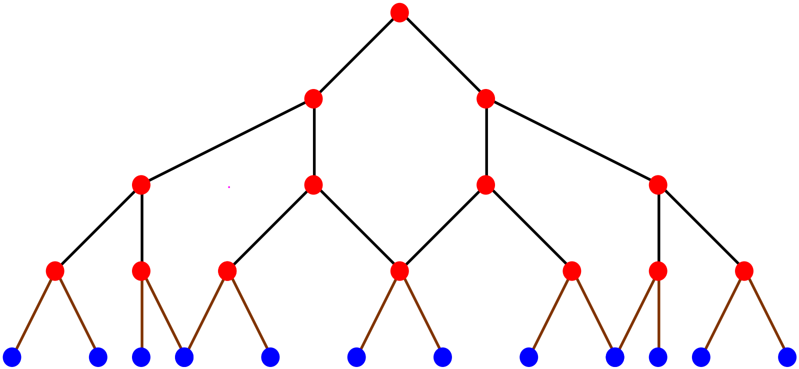
Construction of P_{23}



Construction of P_{23}



Construction of P_{23}



Some results for any i, b

The **rank** of an element $t \in P_{ib}$ is the length of a chain from $\hat{1}$ to t, so $\mathrm{rank}(\hat{1}) = 0$.

$\boldsymbol{p_{ib}(n)}$: number of elements of P_{ib} of rank n

Some results for any i, b

The **rank** of an element $t \in P_{ib}$ is the length of a chain from $\hat{1}$ to t, so $\mathrm{rank}(\hat{1}) = 0$.

$\boldsymbol{p_{ib}(n)}$: number of elements of P_{ib} of rank n

Elementary counting argument gives

$$p_{ib}(n) = ip_{ib}(n-1) - (i-1)p_{ib}(n-b).$$

Some results for any i, b

The **rank** of an element $t \in P_{ib}$ is the length of a chain from $\hat{1}$ to t, so $\text{rank}(\hat{1}) = 0$.

$\boldsymbol{p_{ib}(n)}$: number of elements of P_{ib} of rank n

Elementary counting argument gives

$$p_{ib}(n) = ip_{ib}(n-1) - (i-1)p_{ib}(n-b).$$

Initial conditions: $p_{ib}(n) = i^n$, $0 \leq n \leq b-1$

$$\Rightarrow \sum_{n \geq 0} p_{ib}(n)x^n = \frac{1}{1 - ix + (i-1)x^b}.$$

Some results for any i, b

The **rank** of an element $t \in P_{ib}$ is the length of a chain from $\hat{1}$ to t, so $\mathrm{rank}(\hat{1}) = 0$.

$\boldsymbol{p_{ib}(n)}$: number of elements of P_{ib} of rank n

Elementary counting argument gives

$$p_{ib}(n) = ip_{ib}(n-1) - (i-1)p_{ib}(n-b).$$

Initial conditions: $p_{ib}(n) = i^n$, $0 \le n \le b-1$

$$\Rightarrow \sum_{n \ge 0} p_{ib}(n)x^n = \frac{1}{1 - ix + (i-1)x^b}.$$

Note. Thus $p_{ib}(n)$ grows exponentially except for $(i, b) = (2, 2)$.

The numbers $e(t)$

For $t \in P_{ib}$, let $\boldsymbol{e(t)}$ be the number of paths (saturated chains) from $\hat{1}$ to t. Equivalently, $e(\hat{1}) = 1$ and for $t < \hat{1}$,

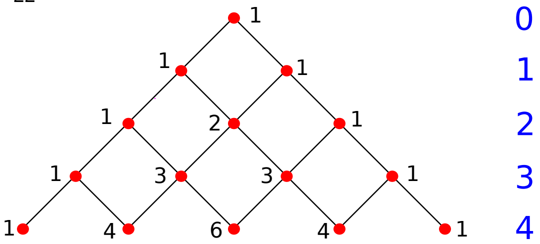
$$e(t) = \sum_{s \text{ covers } t} e(s).$$

The numbers $e(t)$

For $t \in P_{ib}$, let $\boldsymbol{e(t)}$ be the number of paths (saturated chains) from $\hat{1}$ to t. Equivalently, $e(\hat{1}) = 1$ and for $t < \hat{1}$,

$$e(t) = \sum_{s \text{ covers } t} e(s).$$

Example. P_{22}

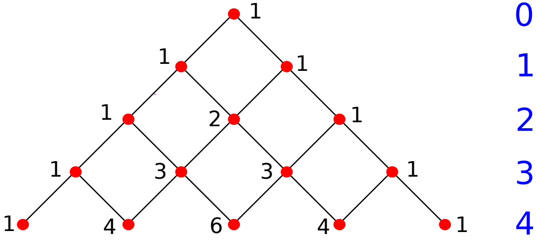


The numbers $e(t)$

For $t \in P_{ib}$, let $\boldsymbol{e(t)}$ be the number of paths (saturated chains) from $\hat{1}$ to t. Equivalently, $e(\hat{1}) = 1$ and for $t < \hat{1}$,

$$e(t) = \sum_{s \text{ covers } t} e(s).$$

Example. P_{22}



Pascal's triangle

A generating function for the $e(t)$'s

Fix i and b.

$\boldsymbol{t_{nk}}$: kth element from left in the nth row of P_{ib}, beginning with $k = 0$.

$$\left\langle \begin{matrix} \boldsymbol{n} \\ \boldsymbol{k} \end{matrix} \right\rangle = \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_{i,b} = e(t_{nk})$$

$\boldsymbol{q_n}$: number of elements of P_{ib} of rank n

$$\boldsymbol{r_n} = \frac{q_n - q_{n-1}}{i-1} \in \mathbb{P} = \{1, 2, \dots\}$$

A generating function for the $e(t)$'s

Fix i and b.

