Contemporary Mathematics
Volume **138**, 1992

# Some Combinatorial Aspects of the Spectra of Normally Distributed Random Matrices

PHILIP J. HANLON, RICHARD P. STANLEY
AND JOHN R. STEMBRIDGE

## Introduction

Let $U$ be a real $n \times n$ matrix whose entries $u_{ij}$ are random variables, and let $A$ and $B$ be fixed $n \times n$ real symmetric matrices. Statisticians (e.g., see **[OU]**) have been interested in the distribution of the eigenvalues $\theta_1, \ldots, \theta_n$ of the matrix $AUBU^t$, where $^t$ denotes transpose. Of particular interest are the quantities $\operatorname{tr}((AUBU^t)^k) = \sum \theta_i^k$ for $k = 1, 2, \ldots$, since these determine the eigenvalues.

More generally, one may consider the distribution of arbitrary symmetric functions of the eigenvalues of $AUBU^t$. Thus let us regard any symmetric polynomial $f$ (say with real coefficients) in the variables $x_1, \ldots, x_n$ as a function on $n \times n$ matrices by defining $f(U)$ to be the value of $f$ at the eigenvalues of the matrix $U$. In these terms, we have

$$\operatorname{tr}((AUBU^t)^k) = p_k(AUBU^t),$$

where $p_k$ denotes the $k$th power-sum symmetric function.

We will follow the symmetric function notation and terminology of **[M1]**. In particular, if $\lambda = (\lambda_1, \lambda_2, \ldots)$ is any partition of $n$ (denoted $\lambda \vdash n$ or $|\lambda| = n$), $p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots$ will denote the product of power sums indexed by the parts of $\lambda$. The $p_\lambda$'s are a linear basis for the symmetric polynomials $f$.

Let $\mathcal{E}_U$ denote expectation with respect to a random $n \times n$ matrix $U$ of independent standard normal variables. Given that $A$ and $B$ are fixed real symmetric matrices, it is easy to show that $\mathcal{E}_U(f(AUBU^t))$ is a symmetric polynomial function of the eigenvalues of $A$ and $B$ separately, and thus is a linear combination of $p_\mu(A)p_\nu(B)$. The main result of this paper (Theorem 3.5) is an explicit expansion for $\mathcal{E}_U(p_\lambda(AUBU^t))$ in terms of $p_\mu(A)p_\nu(B)$, together with a combinatorial

---

1991 *Mathematics Subject Classification*. 05E05, 15A52.
Research supported by grants from the NSF.
This paper is in final form and no version of it will be submitted for publication elsewhere.

interpretation of the coefficients involving perfect matchings on $2k$ points, where $|\lambda| = k$. We will also give analogous results for complex and quaternionic matrices (Theorems 2.3 and 4.1). Tables of these polynomials are provided in the Appendix.

Let $I_n$ and $O_n$ denote the $n \times n$ identity and zero matrices, respectively. Of particular interest is the special case in which $A = I_m \oplus O_{n-m}$ and $B = I_n$. Under these circumstances, one has

$$AUBU^t = \begin{bmatrix} \tilde{U}\tilde{U}^t & * \\ 0 & O_{n-m} \end{bmatrix},$$

where $\tilde{U}$ consists of the first $m$ rows of $U$. Thus the eigenvalues of $AUBU^t$ are just the squares of the singular values of the random $m \times n$ matrix $\tilde{U}$, together with $n - m$ irrelevant zeroes. In this special case, the quantity $\mathcal{E}_U(p_\lambda(AUBU^t))$ turns out to be a polynomial $Q_\lambda(m, n)$ with nonnegative integral coefficients which sum to $1 \cdot 3 \cdot 5 \cdots (2k-1)$ (where $|\lambda| = k$), and our combinatorial interpretation of $\mathcal{E}_U(p_\lambda(AUBU^t))$ simplifies considerably (see Corollary 3.6). Again there are similar results for complex and quaternionic matrices (Corollaries 2.4 and 4.2). We will also derive a more explicit formula (Theorem 5.4) for $Q_\lambda(m, n)$ in the case $\lambda = (k)$ that is more efficient for computational purposes.

The paper is organized as follows. In the first section, we review the theory of zonal spherical functions of Hermitian matrices over finite-dimensional real division algebras. The spherical functions are essentially Jack symmetric functions for special choices of the parameter $\alpha$. The crucial point of this section is Theorem 1.1, which reduces the problem of computing the expectation of $p_\lambda(AUBU^t)$ to the theory of (Jack) symmetric functions. In the next three sections, we analyze the individual details of the complex, real and quaternionic cases. We do the complex case first, primarily because the spherical functions for complex Hermitian matrices (i.e., Schur functions) are easier to work with than the spherical functions for real symmetric matrices (i.e., zonal polynomials). Although the quaternionic case must be formulated more carefully than the others, the corresponding results can be deduced easily from the real case, since the spherical functions for quaternionic Hermitian matrices are closely related to the real symmetric case. In the final section, we present an (incomplete) combinatorial approach to computing the expectation of $p_\lambda(AUBU^t)$. If successful, this would yield a completely elementary method of deriving the main results of this paper without the theory of spherical functions. We will also point out how some recent work of Goulden and Jackson [GJ] provides some partial success for this approach in the complex case.

