Alternating Permutations

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A sequence $a_1, a_2, \ldots, a_k$ of distinct integers is **alternating** if

$$a_1 > a_2 < a_3 > a_4 < \cdots ,$$

and **reverse alternating** if

$$a_1 < a_2 > a_3 < a_4 > \cdots .$$

$\boldsymbol{E_n} = \#\{w \in \mathfrak{S}_n : w \text{ is alternating}\}$.
E.g., $E_4 = 5$: 2143, 3142, 3241, 4132, 4231

Theorem (D. André, 1879)

$$y := \sum_{n\geq 0} E_n \frac{x^n}{n!} = \sec x + \tan x$$

($\Rightarrow$ **combinatorial trigonometry**)

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Naive proof.

$$2E_{n+1} = \sum_{k=0}^{n} \binom{n}{k} E_k E_{n-k}, \quad n \geq 1$$

$$\Rightarrow 2y' = 1 + y^2, \quad \text{etc.}$$

$\exists$ more sophisticated approaches.

E.g., let

$$\boldsymbol{f_k(n)} = \#\{w \in \mathfrak{S}_n \ : \ w(r) < w(s) \ \Leftrightarrow k|r\}$$

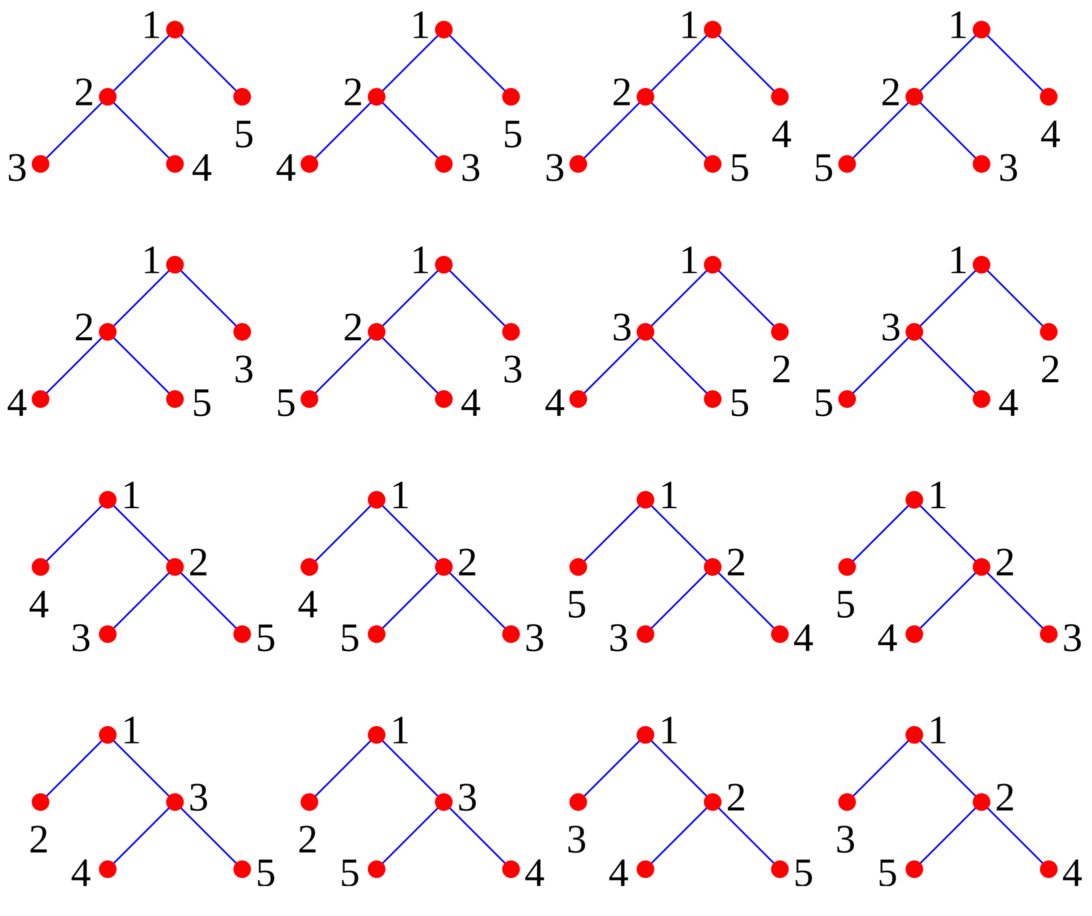
$$f_2(n) = E_n$$

Theorem.