$\boldsymbol{t_{nk}}$: kth element from left in the nth row of P_{ib}, beginning with $k = 0$.

$$\left\langle {\boldsymbol{n} \atop \boldsymbol{k}} \right\rangle = \left\langle {n \atop k} \right\rangle_{i,b} = e(t_{nk})$$

$\boldsymbol{q_n}$: number of elements of P_{ib} of rank n

$$\boldsymbol{r_n} = \frac{q_n - q_{n-1}}{i-1} \in \mathbb{P} = \{1, 2, \ldots\}$$

Theorem. $\displaystyle\sum_k \left\langle {n \atop k} \right\rangle x^k = \prod_{j=1}^{n} \left(1 + x^{r_j} + x^{2r_j} + \cdots + x^{(i-1)r_j}\right)$

(analogue of binomial theorem, the case $i = b = 2$)

Stability

Theorem (repeated).

$$\sum_k \left\langle {n \atop k} \right\rangle x^k = \prod_{j=1}^{n} \left(1 + x^{r_j} + x^{2r_j} + \cdots + x^{(i-1)r_j}\right)$$

Stability

Theorem (repeated).

$$\sum_k \left\langle {n \atop k} \right\rangle x^k = \prod_{j=1}^{n} \left(1 + x^{r_j} + x^{2r_j} + \cdots + x^{(i-1)r_j}\right)$$

For all $(i, b) \neq (2, 2)$, we have $r_n \to \infty$ as $n \to \infty$.

$\Rightarrow$ For fixed k, $e(t_{0k}), e(t_{1k}), e(t_{2k}), \ldots$ eventually becomes constant, say $\overline{e}_k$. Then

$$\sum_{k \geq 0} \overline{e}_k x^k = \prod_{j=1}^{\infty} \left(1 + x^{r_j} + x^{2r_j} + \cdots + x^{(i-1)r_j}\right).$$

Sums of powers

$$\sum_k \binom{n}{k}^2 = \binom{2n}{n}$$

Sums of powers

$$\sum_k \binom{n}{k}^2 = \binom{2n}{n}$$

$$\sum_{n\geq 0} \binom{2n}{n} x^n = \frac{1}{\sqrt{1-4x}},$$

not a rational function (quotient of two polynomials)

Sums of powers

$$\sum_k \binom{n}{k}^2 = \binom{2n}{n}$$

$$\sum_{n\geq 0} \binom{2n}{n} x^n = \frac{1}{\sqrt{1-4x}},$$

not a rational function (quotient of two polynomials)

$$\sum_k \binom{n}{k}^3 = ??$$

Even worse! Generating function is not algebraic.

Sums of powers

$$\sum_k \binom{n}{k}^2 = \binom{2n}{n}$$

$$\sum_{n\geq 0} \binom{2n}{n} x^n = \frac{1}{\sqrt{1-4x}},$$

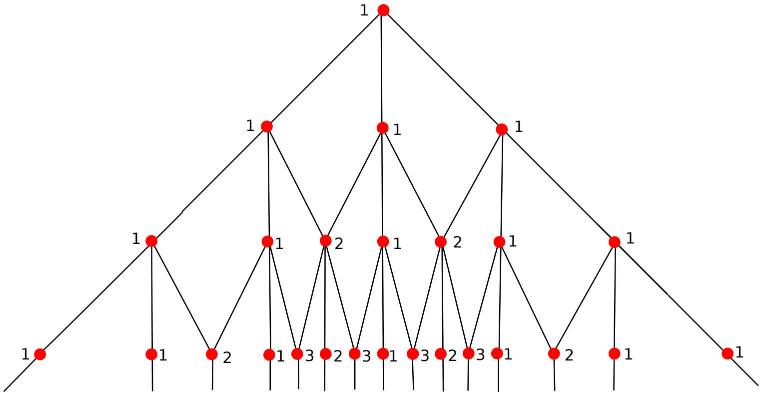
not a rational function (quotient of two polynomials)

$$\sum_k \binom{n}{k}^3 = ??$$

Even worse! Generating function is not algebraic.

Much of this behavior is **atypical**. Different for $(i, b) \neq (2, 2)$.

The poset P_{32} (Stern poset)



Very different behavior from P_{22}.

Stern's triangle

Similar to Pascal's triangle, but we also "bring down" (copy) each number from one row to the next.

Stern's triangle

Similar to Pascal's triangle, but we also "bring down" (copy) each number from one row to the next.

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 1 & & 1 & & \\
 & 1 & & & & 1 & \\
1 & & & & & & 1 \\
 & & & \vdots & & &
\end{array}
$$

Stern's triangle

Similar to Pascal's triangle, but we also "bring down" (copy) each number from one row to the next.

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & 1 & & 1 & & 1 & & \\
 & 1 & & & & & & 1 & \\
1 & & & & & & & & 1 \\
 & & & & \vdots & & & &
\end{array}
$$

Stern's triangle

Similar to Pascal's triangle, but we also "bring down" (copy) each number from one row to the next.

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & 1 & & 1 & & 1 & & \\
 & 1 & & 2 & & 2 & & 1 & \\
1 & & & & & & & & 1 \\
 & & & & \vdots & & & &
\end{array}
$$

Stern's triangle

Similar to Pascal's triangle, but we also "bring down" (copy) each number from one row to the next.

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & 1 & & & 1 & & & 1 & & & \\
 & & 1 & & 1 & 2 & 1 & 2 & 1 & & 1 & & \\
 & 1 & & & & & & & & & & 1 & \\
 & & & & & & \vdots & & & & & &
\end{array}
$$

Stern's triangle

Similar to Pascal's triangle, but we also "bring down" (copy) each number from one row to the next.

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & 1 & & & & 1 & & & & 1 & & & \\
 & 1 & & 1 & & 2 & & 1 & & 2 & & 1 & & 1 & \\
1 & & 2 & & 3 & & 3 & & 3 & & 3 & & 2 & & 1 \\
 & & & & & & & \vdots & & & & & & &
\end{array}
$$

Stern's triangle

Similar to Pascal's triangle, but we also "bring down" (copy) each number from one row to the next.

							1							
			1				1				1			
	1		1		2		1		2		1		1	
1	1	2	1	3	2	3	1	3	2	3	1	2	1	1
							⋮							

Stern's triangle

Similar to Pascal's triangle, but we also "bring down" (copy) each number from one row to the next.

