



Alternating Permutations

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M.I.T.

Definitions

A sequence a_1, a_2, \dots, a_k of distinct integers is **alternating** if

$$a_1 > a_2 < a_3 > a_4 < \dots ,$$

and **reverse alternating** if

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and **reverse alternating** if

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\mathfrak{S}_n : symmetric group of all permutations of
 $1, 2, \dots, n$

E_n

$$\begin{aligned} E_n &= \#\{w \in \mathfrak{S}_n : w \text{ is alternating}\} \\ &= \#\{w \in \mathfrak{S}_n : w \text{ is reverse alternating}\} \end{aligned}$$

via

$$a_1 a_2 \cdots a_n \mapsto n + 1 - a_1, n + 1 - a_2, \dots, n + 1 - a_n$$

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E.g., $E_4 = 5$: 2143, 3142, 3241, 4132, 4231

André's theorem

Theorem (Désiré André, 1879)

$$y := \sum_{n \geq 0} E_n \frac{x^n}{n!} = \sec x + \tan x$$

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E_n is an **Euler number**,
 E_{2n} a **secant number**,
 E_{2n-1} a **tangent number**.

Naive proof

52439**1**826
← →
alternating alternating

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$$\begin{array}{c} 52439\mathbf{1}826 \\ \underbrace{\qquad\qquad\qquad}_{\leftarrow} \quad \underbrace{\qquad\qquad\qquad}_{\rightarrow} \\ \textit{alternating} \quad \textit{alternating} \end{array}$$

Obtain w that is **either** alternating or reverse alternating.

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Obtain w that is **either** alternating or reverse alternating.

$$\Rightarrow 2E_{n+1} = \sum_{k=0}^n \binom{n}{k} E_k E_{n-k}, \quad n \geq 1$$

Completion of proof

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$$2 \sum_{n \geq 0} E_{n+1} \frac{x^n}{n!} = 2y'$$

$$\sum_{n \geq 0} \left(\sum_{k=0}^n \binom{n}{k} E_k E_{n-k} \right) \frac{x^n}{n!} = 1 + y^2$$

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$$\Rightarrow 2y' = 1 + y^2, \text{ etc.}$$

A generalization

∃ more sophisticated approaches.

E.g., let

$$f_k(n) = \#\{w \in \mathfrak{S}_n : w(r) < w(r+1) \Leftrightarrow k|r\}$$

$$f_2(n) = E_n$$

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Theorem.

$$\sum_{k \geq 0} f_k(kn) \frac{x^{kn}}{(kn)!} = \left(\sum_{n \geq 0} (-1)^n \frac{x^{kn}}{(kn)!} \right)^{-1}.$$