$$\sum_{k \geq 0} f_k(kn) \frac{x^{kn}}{(kn)!} = \left(\sum_{n \geq 0} (-1)^n \frac{x^{kn}}{(kn)!} \right)^{-1} .$$

Some occurrences of Euler numbers:

1. E_{2n-1} is the number of complete increasing binary trees on the vertex set $[2n+1] = \{1, 2, \ldots, 2n+1\}$.

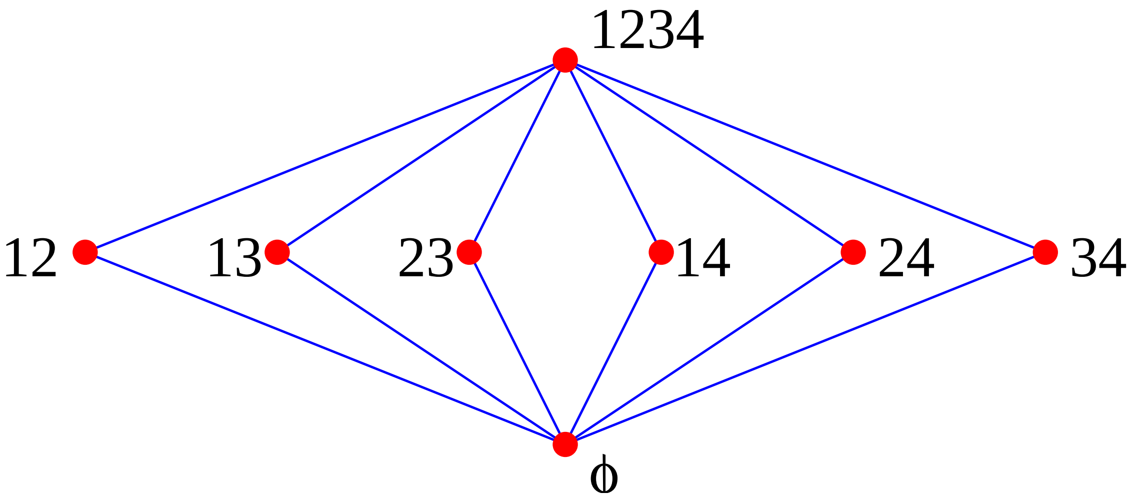


Slightly more complicated for E_{2n}

2. Let

$$P_n = \{S \subseteq [n] \,:\, \#S \text{ even}\},$$

ordered by inclusion. Adjoin $\hat{1}$ if n is odd.



Theorem. $\mu_{P_n}(\hat{0}, \hat{1}) = (-1)^{\lceil n/2 \rceil} E_n$

3. $\mathfrak{S}_n$ acts on Π_n, the lattice of partitions of $[n]$.

Theorem. *The number of orbits of maximal chains is E_{n-1}.*

12−3−4−5	123−4−5	1234−5
12−3−4−5	123−4−5	123−45
12−3−4−5	12−34−5	125−34
12−3−4−5	12−34−5	12−345
12−3−4−5	12−34−5	1234−5

Ulam's problem. Let $\boldsymbol{E(n)}$ be the expected length $\text{is}(w)$ of the longest **increasing** subsequence of w.

$$w = 5\mathbf{24}1\mathbf{7}36 \;\Rightarrow\; \text{is}(w) = 3$$

Long story ...

$$E(n) \sim 2\sqrt{n}$$

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Long story ...

$$E(n) \sim 2\sqrt{n}$$

$$\mathrm{Prob}_{w \in \mathfrak{S}_n}\left(\frac{\mathrm{is}(w) - 2\sqrt{n}}{n^{1/6}} \le t\right) = F(t),$$

the **Tracy-Widom distribution**.

What about **alternating** subsequences?

