A Symmetric Function Arising from a Theta Function of Ramanujan

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(work in progress)

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The function $\phi(\lambda)$

Amdeberhan-Ono-Singh (2024):

$$\phi(\lambda) := (2n)! \cdot \prod_{k=1}^{n} \frac{1}{m_k!} \left(\frac{4^k(4^k - 1)B_{2k}}{(2k)(2k)!} \right)^{m_k},$$

where $\lambda = \langle 1^{m_1}, \ldots, n^{m_n} \rangle \vdash n = \sum i m_i$ (λ is a partition of n with m_i i's) and B_{2k} is a Bernoulli number.

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Original motivation. Express a certain theta function of Ramanujan in terms of Eisenstein series (not explained here).

Euler numbers E_{2n}

Our motivation. Not hard to see that

$$\phi(\lambda) \in \mathbb{Z}, \quad \sum_{\lambda \vdash n} |\phi(\lambda)| = E_{2n},$$

an **Euler number** or **secant number**, defined by

$$\sec x = \sum_{n \geq 0} E_{2n} \frac{x^{2n}}{(2n)!}.$$

Well-known: E_{2n} is equal to the number of **alternating permutations** $a_1 a_2 \cdots a_{2n} \in \mathfrak{S}_{2n}$, i.e.,

$$a_1 > a_2 < a_3 > a_4 < \cdots > a_{2n}.$$

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Question: what does $|\phi(\lambda)|$ count?

A symmetric function

The general form $\phi(\lambda) = (2n)! \prod \frac{1}{m_k!} f_k^{m_k}$ suggests defining a symmetric function in the variables $\boldsymbol{x} = (x_1, x_2, \ldots)$:

$$A_n = A_n(\boldsymbol{x}) = \sum_{\lambda \vdash n} |\phi(\lambda)| \cdot p_\lambda,$$

where $\boldsymbol{p_\lambda}$ is a power sum symmetric function.

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- $A_n = \sum_{\lambda \vdash n} (*) s_\lambda$ (Schur function expansion)

Record partitions

$$\mathfrak{A}_{2n} := \{w \in \mathfrak{S}_{2n} \,:\, w \text{ alternating}\}$$

Recall $\sum_{\lambda \vdash n} |\phi(\lambda)| = E_{2n} = \#\mathfrak{A}_{2n}$.

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If $w = a_1 > a_2 < \cdots > a_{2n} \in \mathfrak{A}_{2n}$ define $\hat{w} = a_1, a_3, \ldots, a_{2n-1}$. Write $\hat{w} = b_1, b_2, , \ldots, b_n$.

record set $\mathrm{rec}(\hat{w})$: set of indices $1 \leq i \leq n$ for which b_i is a left-to-right maximum (or **record**) in $\hat{w}$.

record partition $\boldsymbol{rp}(\hat{w})$: if $\mathrm{rec}(\hat{w}) = \{r_1, r_2, \ldots, r_j\}_<$, then $\boldsymbol{rp}(\hat{w})$ is the partition of n with parts $r_2 - r_1, r_3 - r_2, r_4 - r_3, \ldots, n + 1 - r_j$ (in decreasing order)

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Example. $w = 7,2,5,4,8,3,10,6,9,5 \in \mathfrak{A}_{10}$, $\hat{w} = 7, 5, 8, 10, 9$; $r_1 = 1$, $r_2 = 3$, $r_3 = 4$, $r_2 - r_1 = 2$, $r_3 - r_2 = 1$, $6 - r_3 = 2$, $\mathrm{rp}(\hat{w}) = (2,2,1)$

Combinatorial interpretation of $\phi(\lambda)$

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Note on proof. Recall

$$\phi(\lambda) = (2n)! \cdot \prod_{k=1}^{n} \frac{1}{m_k!} \left(\frac{4^k(4^k-1)B_{2k}}{(2k)(2k)!} \right)^{m_k},$$

where $\lambda = \langle 1^{m_1}, \ldots, n^{m_n} \rangle \vdash \sum i m_i$. To get combinatorics into the picture, use

$$E_{2k-1} = 4^k(4^k-1)\frac{|B_{2k}|}{2k}.$$

Remainder of proof is a bijective argument.