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & 1 & & & & 1 & & & & 1 & & & \\
 & 1 & & 1 & & 2 & & 1 & & 2 & & 1 & & 1 & \\
1 & 1 & 2 & 1 & 3 & 2 & 3 & 1 & 3 & 2 & 3 & 1 & 2 & 1 & 1 \\
 & & & & & & & \vdots & & & & & & &
\end{array}
$$

Stern's triangle

Some properties

- Number of entries in row n (beginning with row 0): $2^{n+1} - 1$

Some properties

- Number of entries in row n (beginning with row 0): $2^{n+1} - 1$
- Sum of entries in row n: 3^n

Some properties

- Number of entries in row n (beginning with row 0): $2^{n+1} - 1$
- Sum of entries in row n: 3^n
- Largest entry in row n: F_{n+1} (Fibonacci number)

Some properties

- Number of entries in row n (beginning with row 0): $2^{n+1} - 1$
- Sum of entries in row n: 3^n
- Largest entry in row n: F_{n+1} (Fibonacci number)
- $\sum_k \left\langle {n \atop k} \right\rangle x^k = \prod_{i=0}^{n-1} \left(1 + x^{2^i} + x^{2 \cdot 2^i}\right)$

Stabilization

$$\sum_{k\geq 0} \bar{e}_k x^k = \prod_{i=0}^{\infty} \left(1 + x^{2^i} + x^{2\cdot 2^i}\right)$$

The sequence $(\bar{e}_0, \bar{e}_1, \dots)$ is **Stern's diatomic sequence** (**Moritz Abraham Stern**, 1807–1894):

$$1\ 1\ 2\ 1\ 3\ 2\ 3\ 1\ 4\ 3\ 5\ 2\ 5\ 3\ 4\ 1\ 5\ 4\ 7\ 3\ 8 \dots,$$

so $\bar{e}_k$ is the number of ways to write k as a sum of powers of 2, where each power of 2 can occur at most **twice**.

Stabilization

$$\sum_{k\geq 0} \bar{e}_k x^k = \prod_{i=0}^{\infty} \left(1 + x^{2^i} + x^{2\cdot 2^i}\right)$$

The sequence $(\bar{e}_0, \bar{e}_1, \dots)$ is **Stern's diatomic sequence** (**Moritz Abraham Stern**, 1807–1894):

$$1\ 1\ 2\ 1\ 3\ 2\ 3\ 1\ 4\ 3\ 5\ 2\ 5\ 3\ 4\ 1\ 5\ 4\ 7\ 3\ 8\ \dots,$$

so $\bar{e}_k$ is the number of ways to write k as a sum of powers of 2, where each power of 2 can occur at most **twice**.

Most amazing property: Every positive rational number occurs exactly once among the numbers $\bar{e}_i/\bar{e}_{i-1}$, $i \geq 1$.

Sums of squares

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & 1 & & & & 1 & & & & 1 & & & \\
 & 1 & & 1 & & 2 & & 1 & & 2 & & 1 & & 1 & \\
1 & 1 & 2 & 1 & 3 & 2 & 3 & 1 & 3 & 2 & 3 & 1 & 2 & 1 & 1 \\
 & & & & & & & \vdots & & & & & & &
\end{array}
$$

$$
u_2(n) := \sum_k \left\langle {n \atop k} \right\rangle^2 = 1,\ 3,\ 13,\ 59,\ 269,\ 1227,\ \ldots
$$

Sums of squares

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & 1 & & & & 1 & & & & 1 & & & \\
 & 1 & & 1 & & 2 & & 1 & & 2 & & 1 & & 1 & \\
1 & 1 & 2 & 1 & 3 & 2 & 3 & 1 & 3 & 2 & 3 & 1 & 2 & 1 & 1 \\
 & & & & & & & \vdots & & & & & & &
\end{array}
$$

$$\boldsymbol{u_2(n)} := \sum_k \left\langle {n \atop k} \right\rangle^2 = 1,\ 3,\ 13,\ 59,\ 269,\ 1227,\ \ldots$$

$$u_2(n+1) = 5u_2(n) - 2u_2(n-1), \quad n \geq 1$$

Sums of squares

$$\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & 1 & & & & 1 & & & & 1 & & & \\
 & 1 & & 1 & & 2 & & 1 & & 2 & & 1 & & 1 & \\
1 & 1 & 2 & 1 & 3 & 2 & 3 & 1 & 3 & 2 & 3 & 1 & 2 & 1 & 1 \\
 & & & & & & & \vdots & & & & & & &
\end{array}$$

$$\boldsymbol{u_2(n)} := \sum_k \left\langle {n \atop k} \right\rangle^2 = 1,\ 3,\ 13,\ 59,\ 269,\ 1227,\ \ldots$$

$$u_2(n+1) = 5u_2(n) - 2u_2(n-1), \quad n \geq 1$$

$$\sum_{n\geq 0} u_2(n)x^n = \frac{1-2x}{1-5x+2x^2}$$

Idea of proof

Define

$$u_{1,1}(n) := \sum_k \left\langle {n \atop k} \right\rangle \left\langle {n \atop k+1} \right\rangle.$$

Idea of proof

Define

$$u_{1,1}(n) := \sum_k \left\langle {n \atop k} \right\rangle \left\langle {n \atop k+1} \right\rangle.$$

Simple combinatorial argument gives

$$\begin{aligned} u_2(n+1) &= 3u_2(n) + 2u_{1,1}(n) \\ u_{1,1}(n) &= 2u_2(n) + 2u_{1,1}(n). \end{aligned}$$

Idea of proof

Define

$$\mathbf{u_{1,1}(n)} := \sum_k \left\langle {n \atop k} \right\rangle \left\langle {n \atop k+1} \right\rangle.$$

Simple combinatorial argument gives

$$\begin{aligned} u_2(n+1) &= 3u_2(n) + 2u_{1,1}(n) \\ u_{1,1}(n) &= 2u_2(n) + 2u_{1,1}(n). \end{aligned}$$

Define $\mathbf{A} := \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}$.