Combinatorial trigonometry

Define $\sec x$ and $\tan x$ by

$$\sec x + \tan x = \sum_{n \geq 0} E_n \frac{x^n}{n!}.$$

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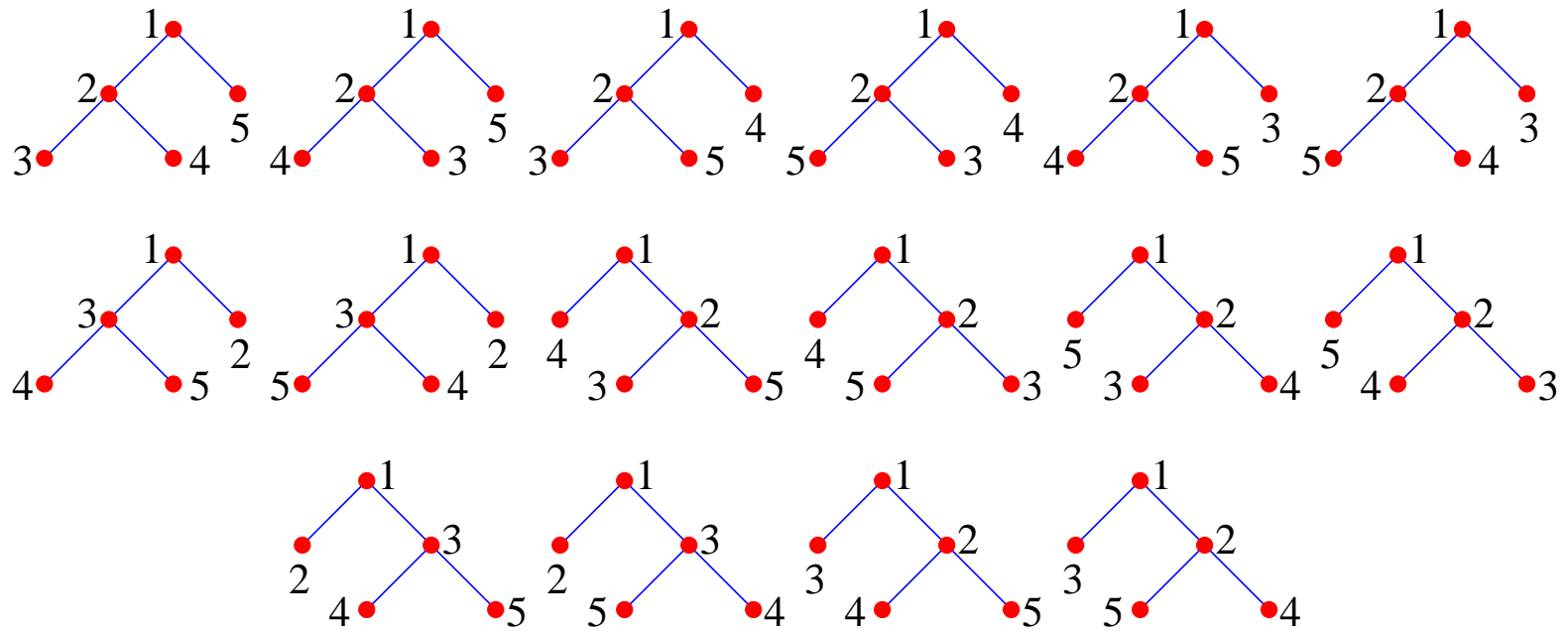
Exercise. $\sec^2 x = 1 + \tan^2 x$