$\mathbf{as}(\boldsymbol{w})$ = length longest alt. subseq. of w

$$\boldsymbol{E'(n)} = \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \mathrm{as}(w) \sim \,?$$

$$w = \mathbf{5}6\mathbf{2}1\mathbf{8}\mathbf{3}\mathbf{4}7 \Rightarrow \mathrm{as}(w) = 5$$

$$
\begin{aligned}
\boldsymbol{a_k(n)} &= \#\{w \in \mathfrak{S}_n \,:\, \mathrm{as}(w) = k\} \\
b_k(n) &= a_1(n) + a_2(n) + \cdots + a_k(n) \\
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MAIN LEMMA. $\forall\, w \in \mathfrak{S}_n \ \exists$ *alternating subsequence of maximal length that contains* n.

$$
\Rightarrow a_k(n) = \sum_{j=1}^{n} \binom{n-1}{j-1}
$$

$$
\sum_{2r+s=k-1} \left(a_{2r}(j-1) + a_{2r+1}(j-1)\right) a_s(n-j)
$$

Let $B(x,t) = \sum_{k,n\geq 0} b_k(n) t^k \frac{x^n}{n!}$. Then

$$B(x,t) = \frac{1+\rho+2te^{\rho x}+(1-\rho)e^{2\rho x}}{1+\rho-t^2+(1-\rho-t^2)e^{2\rho x}},$$

where $\boldsymbol{\rho} = \sqrt{1-t^2}$.

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$$\begin{aligned} \Rightarrow b_1(n) &= 1 \\ b_2(n) &= n \\ b_3(n) &= \tfrac{1}{4}(3^n - 2n + 3) \\ b_4(n) &= \tfrac{1}{8}(4^n - (2n-4)2^n) \\ &\vdots \end{aligned}$$

$$E'(n) = \mathrm{Exp}_{w\in\mathfrak{S}_n}\mathrm{as}(w) = \frac{4n+1}{6},\ n \geq 2$$

$$\mathrm{Var}_{w\in\mathfrak{S}_n}\mathrm{as}(w) = \frac{8}{45}n - \frac{13}{180},\ n \geq 4,\ \text{etc.}$$

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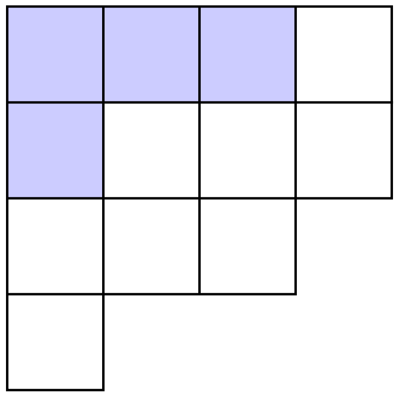
THEOREM (Pemantle, Widom, (Wilf)).

$$\lim_{n\to\infty} \mathrm{Prob}_{w\in\mathfrak{S}_n}\left(\frac{\mathrm{as}(w) - 2n/3}{\sqrt{n}} \leq t\right)$$

$$= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{t\sqrt{45}/4} e^{-s^2}\,ds$$

(Gaussian)

ENUMERATION. Basic tool: symmetric functions.



skew shape:

$$\boldsymbol{\lambda/\mu} = 4431/31, \quad |\lambda/\mu| = \boldsymbol{n} = 8$$

skew Schur function:

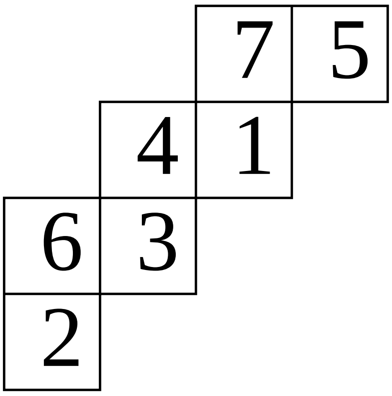
$$\boldsymbol{s_{\lambda/\mu}} = s_{\lambda/\mu}(x_1, x_2, \dots)$$

$\mathfrak{S}_n$ skew character:

$$\chi^{\lambda/\mu} : \mathfrak{S}_n \to \mathbb{Z}$$

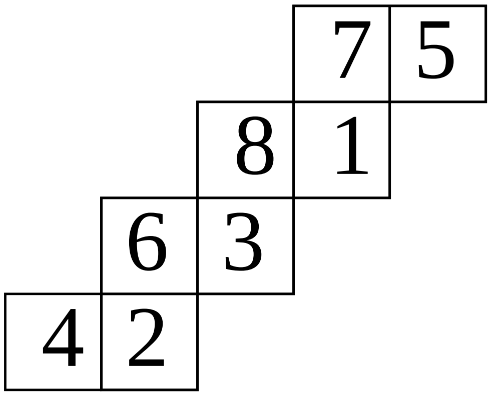
$$\begin{aligned} s_{\lambda/\mu} &= \operatorname{ch} \chi^{\lambda/\mu} \\ &= \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \chi^{\lambda/\mu}(w) p_{\rho(w)} \end{aligned}$$

$$\dim \chi^{\lambda/\mu} = \#\text{SYT of shape } \lambda/\mu$$



τ_7

5714362



τ_8

57183624

$$\dim \chi^{\tau_n} = E_n$$

H. O. Foulkes: $\chi^{\tau_n}(w) = 0, \pm E_r$

For a symmetric function f write

$$\boldsymbol{f[p_1, p_2, \ldots]}$$

for f as a polynomial in $p_1, p_2, \ldots$.