Examples.

$$
\begin{aligned}
A_1 &= p_1 \\
A_2 &= 3p_1^2 + 2p_2 \\
A_3 &= 15p_1^3 + 30p_2p_1 + 16p_3 \\
A_4 &= 105p1^4 + 420p_2p_1^2 + 140p_2^2 + 448p_3p_1 + 272p_4
\end{aligned}
$$

A_2:

w	$\hat{w}$	$\mathrm{rp}(\hat{w})$
2143	24	11
3142	34	11
3241	34	11
4132	43	2
4231	43	2

A generating function

$$\boldsymbol{F(x,t)} := \sum_{n\geq 0} \frac{A_n(\boldsymbol{x})t^n}{(2n)!}$$

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Follows from

$$\begin{aligned} |\phi(\lambda)| &= (2n)! \cdot \prod_{k=1}^{n} \frac{1}{m_k!} \left(\frac{4^k(4^k-1)|B_{2k}|}{(2k)(2k)!} \right)^{m_k} \\ &= (2n)! \cdot \prod_{k=1}^{n} \frac{1}{m_k!} \left(\frac{E_{2k-1}}{(2k)!} \right)^{m_k} \end{aligned}$$

that

$$F(\boldsymbol{x},t) = \exp\left(\sum_{k\geq 1} \frac{E_{2k-1} p_k t^k}{(2k)!} \right).$$

A "shifted" generating function

$$
\begin{aligned}
\sum_k \frac{E_{2k-1}x^{2k}}{(2k)!} &= \int \left(\sum_k \frac{E_{2k-1}x^{2k-1}}{(2k-1)!} \right) dx \\
&= \int \tan(x) dx \\
&= \log(\sec(x))
\end{aligned}
$$

Product formula

$$
\begin{aligned}
\Rightarrow F(\boldsymbol{x}, t) &= \exp\left(\sum_{k\geq 1} \frac{E_{2k-1} p_k t^k}{(2k)!}\right) \\
&= \prod_i \exp\left(\sum_{k\geq 1} \frac{E_{2k-1} x_i^k t^k}{(2k)!}\right) \\
&= \prod_i \exp(\log \sec(\sqrt{x_i t})) \\
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Theorem. $F(\boldsymbol{x}, t) = \prod_{i\geq 1} \sec(\sqrt{x_i t})$

h-positivity

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Proof. Weierstrass product formula $\cos(t) = \prod_{k\geq 1}\left(1 - \frac{4t^2}{\pi^2(2k-1)^2}\right)$ implies:

$$\begin{aligned} F(\boldsymbol{x}, t) &= \prod_{i\geq 1}\prod_{k\geq 1}\left(1 - \frac{4x_i t}{\pi^2(2k-1)^2}\right)^{-1} \\ &= \prod_{k\geq 1}\left(\sum_{n\geq 0}\left(\frac{4}{\pi^2(2k-1)^2}\right)^n t^n h_n(\boldsymbol{x})\right). \quad \square \end{aligned}$$

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Very noncombinatorial formula for the coefficients!

Some data

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A_1 &= h_1 \\
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A_3 &= h_1^3 + 12h_2h_1 + 48h_3 \\
A_4 &= h_1^4 + 24h_2h_1^2 + 256h_3h_1 + 16h_2^2 + 1088h_4 \\
A_5 &= h_1^5 + 40h_2h_1^3 + 800h_3h_1^2 + 80h_2^2h_1 + 9280h_4h_1 \\
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Note. Coefficient of h_n is nE_{2n-1}, the number of "cyclically alternating" permutations in $\mathfrak{S}_{2n}$.