Exercise. $\tan(x + y) = \frac{\tan x + \tan y}{1 - (\tan x)(\tan y)}$

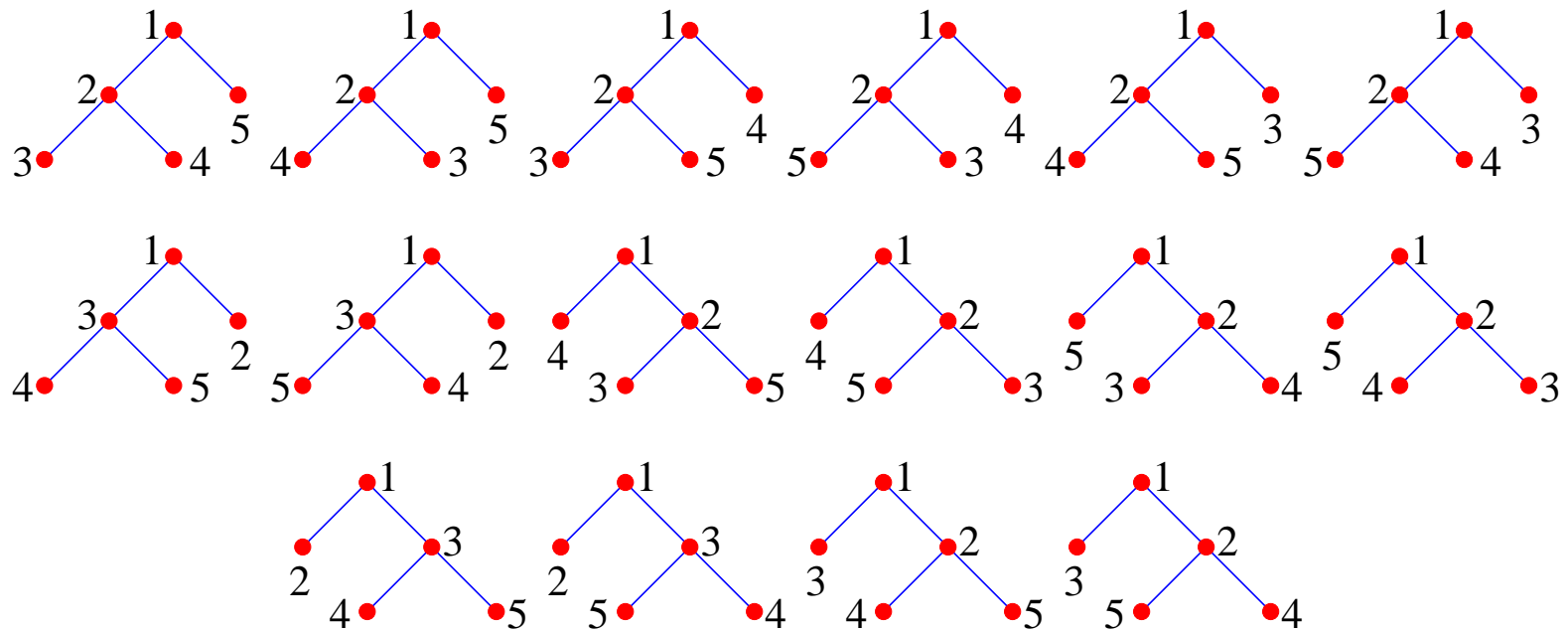
Some occurrences of Euler numbers

(1) E_{2n-1} is the number of complete increasing binary trees on the vertex set $[2n + 1] = \{1, 2, \dots, 2n + 1\}$.

Five vertices



Five vertices



Slightly more complicated for E_{2n}

Simsun permutations

Definition. A permutation $w = a_1 \cdots a_n \in \mathfrak{S}_n$ is **simsun** if for all $1 \leq k \leq n$, the restriction $b_1 \cdots b_k$ of w to $1, 2, \dots, k$ has no **double descents** $b_{i-1} > b_i > b_{i+1}$.

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Theorem (R. **Simion** and S. **Sundaram**) *The number of simsun permutations $w \in \mathfrak{S}_n$ is E_n .*

Chains of partitions

(2) Start with n one-element sets $\{1\}, \dots, \{n\}$.

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Theorem. *The number of \mathfrak{S}_n -orbits is E_{n-1} .*

Orbit representatives for $n = 5$

$12-3-4-5$	$123-4-5$	$1234-5$
$12-3-4-5$	$123-4-5$	$123-45$
$12-3-4-5$	$12-34-5$	$125-34$
$12-3-4-5$	$12-34-5$	$12-345$
$12-3-4-5$	$12-34-5$	$1234-5$

A polytope volume

Let \mathcal{P}_n be the convex polytope in \mathbb{R}^n with equations

$$\begin{aligned}x_i &\geq 0, & 1 \leq i \leq n \\x_i + x_{i+1} &\leq 1, & 1 \leq i \leq n - 1.\end{aligned}$$

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Theorem. $\text{vol}(\mathcal{P}_n) = \frac{1}{n!} E_n$

Ulam's problem

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$$w = 5\mathbf{2}41\mathbf{7}36 \Rightarrow \text{is}(w) = 3$$

Two highlights

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$$E(n) \sim 2\sqrt{n}$$

Write $\mathbf{is}_n(w)$ for $\text{is}(w)$ when $w \in \mathfrak{S}_n$.

Baik-Deift-Johansson:

$$\lim_{n \rightarrow \infty} \text{Prob} \left(\frac{\mathbf{is}_n(w) - 2\sqrt{n}}{n^{1/6}} \leq t \right) = F(t),$$

the **Tracy-Widom distribution**.

Alternating subsequences?