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Umbral notation: evaluate as polynomial in E, and replace E^k by E_k at end.

$$\begin{aligned}(1+E^2)^3 &= 1 + 3E^2 + 3E^4 + E^6 \\ &= 1 + 3E_2 + 3E_4 + E_6 \\ &= 1 + 3\cdot 1 + 3 \cdot 5 + 61 \\ &= 80\end{aligned}$$

$$
\begin{aligned}
(1+t)^E &= 1 + Et + \binom{E}{2} t^2 + \binom{E}{3} t^3 + \cdots \\
&= 1 + Et + \frac{1}{2}(E^2 - E)t^2 \\
&\quad + \frac{1}{6}(E^3 - 3E^2 + 2E)t^3 + \cdots \\
&= 1 + E_1 t + \frac{1}{2}(E_2 - E_1)t^2 \\
&\quad + \frac{1}{6}(E_3 - 3E_2 + 2E_1)t^3 + \cdots \\
&= 1 + t + \frac{1}{6} t^3 + \cdots .
\end{aligned}
$$

$$
\begin{aligned}
e_3[E, 0, -E] &= \frac{1}{6}(p_1^3 - 3p_1p_2 + 2p_3)[E, 0, -E] \\
&= \frac{1}{6}(E^3 - 2E) \\
&= \frac{1}{6}(E_3 - 2E_1) \\
&= 0
\end{aligned}
$$

Corollary (restatement of Foulkes).

n odd $\Rightarrow$

$$\langle f, s_{\tau_n} \rangle = f[E, 0, -E, 0, E, 0, -E, 0, \ldots]$$

n even $\Rightarrow$

$$\langle f, s_{\tau_n} \rangle = f[E, 1, -E, -1, E, 1, -E, -1, \ldots].$$

(homomorphisms)

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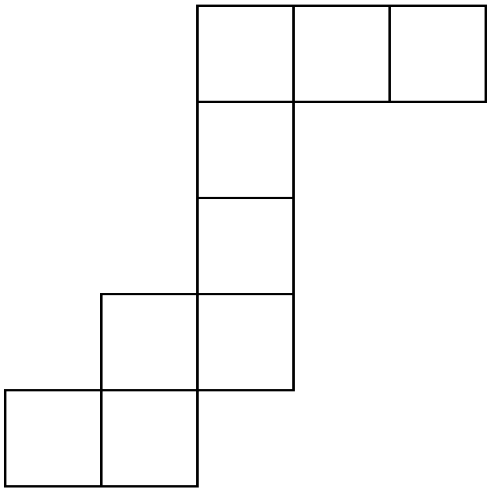
(homomorphisms)

Example.

$$s_{31} = \frac{1}{8}(p_1^4 + 2p_1^2 p_2 - p_2^2 - 2p_4)$$

$$\langle s_{31}, \tau_4 \rangle = \frac{1}{8}(E_4 + 2E_2 - 1 + 2) = 1$$

$$\alpha = (3, 1, 1, 2, 2) \in \mathrm{Comp}(9)$$



B_{31122}

$$\tau_7 = B_{2221}$$

Gessel-Reutenauer: $\exists$ (known) symmetric function L_λ such that

$$\langle L_\lambda, s_{B_\alpha}\rangle = \#\{w \in \mathfrak{S}_n :$$

$$\text{cycle type}\,\lambda,\ D(w) = \{\alpha_1, \alpha_1+\alpha_2+\cdots\}\},$$

where $D(w)$ is the **descent set** of w.

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where $D(w)$ is the **descent set** of w.