Then

$$A \begin{bmatrix} u_2(n) \\ u_{1,1}(n) \end{bmatrix} = \begin{bmatrix} u_2(n+1) \\ u_{1,1}(n+1) \end{bmatrix}.$$

Idea of proof

Define

$$\boldsymbol{u_{1,1}(n)} := \sum_k \left\langle {n \atop k} \right\rangle \left\langle {n \atop k+1} \right\rangle.$$

Simple combinatorial argument gives

$$\begin{aligned} u_2(n+1) &= 3u_2(n) + 2u_{1,1}(n) \\ u_{1,1}(n) &= 2u_2(n) + 2u_{1,1}(n). \end{aligned}$$

Define $\boldsymbol{A} := \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}$.

Then

$$A \begin{bmatrix} u_2(n) \\ u_{1,1}(n) \end{bmatrix} = \begin{bmatrix} u_2(n+1) \\ u_{1,1}(n+1) \end{bmatrix}.$$

$$\Rightarrow A^n \begin{bmatrix} u_2(1) \\ u_{1,1}(1) \end{bmatrix} = \begin{bmatrix} u_2(n) \\ u_{1,1}(n) \end{bmatrix}$$

Idea of proof (concluded)

Characteristic (or minimum) polynomial of A: $x^2 - 5x + 2$

Idea of proof (concluded)

Characteristic (or minimum) polynomial of A: $x^2 - 5x + 2$

$$\begin{aligned} (A^2 - 5A + 2I)A^{n-1} &= 0_{2\times 2} \\ \Rightarrow u_2(n+1) &= 5u_2(n) - 2u_2(n-1) \end{aligned}$$

Idea of proof (concluded)

Characteristic (or minimum) polynomial of A: $x^2 - 5x + 2$

$$\begin{aligned}(A^2 - 5A + 2I)A^{n-1} &= 0_{2\times 2} \\ \Rightarrow u_2(n+1) &= 5u_2(n) - 2u_2(n-1)\end{aligned}$$

Also $u_{1,1}(n+1) = 5u_{1,1}(n) - 2u_{1,1}(n-1)$.

Sums of cubes

$$u_3(n) := \sum_k \left\langle {n \atop k} \right\rangle^3 = 1,\ 3,\ 21,\ 147,\ 1029,\ 7203,\ \ldots$$

Sums of cubes

$$u_3(n) := \sum_k \left\langle {n \atop k} \right\rangle^3 = 1,\ 3,\ 21,\ 147,\ 1029,\ 7203,\ \ldots$$

$$u_3(n) = 3 \cdot 7^{n-1}, \quad n \geq 1$$

Sums of cubes

$$u_3(n) := \sum_k \left\langle {n \atop k} \right\rangle^3 = 1,\ 3,\ 21,\ 147,\ 1029,\ 7203,\ \ldots$$

$$u_3(n) = 3 \cdot 7^{n-1}, \quad n \geq 1$$

Equivalently, if $\prod_{i=0}^{n-1} \left(1 + x^{2^i} + x^{2 \cdot 2^i}\right) = \sum a_j x^j$, then

$$\sum a_j^3 = 3 \cdot 7^{n-1}.$$

Why so simple?

Same method gives the matrix $\begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}$.

Why so simple?

Same method gives the matrix $\begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}$.

Characteristic polynomial: $x(x-7)$ (zero eigenvalue!)

Why so simple?

Same method gives the matrix $\begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}$.

Characteristic polynomial: $x(x-7)$ (zero eigenvalue!)

Thus $u_3(n+1) = 7u_3(n),\ \ n \geq 1$ (not $n \geq 0$).

Why so simple?

Same method gives the matrix $\begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}$.

Characteristic polynomial: $x(x-7)$ (zero eigenvalue!)

Thus $u_3(n+1) = 7u_3(n), \ n \geq 1$ $(\text{not } n \geq 0)$.

Much nicer than $\sum_k \binom{n}{k}^3$

What about $u_r(n)$ for general $r \geq 1$?

By the same technique, can show that

$$\sum_{n \geq 0} u_r(n) x^n$$

is rational.

What about $u_r(n)$ for general $r \geq 1$?

By the same technique, can show that

$$\sum_{n \geq 0} u_r(n) x^n$$

is rational.

Example. $$\sum_{n \geq 0} u_4(n) x^n = \frac{1 - 7x - 2x^2}{1 - 10x - 9x^2 + 2x^3}$$

What about $u_r(n)$ for general $r \geq 1$?

By the same technique, can show that

$$\sum_{n \geq 0} u_r(n) x^n$$

is rational.

Example. $$\sum_{n \geq 0} u_4(n) x^n = \frac{1 - 7x - 2x^2}{1 - 10x - 9x^2 + 2x^3}$$

Much more can be said!

What about $u_r(n)$ for general $r \geq 1$?

By the same technique, can show that

$$\sum_{n\geq 0} u_r(n)x^n$$

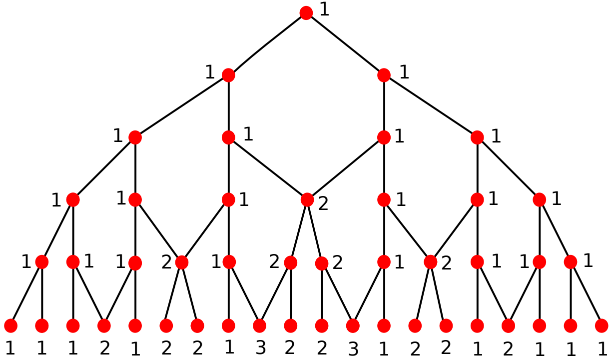
is rational.

Example. $$\sum_{n\geq 0} u_4(n)x^n = \frac{1-7x-2x^2}{1-10x-9x^2+2x^3}$$

Much more can be said!