## 1. Zonal Spherical Functions

Let $F$ be a finite dimensional real division algebra; i.e., $F = \mathbf{R}$, $\mathbf{C}$, or the quaternions $\mathbf{H}$, and let $x \mapsto \bar{x}$ denote the usual conjugation on $F$. (In particular, $\bar{x} = x$ when $F = \mathbf{R}$.) Let $\Re(x) = (x+\bar{x})/2$ denote the real part of $x$. An $F$-valued random variable $u$ will be said to be *standard normal* if $\mathcal{E}(u) = 0$, $\mathcal{E}(u\bar{u}) = 1$, and the distribution is Gaussian.

Let $\mathcal{S}_n(F)$ denote the real vector space of $n \times n$ "Hermitian" matrices $A$ satisfying $A = A^*$, where $A^*$ denotes the conjugated transpose. The general linear group $GL_n(F)$ acts on $\mathcal{S}_n(F)$ via $A \mapsto XAX^*$. Since any Hermitian $A$ can be diagonalized by a member of the "unitary" group $K_n = \{X \in GL_n(F) : XX^* = I_n\}$, it follows that the $K_n$-invariant polynomials on $\mathcal{S}_n(F)$ (i.e., the polynomials $f$ satisfying $f(XAX^*) = f(A)$ for all $X \in K_n$, $A \in \mathcal{S}_n(F)$) can be identified with the symmetric polynomial functions of $n$ (real) variables. For example, the polynomial function corresponding to the $k$th power sum $p_k$ is

$$p_k(A) = \mathrm{tr}(A^k) \quad (A \in \mathcal{S}_n(F)), \tag{1.1}$$

and these particular functions generate the entire algebra of $K_n$-invariants.

Since $(GL_n(F), K_n)$ is a Gelfand pair, it follows that each irreducible $GL_n(F)$-invariant subspace of polynomial functions on $\mathcal{S}_n(F)$ contains a unique (up to scalar multiples) $K_n$-invariant polynomial, known as a *(zonal) spherical function* for the pair $(GL_n(F), K_n)$. Furthermore, these spherical functions form a basis for the space of all $K_n$-invariant polynomials. (For further details, e.g., see [**GR**]).

Macdonald [**M2**] has observed that in each of the cases $F = \mathbf{R}$, $\mathbf{C}$ and $\mathbf{H}$, the spherical functions for the pair $(GL_n(F), K_n)$ may be uniformly described as Jack symmetric functions $J_\lambda(x; \alpha)$ for special instances of the parameter $\alpha$ (depending on $F$). Although we do not intend to define these functions here,[1] we should explain that for any positive real number $\alpha$, the Jack symmetric functions $J_\lambda(x; \alpha)$ form a basis for the homogeneous symmetric polynomials of degree $k$ in the variables $x = (x_1, x_2, \dots)$, where $\lambda$ ranges over the partitions of $k$. Furthermore, when $J_\lambda$ is restricted to $n$ variables (i.e., $x_m = 0$ for all $m > n$), one has $J_\lambda = 0$ if and only if $\lambda$ has more than $n$ parts, and the remaining nonzero Jack symmetric functions form a basis for the symmetric polynomial functions of $(x_1, \dots, x_n)$. In these terms, the spherical functions for the pair $(GL_n(F), K_n)$ can be obtained by setting $\alpha = 2$, $1$, or $1/2$, according to whether $F = \mathbf{R}$, $\mathbf{C}$, or $\mathbf{H}$.

As a second remark, we should note that although the Jack symmetric functions are well-defined only up to scalar multiplication, we will follow the usual convention and insist that $J_\lambda$ be normalized so that the coefficient of $x_1 \cdots x_k$ in $J_\lambda(x; \alpha)$ is $k!$, assuming $|\lambda| = k$.

In what follows, we need to extend any $K_n$-invariant polynomial $f$ on $\mathcal{S}_n(F)$ (or equivalently, any symmetric function) to the full matrix algebra $M_n(F)$, so that expressions such as $f(AUBU^*)$ are well-defined. For this it is enough to decide how to extend the power sums, since these generate all symmetric functions. In the real and complex cases, let us use the obvious extension of (1.1)

[1] For this and other details about Jack symmetric functions, see [**St**].

to all of $M_n(F)$; this is consistent with the conventions of the Introduction, where we defined $f(U)$ to be the value of $f$ at the diagonal matrix of eigenvalues of $U$. In the quaternionic case, more delicacy is required. Indeed, if we attempted to use (1.1) for all of $M_n(\mathbf{H})$, then the value of an arbitrary symmetric function $f$ on $M_n(\mathbf{H})$ would not be well-defined (e.g., $p_r(U)$ and $p_s(U)$ need not commute). To avoid this difficulty, we define

$$p_k(U) = \Re(\mathrm{tr}(U^k)) \tag{1.2}$$

for all $u \in M_n(\mathbf{H})$. This convention is not entirely *ad hoc*; it derives from the fact that $\Re(\mathrm{tr}(U))$ is one-half the trace of $U$ in the standard embedding $M_n(\mathbf{H}) \hookrightarrow M_{2n}(\mathbf{C})$. In particular, since $2\Re(\mathrm{tr}(\,\cdot\,))$ is therefore a trace over a *complex* matrix algebra, it follows that $f(UV) = f(VU)$ for all symmetric functions $f$ and all $U, V \in M_n(F)$, even in the quaternionic case.