$\text{as}(w)$ = length of longest alternating subseq. of w

$$w = 56218347 \Rightarrow \text{as}(w) = 5$$

The main lemma

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$$a_k(n) = \#\{w \in \mathfrak{S}_n : \text{as}(w) = k\}$$

$$\begin{aligned} b_k(n) &= a_1(n) + a_2(n) + \cdots + a_k(n) \\ &= \#\{w \in \mathfrak{S}_n : \text{as}(w) \leq k\}. \end{aligned}$$

Recurrence for $a_k(n)$

$$\Rightarrow a_k(n) = \sum_{j=1}^n \binom{n-1}{j-1}$$

$$\sum_{2r+s=k-1} (a_{2r}(j-1) + a_{2r+1}(j-1)) a_s(n-j)$$

The main generating function

Define $B(x, t) = \sum_{k, n \geq 0} b_k(n) t^k \frac{x^n}{n!}$

Theorem.

$$B(x, t) = \frac{2/\rho}{1 - \frac{1-\rho}{t} e^{\rho x}} - \frac{1}{\rho},$$

where $\rho = \sqrt{1 - t^2}$.

Formulas for $b_k(n)$

Corollary.

$$\Rightarrow b_1(n) = 1$$

$$b_2(n) = n$$

$$b_3(n) = \frac{1}{4}(3^n - 2n + 3)$$

$$b_4(n) = \frac{1}{8}(4^n - (2n - 4)2^n)$$

⋮

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no such formulas for longest **increasing** subsequences

Mean (expectation) of $as(w)$

$$D(n) = \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} as(w),$$

the **expectation** of $as(n)$ for $w \in \mathfrak{S}_n$

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Let

$$\begin{aligned} A(x, t) &= \sum_{k, n \geq 0} a_k(n) t^k \frac{x^n}{n!} = (1-t)B(x, t) \\ &= (1-t) \left(\frac{2/\rho}{1 - \frac{1-\rho}{t} e^{\rho x}} - \frac{1}{\rho} \right). \end{aligned}$$

Formula for $D(n)$

$$\begin{aligned}\sum_{n \geq 0} D(n)x^n &= \frac{\partial}{\partial t} A(x, 1) \\ &= \frac{6x - 3x^2 + x^3}{6(1-x)^2} \\ &= x + \sum_{n \geq 2} \frac{4n+1}{6} x^n.\end{aligned}$$

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Recall $E(n) \sim 2\sqrt{n}$.

Variance of $as(w)$

$$V(n) = \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \left(as(w) - \frac{4n+1}{6} \right)^2, \quad n \geq 2$$

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$$V(n) = \frac{8}{45}n - \frac{13}{180}, \quad n \geq 4$$

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Corollary.

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similar results for higher moments

A new distribution?

$$P(t) = \lim_{n \rightarrow \infty} \text{Prob}_{w \in \mathfrak{S}_n} \left(\frac{\text{as}(w) - 2n/3}{\sqrt{n}} \leq t \right)$$

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Stanley distribution?

Limiting distribution

Theorem (Pemantle, Widom, (Wilf)).

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$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{t\sqrt{45}/4} e^{-s^2} ds$$

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Umbral enumeration

Umbral formula: involves E^k , where E is an indeterminate (the **umbra**). Replace E^k with the Euler number E_k . (Technique from 19th century, modernized by Rota et al.)

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Example.

$$\begin{aligned}(1 + E^2)^3 &= 1 + 3E^2 + 3E^4 + E^6 \\ &= 1 + 3E_2 + 3E_4 + E_6 \\ &= 1 + 3 \cdot 1 + 3 \cdot 5 + 61 \\ &= 80\end{aligned}$$

Another example

$$\begin{aligned}(1+t)^E &= 1 + Et + \binom{E}{2}t^2 + \binom{E}{3}t^3 + \dots \\ &= 1 + Et + \frac{1}{2}E(E-1)t^2 + \dots \\ &= 1 + E_1t + \frac{1}{2}(E_2 - E_1)t^2 + \dots \\ &= 1 + t + \frac{1}{2}(1-1)t^2 + \dots \\ &= 1 + t + O(t^3).\end{aligned}$$