RS, *Amer. Math. Monthly* **127** (2020), 99-111
RS, *Europ. J. Combinatorics* **119** (2023), 113359
A. L. B. Yang, arXiv:2006.00400

The Fibonacci poset $\mathfrak{F} = P_{23}$.



Basic properties

q_n (number of elements of rank n): $F_{n+2} - 1$, where $F_1 = F_2 = 1, F_{n+1} = F_n + F_{n-1}$

Basic properties

q_n (number of elements of rank n): $F_{n+2} - 1$, where
$F_1 = F_2 = 1, F_{n+1} = F_n + F_{n-1}$

$$\sum_k \left\langle {n \atop k} \right\rangle x^k = \prod_{i=1}^{n} \left(1 + x^{F_{i+1}}\right) := I_n(x)$$

Basic properties

$\boldsymbol{q_n}$ (number of elements of rank n): $F_{n+2} - 1$, where
$F_1 = F_2 = 1, F_{n+1} = F_n + F_{n-1}$

$$\sum_k \left\langle {n \atop k} \right\rangle x^k = \prod_{i=1}^{n} \left(1 + x^{F_{i+1}}\right) := \boldsymbol{I_n(x)}$$

$$\begin{aligned} I_4(x) &= (1+x)(1+x^2)(1+x^3)(1+x^5) \\ &= 1 + x + x^2 + 2x^3 + x^4 + 2x^5 + 2x^6 + x^7 + 2x^8 + x^9 + x^{10} + x^{11} \end{aligned}$$

$v_2(n) := \sum_k \left\langle {n \atop k} \right\rangle^2$

Can obtain a system of recurrences analogous to

$$\begin{aligned} v_2(n+1) &= 3v_2(n) + 2v_{1,1}(n) \\ v_{1,1}(n+1) &= 2v_2(n) + 2v_{1,1}(n) \end{aligned}$$

for Stern's triangle.

$v_2(n) := \sum_k \left\langle {n \atop k} \right\rangle^2$

Can obtain a system of recurrences analogous to

$$\begin{aligned} v_2(n+1) &= 3v_2(n) + 2v_{1,1}(n) \\ v_{1,1}(n+1) &= 2v_2(n) + 2v_{1,1}(n) \end{aligned}$$

for Stern's triangle.

Quite a bit more complicated (automated by **D. Zeilberger**).

$v_2(n) := \sum_k \left\langle {n \atop k} \right\rangle^2$

Can obtain a system of recurrences analogous to

$$\begin{aligned} v_2(n+1) &= 3v_2(n) + 2v_{1,1}(n) \\ v_{1,1}(n+1) &= 2v_2(n) + 2v_{1,1}(n) \end{aligned}$$

for Stern's triangle.

Quite a bit more complicated (automated by **D. Zeilberger**).

Theorem. $\displaystyle\sum_{n\geq 0} v_2(n)x^n = \frac{1-2x^2}{1-2x-2x^2+2x^3}$

Reminder: $v_2(n) = \sum a_i^2$, where

$$\prod_{i=1}^{n} \left(1 + x^{F_{i+1}}\right) = \sum_i a_i x^i.$$

Higher powers

$\boldsymbol{v_r(n)}$: sum of rth powers of coefficients of $I_n(x)$

$$\boldsymbol{V_r(x)} := \sum_{n \geq 0} v_r(n) x^n$$

Higher powers

$\boldsymbol{v_r(n)}$: sum of rth powers of coefficients of $I_n(x)$

$$\boldsymbol{V_r(x)} := \sum_{n\geq 0} v_r(n)x^n$$

$V_r(x)$ is a rational function.

$V_r(x)$ for $r \leq 6$

Theorem. $V_1(x) = \dfrac{1}{1-2x}$ (clear)

$$V_2(x) = \frac{1-2x^2}{1-2x-2x^2+2x^3}$$

$$V_3(x) = \frac{1-4x^2}{1-2x-4x^2+2x^3}$$

$$V_4(x) = \frac{1-7x^2-2x^4}{1-2x-7x^2-2x^4+2x^5}$$

$$V_5(x) = \frac{1-11x^2-20x^4}{1-2x-11x^2-8x^3-20x^4+10x^5}$$

$$V_6(x) = \frac{1-17x^2-88x^4-4x^6}{1-2x-17x^2-28x^3-88x^4+26x^5-4x^6+4x^7}$$

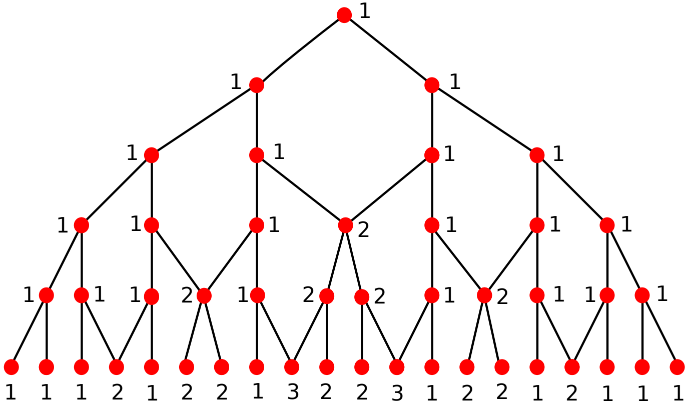
$V_r(x)$ for $r \leq 6$

Theorem. $V_1(x) = \dfrac{1}{1-2x}$ (clear)