Corollary. $\langle L_\lambda, s_{\tau_n}\rangle$

$$= \#\{w \in \mathfrak{S}_n \ : \ \text{cycle type}\,\lambda,\ \text{alternating}\}$$

$$L_n = \frac{1}{n} \sum_{d|n} \mu(d) p_d^{n/d}$$

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Corollary. *Let* ***b(n)*** *be the number of alternating n-cycles in* $\mathfrak{S}_n$. *Then if n is odd,*

$$\begin{aligned} b(n) &= L_n[E, 0, -E, 0, \dots] \\ &= \frac{1}{n}\sum_{d|n} \mu(d) \left[(-1)^{(d-1)/2} E\right]^{n/d} \\ &= \frac{1}{n}\sum_{d|n} \mu(d)(-1)^{(d-1)/2} E_{n/d}. \end{aligned}$$

Combinatorial proof, especially if n is prime?

Let $\boldsymbol{f(n)}$ be the number of alternating fixed-point free involutions in $\mathfrak{S}_{2n}$.

$$n = 3 : \quad 214365, 645231$$

$$\sum_{n\geq 0} L_{\langle 2^n\rangle} t^n = \exp \sum_{k\geq 1} \frac{1}{2k}(p_k^2 - p_{2k})t^k$$

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$\Rightarrow$ **Theorem.**

$$\begin{aligned} \boldsymbol{F(x)} &= \sum_{n\geq 0} f(n)x^n \\ &= \left(\frac{1+x}{1-x}\right)^{(E^2+1)/4} \end{aligned}$$

Theorem (Ramanujan, Berndt, implicitly) *As* $x \to 0+$,

$$2 \sum_{n \geq 0} \left(\frac{1-x}{1+x} \right)^{n(n+1)} \sim \sum_{k \geq 0} f(k) x^k.$$

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Corollary (via Ramanujan, Andrews).

$$F(x) = 2 \sum_{n \geq 0} q^n \frac{\prod_{j=1}^{n} (1 - q^{2j-1})}{\prod_{j=1}^{2n+1} (1 + q^j)},$$

where $\boldsymbol{q} = \left(\frac{1-x}{1+x} \right)^{2/3}$, *a* ***formal*** *identity.*

FIXED POINTS

$$d_k(n) = \#\{\text{alt. } w \in \mathfrak{S}_n \ : \ k \text{ fixed points}\}$$

$$\mathcal{O}_t \sum a_n t^n = \sum a_{2n+1} t^{2n+1}$$

Main result.

$$\sum_{k,n\geq 0} d_k(2n+1) q^k t^{2n+1}$$

$$\mathcal{O}_t \frac{\exp(E(\tan^{-1} qt - \tan^{-1} t))}{1 - Et}$$

(similar for $d_k(2n)$)

Sample applications:

Theorem. $d_0(n) = d_1(n)$ *for* $n \geq 1$

Combinatorial proof?

Let $\boldsymbol{H(n)}$ be the expected number of fixed points of an alternating permutation $w \in \mathfrak{S}_n$.

Let $\boldsymbol{H^*(n)}$ be the expected number of fixed points of a **reverse** alternating permutation $w \in \mathfrak{S}_n$.

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Theorem.

$$H(2n+1) = \frac{1}{E_{2n+1}}(E_{2n+1} - E_{2n-3} + E_{2n-5} - \cdots \pm E_1)$$

$$H(2n) = \frac{1}{E_{2n}}(E_{2n} - 2E_{2n-2} + 2E_{2n-4} + \cdots \pm 2E_2 \mp E_0).$$

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$$H(2n) = \frac{1}{E_{2n}}(E_{2n} - 2E_{2n-2} + 2E_{2n-4} + \cdots \pm 2E_2 \mp E_0)$$

$$H^*(2n+1) = H(2n+1)$$

$$H^*(2n) = \frac{1}{E_{2n}}(E_{2n} - (-1)^n).$$

Combinatorial proof?

For n odd,

$$d_0(n) \sim \frac{1}{e} E_n.$$

Compare $D_n \sim \frac{1}{e} n!$.

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In fact,

$$\begin{aligned} d_0(n) &\sim \frac{1}{e}\left(E_n + a_1 E_{n-2} + a_2 E_{n-4} + \cdots\right) \\ &= \frac{1}{e}\left(E_n + \frac{1}{3} E_{n-2} - \frac{13}{90} E_{n-4} + \cdots\right), \end{aligned}$$

where

$$\sum_{k \geq 0} a_k x^{2k} = \exp\left(1 - \frac{1}{x}\tan^{-1} x\right).$$

Similar result for n even.