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Let $f(n)$ be the number of **alternating** fixed-point free involutions in \mathfrak{S}_{2n} .

$$n = 3 : \quad 214365 = (1, 2)(3, 4)(5, 6)$$

$$645231 = (1, 6)(2, 4)(3, 5)$$

$$f(3) = 2$$

An umbral theorem

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$$F(x) = \sum_{n \geq 0} f(n)x^n$$

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Proof. Uses representation theory of the symmetric group \mathfrak{S}_n . In particular, there is a “nice” representation of dimension E_n .

Ramanujan's Lost Notebook

Theorem (Ramanujan, Berndt, implicitly) As $x \rightarrow 0+$,

$$2 \sum_{n \geq 0} \left(\frac{1-x}{1+x} \right)^{n(n+1)} \sim \sum_{k \geq 0} f(k) x^k = F(x),$$

an **analytic** (non-formal) identity.

A formal identity

Corollary (via Ramanujan, Andrews).

$$F(x) = 2 \sum_{n \geq 0} q^n \frac{\prod_{j=1}^n (1 - q^{2j-1})}{\prod_{j=1}^{2n+1} (1 + q^j)},$$

where $q = \left(\frac{1-x}{1+x}\right)^{2/3}$, a **formal** identity.

Simple result, hard proof

Recall: number of n -cycles in \mathfrak{S}_n is $(n - 1)!$.

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Theorem. Let $b(n)$ be the number of **alternating** n -cycles in \mathfrak{S}_n . Then if n is odd,

$$b(n) = \frac{1}{n} \sum_{d|n} \mu(d) (-1)^{(d-1)/2} E_{n/d}.$$

Special case

Corollary. *Let p be an odd prime. Then*

$$b(p) = \frac{1}{p} \left(E_p - (-1)^{(p-1)/2} \right).$$

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$J(n)$ = expected number of fixed points of **reverse alternating** $w \in \mathfrak{S}_n$

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Similar (but more complicated) for n odd or for alternating permutations.

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Combinatorial proof?

Descent sets

Let $w = a_1 a_2 \cdots a_n \in \mathfrak{S}_n$. **Descent set** of w :

$$D(w) = \{i : a_i > a_{i+1}\} \subseteq \{1, \dots, n-1\}$$

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$$D(4157623) = \{1, 4, 5\}$$

$$D(4152736) = \{1, 3, 5\} \text{ (alternating)}$$

$$D(4736152) = \{2, 4, 6\} \text{ (reverse alternating)}$$

$\beta_n(S)$

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w	$D(w)$
123	\emptyset
213	$\{1\}$
312	$\{1\}$
132	$\{2\}$
231	$\{2\}$
321	$\{1, 2\}$

$$\beta_3(\emptyset) = 1, \quad \beta_3(1) = 2$$

$$\beta_3(2) = 2, \quad \beta_3(1, 2) = 1$$

Fix n . Let $S \subseteq \{1, \dots, n-1\}$. Let $u_S = t_1 \cdots t_{n-1}$, where

$$t_i = \begin{cases} a, & i \notin S \\ b, & i \in S. \end{cases}$$

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Example. $n = 8$, $S = \{2, 5, 6\} \subseteq \{1, \dots, 7\}$

$$u_S = abaabba$$

A noncommutative gen. function

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Thus

$$\Psi_3(a, b) = aa + 2ab + 2ba + bb$$

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Thus

$$\begin{aligned} \Psi_3(a, b) &= aa + 2ab + 2ba + bb \\ &= (a + b)^2 + (ab + ba) \end{aligned}$$

The cd -index

Theorem. *There exists a noncommutative polynomial $\Phi_n(c, d)$, called the **cd -index** of \mathfrak{S}_n , with **nonnegative** integer coefficients, such that*