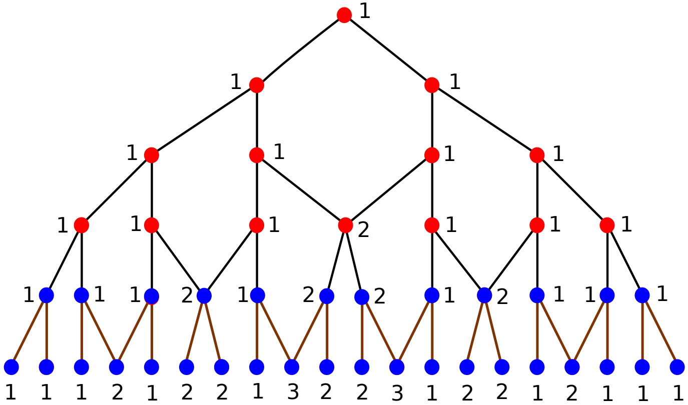
$$
\begin{aligned}
V_2(x) &= \frac{1-2x^2}{1-2x-2x^2+2x^3} \\
V_3(x) &= \frac{1-4x^2}{1-2x-4x^2+2x^3} \\
V_4(x) &= \frac{1-7x^2-2x^4}{1-2x-7x^2-2x^4+2x^5} \\
V_5(x) &= \frac{1-11x^2-20x^4}{1-2x-11x^2-8x^3-20x^4+10x^5} \\
V_6(x) &= \frac{1-17x^2-88x^4-4x^6}{1-2x-17x^2-28x^3-88x^4+26x^5-4x^6+4x^7}
\end{aligned}
$$

Note. Numerator is "even part" of denominator (**I. Bogdanov** MO457900, 2023)

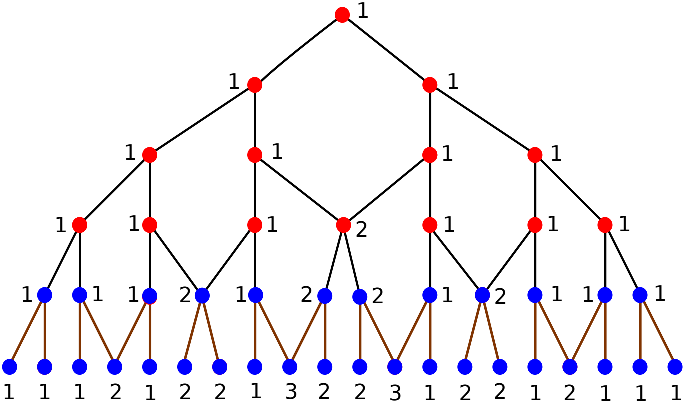
Strings of size two and three



Strings of size two and three



Strings of size two and three



What is the sequence of string sizes on each level? E.g., on level 5, the sequence $2, 3, 2, 3, 3, 2, 3, 2$.

The limiting sequence

As $n \to \infty$, we get a "limiting sequence"

$$2, 3, 2, 3, 3, 2, 3, 2, 3, 3, 2, 3, 3, 2, 3, 2, 3, 3, 2, 3, \ldots.$$

The limiting sequence

As $n \to \infty$, we get a "limiting sequence"

$$2, 3, 2, 3, 3, 2, 3, 2, 3, 3, 2, 3, 3, 2, 3, 2, 3, 3, 2, 3, \ldots.$$

Let $\phi = (1 + \sqrt{5})/2$, the golden mean.

The limiting sequence

As $n \to \infty$, we get a "limiting sequence"

$$2, 3, 2, 3, 3, 2, 3, 2, 3, 3, 2, 3, 3, 2, 3, 2, 3, 3, 2, 3, \ldots .$$

Let $\phi = (1 + \sqrt{5})/2$, the golden mean.

Theorem. *The limiting sequence* $(c_1, c_2, \ldots)$ *is given by*

$$c_n = 1 + \lfloor n\phi \rfloor - \lfloor (n-1)\phi \rfloor .$$

Two properties of $c_n = 1 + \lfloor n\phi \rfloor - \lfloor (n-1)\phi \rfloor$

$$2, 3, 2, 3, 3, 2, 3, 2, 3, 3, 2, 3, 3, 2, 3, 2, 3, 3, 2, 3, \dots.$$

- $\gamma = (c_2, c_3, \dots)$ characterized by invariance under $2 \to 3$, $3 \to 32$ (**Fibonacci word** in the letters 2,3).

Two properties of $c_n = 1 + \lfloor n\phi \rfloor - \lfloor (n-1)\phi \rfloor$

$$2, 3, 2, 3, 3, 2, 3, 2, 3, 3, 2, 3, 3, 2, 3, 2, 3, 3, 2, 3, \ldots.$$

- $\gamma = (c_2, c_3, \ldots)$ characterized by invariance under $2 \to 3$, $3 \to 32$ (**Fibonacci word** in the letters 2,3).
- $\gamma = z_1 z_2 \ldots$ (concatenation), where $z_1 = 3$, $z_2 = 23$, $z_k = z_{k-2} z_{k-1}$

$$3 \cdot 23 \cdot 323 \cdot 23323 \cdot 32323323 \cdots$$

Two properties of $c_n = 1 + \lfloor n\phi \rfloor - \lfloor (n-1)\phi \rfloor$

$$2, 3, 2, 3, 3, 2, 3, 2, 3, 3, 2, 3, 3, 2, 3, 2, 3, 3, 2, 3, \ldots.$$

- $\gamma = (c_2, c_3, \ldots)$ characterized by invariance under $2 \to 3$, $3 \to 32$ (**Fibonacci word** in the letters 2,3).
- $\gamma = z_1 z_2 \ldots$ (concatenation), where $z_1 = 3$, $z_2 = 23$, $z_k = z_{k-2} z_{k-1}$

$$3 \cdot 23 \cdot 323 \cdot 23323 \cdot 32323323 \cdots$$

Two properties of $c_n = 1 + \lfloor n\phi \rfloor - \lfloor (n-1)\phi \rfloor$

$$2, 3, 2, 3, 3, 2, 3, 2, 3, 3, 2, 3, 3, 2, 3, 2, 3, 3, 2, 3, \dots.$$

- $\gamma = (c_2, c_3, \dots)$ characterized by invariance under $2 \to 3$, $3 \to 32$ (**Fibonacci word** in the letters 2,3).
- $\gamma = z_1 z_2 \dots$ (concatenation), where $z_1 = 3$, $z_2 = 23$, $z_k = z_{k-2} z_{k-1}$

$$3 \cdot 23 \cdot 323 \cdot 23323 \cdot 32323323 \cdots$$

Further work by **Teresa Xueshan Li**.