Chromatic symmetric functions

G: finite simple graph on vertex set $V(G) = \{v_1, v_2, \ldots, v_p\}$

$$X_G(\boldsymbol{x}) := \sum_{\substack{\kappa\colon V(G)\to\mathbb{P} \\ uv\in E(G)\Rightarrow\kappa(u)\neq\kappa(v)}} x_{\kappa(v_1)} x_{\kappa(v_2)} \cdots x_{\kappa(v_p)}$$

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$$\begin{aligned} X_{\overline{K_p}}(\boldsymbol{x}) &= (x_1 + x_2 + \cdots)^p = e_1^p \\ X_{K_p}(\boldsymbol{x}) &= p!\, e_p \end{aligned}$$

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$$X_G(\underbrace{1, 1, \dots, 1}_{m\ 1\text{'s}}, 0, 0, \dots) = \chi_G(m),$$

the **chromatic polynomial** of G.

Interval orders

$\mathcal{I} = \{[a_1, b_1], \ldots, [a_n, b_n]\}$, a collection of closed intervals in $\mathbb{R}$, so $a_i < b_i$.

$G_{\mathcal{I}}$: graph with vertex set $\mathcal{I}$, with $[a_i, b_i]$ adjacent to $[a_j, b_j]$ if $[a_i, b_i] \cap [a_j, b_j] \neq \emptyset$ (incomparability graph of the corresponding interval order: $[a_i, b_i] < [a_j, b_j]$ if $b_i < a_j$]).

M: a complete matching $a_1 b_1, a_2 b_2, \ldots, a_n b_n$ on $[2n] := \{1, 2, \ldots, 2n\}$, with $a_i < b_i$ (so $\{a_1, b_1, \ldots, a_n, b_n\} = [2n]$)

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$\mathcal{I}(M) := \{[a_1, b_1], \dots, [a_n, b_n]\}$

Theorem. *$\omega A_n = \sum_{M \in \mathcal{M}_n} X_{G_{\mathcal{I}(M)}}$, where $\mathcal{M}_n$ is the set of all $(2n-1)!!$ complete matchings on $[2n]$, and $X_{G_{\mathcal{I}(M)}}$ is the chromatic symmetric function of the graph $G_{\mathcal{I}(M)}$.*

The case $n = 2$

matching M	graph $G_{\mathcal{I}(M)}$	$X_{G_{\mathcal{I}(M)}}$
12, 34	$\bullet \qquad \bullet$	e_1^2
13, 24	$\bullet\!-\!\!-\!\!-\!\!-\!\bullet$	$2e_2$
14, 23	$\bullet\!-\!\!-\!\!-\!\!-\!\bullet$	$2e_2$

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Equivalently, $A_2 = h_1^2 + 4h_2$.

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Warning! The interval orders $\mathcal{I}(M)$ need not be **unit** interval orders, so $X_{G_{\mathcal{I}(M)}}$ need not be e-positive. Thus the theorem does not give another proof that A_n is h-positive. (By a recent result of **Tatsuyuki Hikita**, X_G is e-positive for incomparability graphs of unit interval orders.)

Two problems

Problem 1. What can be said about the structure of the interval orders $\mathcal{I}(M)$ for $M \in \mathcal{M}_n$?

How many are semiorders? (For $1 \leq n \leq 4$: $1, 2, 5, 10$.)

How many times does a given interval order occur (up to isomorphism)? E.g., an antichain occurs $n!$ times.

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Problem 2. Are there other "nice" examples of sums (or linear combinations) of X_G's being e-positive?