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Example. Recall

$$\Psi_3(a, b) = aa + 2ab + 2ba + b^2 = (a + b)^2 + (ab + ba).$$

Therefore

$$\Phi_3(c, d) = c^2 + d.$$

Small values of $\Phi_n(c, d)$

$$\Phi_1 = 1$$

$$\Phi_2 = c$$

$$\Phi_3 = c^2 + d$$

$$\Phi_4 = c^3 + 2cd + 2dc$$

$$\Phi_5 = c^4 + 3c^2d + 5cdc + 3dc^2 + 4d^2$$

$$\Phi_6 = c^5 + 4c^3d + 9c^2dc + 9cdc^2 + 4dc^3 \\ + 12cd^2 + 10dcd + 12d^2c.$$

The set \mathcal{S}_μ

Recall: $w = a_1 \cdots a_n \in \mathfrak{S}_n$ is **simsum** if for all $1 \leq k \leq n$, the restriction $b_1 \cdots b_k$ of w to $1, 2, \dots, k$ has no **double descents**
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The set S_μ

Recall: $w = a_1 \cdots a_n \in \mathfrak{S}_n$ is **simsum** if for all $1 \leq k \leq n$, the restriction $b_1 \cdots b_k$ of w to $1, 2, \dots, k$ has no **double descents** $b_{i-1} > b_i > b_{i+1}$.

Given a cd -monomial μ , let $c \rightarrow 0$, $d \rightarrow 10$, and remove final 0.

Example.

$$\mu = cd^2c^2d \rightarrow 01010001,$$

the characteristic vector of $S_\mu = \{2, 4, 8\}$.

The coefficients of $\Phi_n(c, d)$

Theorem (Simion-Sundaram) *Let μ be a cd -monomial of degree $n - 1$. The coefficient of μ in $\Phi_n(c, d)$ is the number of simsun permutations $w \in \mathfrak{S}_{n-1}$ with descent set S_μ .*

The case $n = 4$

w	$D(w)$	μ
123	\emptyset	$c^3 \rightarrow 00$
213	1	$dc \rightarrow 10$
312	1	$dc \rightarrow 10$
132	2	$cd \rightarrow 01$
231	2	$cd \rightarrow 01$

$$\Rightarrow \Phi_4(c, d) = c^3 + 2dc + 2cd$$

Expansion of cd -monomials

Let $\mathbf{m} = m(c, d)$ be a cd -monomial, e.g., c^2d^2cd .
Expand $m(a + b, ab + ba)$:

$$(a + b)^2(ab + ba)^2(a + b)(ab + ba) = a^9 + ba^8 + \dots$$

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Key observation: $m(a + b, ab + ba)$ is a sum of distinct monomials, always including $ababab\dots$ and $bababa\dots$.

Alternating permutations

The coefficient of $ababab \dots$ and of $bababa \dots$ in $\Psi_n(a, b)$ is

$$\beta_n(2, 4, 6, \dots) = \beta_n(1, 3, 5, \dots) = E_n.$$

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The coefficient of $ababab \dots$ and of $bababa \dots$ in $\Psi_n(a, b)$ is

$$\beta_n(2, 4, 6, \dots) = \beta_n(1, 3, 5, \dots) = E_n.$$

Hence:

Theorem. (a) $\Phi_n(1, 1) = E_n$ (can interpret coefficients combinatorially).

(b) For all $S \subseteq \{1, \dots, n-1\}$,

$$\beta_n(S) \leq \beta_n(1, 3, 5, \dots) = E_n.$$

An example

$$\Phi_5 = 1c^4 + 3c^2d + 5cdc + 3dc^2 + 4d^2$$

$$1 + 3 + 5 + 3 + 4 = 16 = E_5$$

Comments

NOTE. (b) is due originally to Niven and later de Bruijn (different proofs).

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NOTE. Not hard to show that

$$\beta_n(S) < E_n$$

unless $S = \{1, 3, 5, \dots\}$ or $S = \{2, 4, 6, \dots\}$.