Coefficients of $I_n(x)$

$$I_n(x) = \prod_{i=1}^{n} \left(1 + x^{F_{i+1}}\right)$$

Coefficient of x^m: number of ways to write m as a sum of distinct Fibonacci numbers from $\{F_2, F_3, \ldots, F_{n+1}\}$.

Coefficients of $I_n(x)$

$$I_n(x) = \prod_{i=1}^{n} \left(1 + x^{F_{i+1}}\right)$$

Coefficient of x^m: number of ways to write m as a sum of distinct Fibonacci numbers from $\{F_2, F_3, \ldots, F_{n+1}\}$.

Example. Coefficient of x^8 in $(1+x)(1+x^2)(1+x^3)(1+x^5)(1+x^8)$ is 3:

$$8 = 5 + 3 = 5 + 2 + 1.$$

Coefficients of $I_n(x)$

$$I_n(x) = \prod_{i=1}^{n} \left(1 + x^{F_{i+1}}\right)$$

Coefficient of x^m: number of ways to write m as a sum of distinct Fibonacci numbers from $\{F_2, F_3, \ldots, F_{n+1}\}$.

Example. Coefficient of x^8 in $(1+x)(1+x^2)(1+x^3)(1+x^5)(1+x^8)$ is 3:

$$8 = 5 + 3 = 5 + 2 + 1.$$

Can we see these sums from $\mathfrak{F}$? Each path from the top to a point $t \in \mathfrak{F}$ should correspond to a sum.

An edge labeling of $\mathfrak{F}$

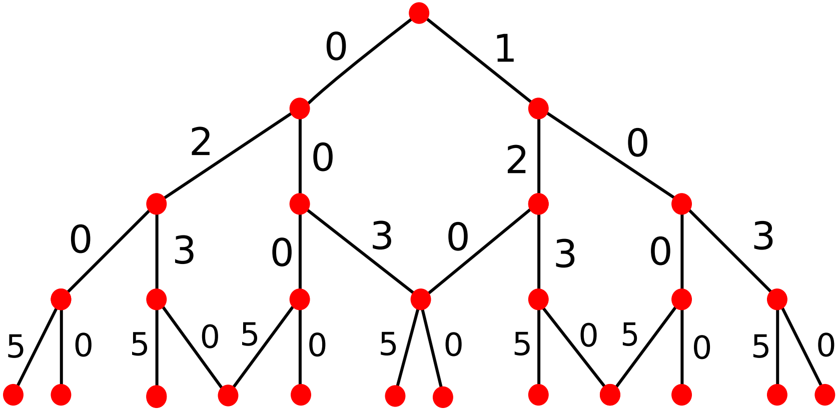
The edges between ranks $2k$ and $2k+1$ are labelled alternately $0, F_{2k+2}, 0, F_{2k+2}, \ldots$ from left to right.

An edge labeling of $\mathfrak{F}$

The edges between ranks $2k$ and $2k+1$ are labelled alternately $0, F_{2k+2}, 0, F_{2k+2}, \ldots$ from left to right.

The edges between ranks $2k-1$ and $2k$ are labelled alternately $F_{2k+1}, 0, F_{2k+1}, 0, \ldots$ from left to right.

Diagram of the edge labeling



Connection with sums of Fibonacci numbers

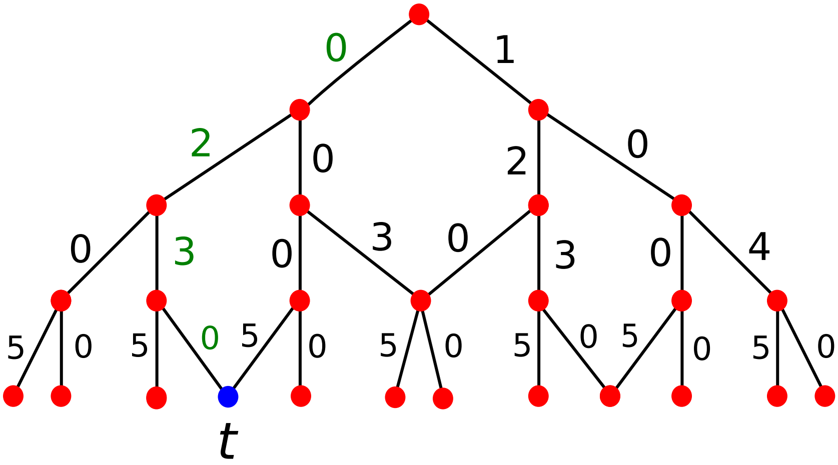
Let $\boldsymbol{t} \in \mathfrak{F}$. All paths (saturated chains) from the top to t have the same sum of their elements $\boldsymbol{\sigma(t)}$.

Connection with sums of Fibonacci numbers

Let $\boldsymbol{t} \in \mathfrak{F}$. All paths (saturated chains) from the top to t have the same sum of their elements $\boldsymbol{\sigma(t)}$.

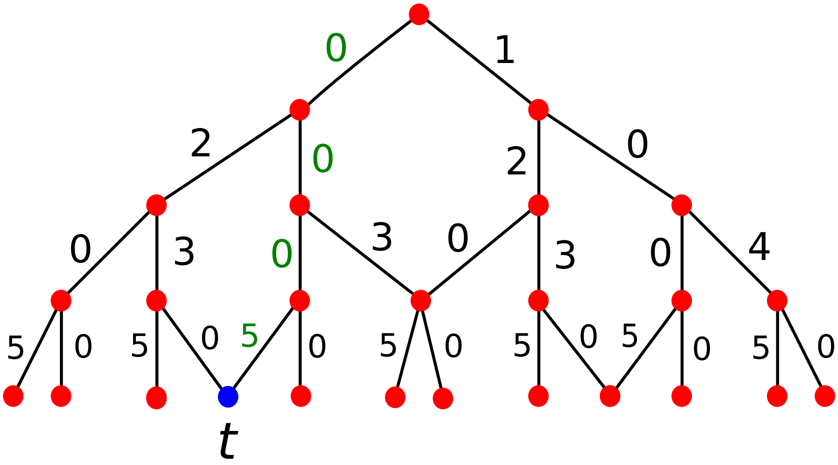
If $\mathrm{rank}(t) = n$, this gives all ways to write $\sigma(t)$ as a sum of distinct Fibonacci numbers from $\{F_2, F_3, \ldots, F_{n+1}\}$.