Monomial symmetric functions

Example. Coefficient of m_{311} in A_5 is the number of $w = a_1, \ldots, a_{10} \in \mathfrak{S}_{10}$ satisfying

$$\underbrace{a_1 > a_2 < a_3 > a_4 < a_5 > a_6}_{\text{length } 6=2\lambda_1}\ \underbrace{a_7 > a_8}_{2=2\lambda_2}\ \underbrace{a_9 > a_{10}}_{2=2\lambda_3}.$$

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Proof sketch. Expand

$$\sum_n A_n(\boldsymbol{x}) = \prod_i \sec(\sqrt{x_i}\,t) = \prod_i \left(\sum_n E_{2n} \frac{x_i^n t^n}{(2n)!} \right)$$

$$[m_\lambda]A_n(\boldsymbol{x}) = \binom{2n}{2\lambda_1, 2\lambda_2, \dots} E_{2\lambda_1} E_{2\lambda_2} \cdots ,$$

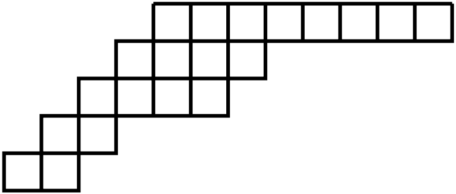
etc.

Fundamental quasisymmetric expansion

Note (for quasisymmetric aficiandos). Can easily deduce the expansion of A_n in terms of fundamental quasisymmetric functions L_S, $S \subseteq [n-1]$, from the monomial expansion. Details omitted.

Schur function expansion

Example. To get the coefficient of s_{5311} in $A_{10}(\boldsymbol{x})$, take the conjugate partition 42211 and double each part: $\mu = 84422$. Form the skew shape ρ_{5311}:



Row lengths are the parts of μ.
Each row begins one square to the left of the row above.

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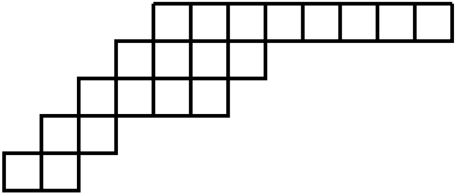

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Theorem. *For general $\lambda \vdash n$, the coefficient of s_λ in A_n is the number of standard Young tableaux of (skew) shape ρ_λ.*
(Well-known determinantal formula.)

First generalization

Note $\sec t = \left(\sum_{n\geq 0}(-1)^n \frac{t^{2n}}{(2n)!}\right)^{-1}$.

Recall $F(\boldsymbol{x}, t) = \sum_{n\geq 0} \frac{A_n(\boldsymbol{x})t^n}{(2n)!} = \prod_{i\geq 1} \sec(\sqrt{x_i t})$.

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p-expansion of $A_{n,d}$

Recall for $d = 2$:

$$A_n = A_{n,2} = \sum_{\lambda \vdash n} \#\{w \in \mathfrak{A}_{2n} : \mathrm{rp}(\hat{w}) = \lambda\} p_\lambda,$$

where $\mathfrak{A}_{2n} = \{w \in \mathfrak{S}_{2n} : w \text{ alternating}\}$ and $\hat{w} = a_1, a_3, \ldots, a_{2n-1}$.

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For $A_{n,d}$, replace $\mathfrak{A}_{2n}$ with

$$\{w = a_1 \cdots a_{dn} \in \mathfrak{S}_{dn} \,:\, \mathrm{Asc}(w) = \{d, 2d, \ldots, (n-1)d\}.$$

Replace $\hat{w}$ with $a_1, a_{d+1}, a_{2d+1}, \ldots, a_{(n-1)d+1}$.

h-expansion of $A_{n,d}$

For $d \geq 3$ we don't know the complex zeros of

$$G_d(t) := \sum_{n \geq 0} \frac{(-1)^n t^n}{(dn)!}$$

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However, $G_d(t)$ is a special case of a **Mittag-Leffler function**. It is known that that there are real numbers $0 < \alpha_1 < \alpha_2 < \cdots$ such that

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Can also be proved using the theory of total positivity (**Edrei-Thoma** theorem).