An example



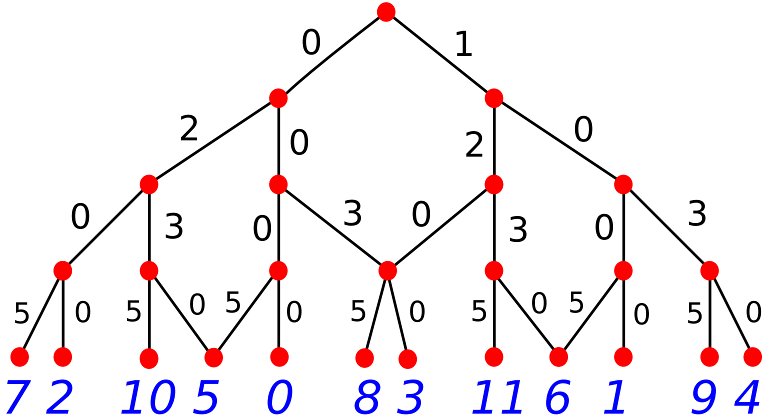
$$2 + 3 = F_3 + F_4$$

An example



$5 = F_5$

An ordering of $\mathbb{N}$



In the limit as rank $\to \infty$, get an interesting dense linear ordering $\prec$ of $\mathbb{N}$.

Special case of $\prec$

Every nonnegative integer has a unique representation as a sum of nonconsecutive Fibonacci numbers, where a summand equal to 1 is always taken to be F_2 (**Zeckendorf's theorem**).

$$n = F_{j_1} + \cdots + F_{j_s}, \quad j_1 < \cdots < j_s$$

Special case of $\prec$

Every nonnegative integer has a unique representation as a sum of nonconsecutive Fibonacci numbers, where a summand equal to 1 is always taken to be F_2 (**Zeckendorf's theorem**).

$$n = F_{j_1} + \cdots + F_{j_s}, \quad j_1 < \cdots < j_s$$

Then $n \prec 0$ if and only if j_1 is odd.

Congruence properties

$\boldsymbol{h_{m,a}(n)}$: number of coefficients of $I_n(x)$ that are $\equiv a \pmod{m}$.

$$\boldsymbol{H_{m,a}(x)} := \sum_{n\geq 0} h_{m,a}(n)x^n.$$

Congruence properties

$\mathbf{h_{m,a}(n)}$: number of coefficients of $I_n(x)$ that are $\equiv a \pmod{m}$.

$$\mathbf{H_{m,a}(x)} := \sum_{n\geq 0} h_{m,a}(n)x^n.$$

Can show that $H_{m,a}(x)$ is a rational function.

$n = 2, 3$

$$H_{2,0}(x) = \frac{x^3(1-2x^2)}{(1-x)(1-x-x^2)(1-2x+2x^2-2x^3)}$$

$$H_{2,1}(x) = \frac{1+2x^2}{1-2x+2x^2-2x^3}$$

$$H_{3,0}(x) = \frac{2x^5(1-2x^2)}{(1-x)(1-x-x^2)(1-2x+2x^2-3x^3+4x^4-4x^5)}$$

$$H_{3,1}(x) = \frac{1-2x+4x^2-6x^3+8x^4-10x^5+8x^6-6x^7}{(1-x)(1-x+x^2)(1-2x+2x^2-3x^3+4x^4-4x^5)}$$

$$H_{3,2}(x) = \frac{x^3(1+2x^4)}{(1-x)(1-x+x^2)(1-2x+2x^2-3x^3+4x^4-4x^5)}$$

$n = 4$

$$
\begin{aligned}
H_{4,0}(x) &= \frac{x^6(1-2x^2)(1-3x^2+4x^3-4x^4)}{(1-x)(1-x-x^2)(1-x^2+2x^4)(1-2x+2x^2-2x^3)^2} \\
H_{4,1}(x) &= \frac{1-2x+5x^2-8x^3+10x^4-12x^5+8x^6-6x^7}{(1-x)(1-2x+2x^2-2x^3)(1-x+2x^2-2x^3+2x^4)} \\
H_{4,2}(x) &= \frac{x^3(1+x^2)(1-2x^2)}{(1-x^2+2x^4)(1-2x+2x^2-2x^3)^2} \\
H_{4,3}(x) &= \frac{2x^5(1+x^2)}{(1-x)(1-2x+2x^2-2x^3)(1-x+2x^2-2x^3+2x^4)}
\end{aligned}
$$

$n = 4$

$$
\begin{aligned}
H_{4,0}(x) &= \frac{x^6(1-2x^2)(1-3x^2+4x^3-4x^4)}{(1-x)(1-x-x^2)(1-x^2+2x^4)(1-2x+2x^2-2x^3)^2} \\
H_{4,1}(x) &= \frac{1-2x+5x^2-8x^3+10x^4-12x^5+8x^6-6x^7}{(1-x)(1-2x+2x^2-2x^3)(1-x+2x^2-2x^3+2x^4)} \\
H_{4,2}(x) &= \frac{x^3(1+x^2)(1-2x^2)}{(1-x^2+2x^4)(1-2x+2x^2-2x^3)^2} \\
H_{4,3}(x) &= \frac{2x^5(1+x^2)}{(1-x)(1-2x+2x^2-2x^3)(1-x+2x^2-2x^3+2x^4)}
\end{aligned}
$$

- Why the factorization of the denominators?
- Why so many numerators with two terms?

References

The Stern triangle: **RS**, *Amer. Math. Monthly* **127** (2020), 99–111; **arXiv:1901.04647**

The Stern triangle: **D. Speyer**, **arXiv:1901:06301**

The Fibonacci triangle (and more): **RS**, **arXiv:2101.02131**

The final slide

The final slide