Monomial expansion of $A_{n,d}$

Recall: Coefficient of m_{311} in A_5 is the number of $w = a_1, \dots, a_{10} \in \mathfrak{S}_{10}$ satisfying

$$\underbrace{a_1 > a_2 < a_3 > a_4 < a_5 > a_6}_{\text{length } 6=2\lambda_1}\ \underbrace{a_7 > a_8}_{2=2\lambda_2}\ \underbrace{a_9 > a_{10}}_{2=2\lambda_3}.$$

Example. Coefficient of m_{21} in $A_{5,3}$ is the number of $w \in \mathfrak{S}_9$ satisfying

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Also nice Schur function expansion.

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Background (**Carlitz-Scoville-Vaughan** 1974, **RS** 1976). Let

$$f_k(n) = \#\{(u_1,\ldots,u_k) \in \mathfrak{S}_n^k \,:\, D(u_1)\cap\cdots\cap D(u_k) = \emptyset\}.$$

Then $\sum_{n\geq 0} f_k(n)\frac{t^n}{n!^k} = \left(\sum_{n\geq 0}(-1)^n \frac{t^n}{n!^k}\right)^{-1}$.

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Recall for denominator $(2n)!$ (the original problem):

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Replace $\mathrm{rp}(\hat{w})$ with $\mathrm{rp}(u_i)$, for any fixed $1 \leq i \leq k$.

h-expansion of $B_{n,k}$

One can show from the theory of hypergeometric series that there are real numbers $0 < \beta_1 < \beta_2 < \cdots$ such that

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Schur function expansion: in progress

q-analogues

Can turn each $r!$ into its **q-analogue**

$$(\boldsymbol{r})! = (\mathbf{1})(\mathbf{2})\cdots(\boldsymbol{r}),$$

where $(\boldsymbol{i}) = 1 + q + q^2 + \cdots + q^{i-1}$ (work in progress). Here we discuss only the simplest case.

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Put $d = 1$.

A triviality and its q-analogue

$$\begin{aligned} F_1(\boldsymbol{x}, t) &= \exp(t(x_1 + x_2 + \cdots)) \\ \Rightarrow A_{n,1}(\boldsymbol{x}) &= p_1^n = h_1^n. \end{aligned}$$

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Utterly trivial, but what about the q-analogue?

$$\begin{aligned} \boldsymbol{A(t)} &= \left(\sum_{n \geq 0} (-1)^n \frac{t^n}{\boldsymbol{(n)!}} \right)^{-1} \\ \prod_{i \geq 1} A(tx_i) &= \sum_{n \geq 0} \boldsymbol{G_n(x)} \frac{t^n}{\boldsymbol{(n)!}} \end{aligned}$$

Schur expansion of $G_n(x)$

Theorem. $G_n(\boldsymbol{x}) = \sum_{\lambda \vdash n} f^\lambda(q) s_\lambda(\boldsymbol{x})$, where

$$\begin{aligned} f^\lambda(q) &= \sum_{\substack{\text{SYT } T \\ \text{shape}(T)=\lambda}} q^{\mathrm{maj}(T)} \\ &= (1-q)(1-q^2)\cdots(1-q^n)s_\lambda(1,q,q^2,\dots) \\ &= \frac{q^{\sum(i-1)\lambda_i}(\boldsymbol{n})!}{\prod_{u\in\lambda}(\mathbf{hl}(\boldsymbol{u}))}, \end{aligned}$$

where ***hl***(u) is the **hook length** at u.

(h, r)-positivity

$G_2(\boldsymbol{x}) = h_1^2 + (q-1)h_2$, so not (h, q)-positive. But let $\boldsymbol{r} = q - 1$. Then $G_2(\boldsymbol{x}) = h_1^2 + rh_2$. Similarly,

$$\begin{aligned} G_3(\boldsymbol{x}) &= h_1^3 + (3r + r^2)h_2h_1 + (2r^2 + r^3)h_3 \\ G_4(\boldsymbol{x}) &= h_1^4 + (6r + 4r^2 + r^3)h_2h_1^2 + \cdots . \end{aligned}$$

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Much more is known, and even more to be done**!**.

The final slide

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