Now fix two matrices $A, B \in \mathcal{S}_n(F)$, and let $U = [u_{ij}]$ be a random $n \times n$ matrix of independent $F$-valued standard normal variables. The following key property of zonal polynomials follows from the theory of spherical functions and goes back to Gelfand [**G**] and Godement [**Go**, Thm. 10] (see also James [**J**, (29)]). The earliest explicit formulation of it we have found in the form below is in [**T**, p. 31, Thm. 3] (the real case) and [**T**, p. 88, Thm. 4] (the complex case). We know of no previous formulation of this particular version of the quaternionic case, although it follows from the general theory in the same way as the real and complex cases.

1.1 THEOREM. *Let $\alpha = 2$, $1$, or $1/2$, according to whether $F = \mathbf{R}$, $\mathbf{C}$ or $\mathbf{H}$. If $\lambda$ is any partition, then*

$$\mathcal{E}_U(J_\lambda(AUBU^*;\alpha)) = J_\lambda(A;\alpha)J_\lambda(B;\alpha).$$

SKETCH OF PROOF. Let $U = RX$ denote the "polar coordinate" decomposition of $U$ in which $X \in K_n$ and $R$ is the unique positive semidefinite square root of $UU^*$. Since the distribution of $U$ is invariant under left and right multiplication by $K_n$, it follows that for fixed $R$, $X$ is uniformly distributed on $K_n$ according to Haar measure. We therefore have

$$\mathcal{E}_U(f(AUBU^*)) = \mathcal{E}_R(\mathcal{E}_X(f(ARXBX^*R))) = \mathcal{E}_R(\mathcal{E}_X(f(RARXBX^*))), \tag{1.3}$$

for any $K_n$-invariant function $f$ on $\mathcal{S}_n(F)$.

However, by the general theory of spherical functions (e.g., [**GR**, Prop. 5.5]), one has

$$\mathcal{E}_X(J_\lambda(AXBX^*;\alpha)) = \frac{J_\lambda(A;\alpha)J_\lambda(B;\alpha)}{J_\lambda(I_n;\alpha)},$$

so (1.3) implies

$$\mathcal{E}_U(J_\lambda(AUBU^*;\alpha)) = \frac{J_\lambda(B;\alpha)}{J_\lambda(I_n;\alpha)}\mathcal{E}_R(J_\lambda(RAR;\alpha)).$$

Now since the expectation of $f(AUBU^*)$ is invariant under interchanging $A$ and $B$ (recall that $f(UV) = f(VU)$), we must therefore have

$$\mathcal{E}_U(J_\lambda(AUBU^*;\alpha)) = c_\lambda J_\lambda(A;\alpha)J_\lambda(B;\alpha) \tag{1.4}$$

for some scalar $c_\lambda$ (namely, $\mathcal{E}_U(J_\lambda(UU^*;\alpha))/J_\lambda(I_n;\alpha)^2$).

To determine this scalar, we first note that by bordering $A$ and $B$ with zeroes, one can show that $c_\lambda$ does not depend on $n$ (cf. [**T**, p. 32]). We may therefore assume $n \geq k$, where $|\lambda| = k$. Now let $A = \operatorname{diag}(a_1, \ldots, a_n)$ and $B = \operatorname{diag}(b_1, \ldots, b_n)$, and consider the coefficient of $a_1 b_1 \cdots a_k b_k$ in (1.4). On the right side, one obtains $(k!)^2 c_\lambda$, by the normalization convention for $J_\lambda$. For the left side, note that among the power sums $p_\mu(x_1, \ldots, x_n)$ with $\mu \vdash k$, the only case for which the coefficient of $x_1 \cdots x_k$ is nonzero is the case $\mu = (1^k)$; in that case, the coefficient is $k!$. Thus when $J_\lambda$ is expanded in terms of power sums, the coefficient of $p_1^k$ is 1.

Finally, observe that the coefficient of $a_1 b_1 \cdots a_k b_k$ in

$$p_1(AUBU^*)^k = \Big(\sum_{i,j=1}^{k} a_i b_j u_{ij} \bar{u}_{ij}\Big)^k$$

will be a sum of $(k!)^2$ terms, each of which is of the form $(u_1\bar{u}_1)\cdots(u_k\bar{u}_k)$, where $u_1, \ldots, u_k$ are independent standard normal variables. Since $\mathcal{E}(u_i\bar{u}_i) = 1$, it will thus follow that the coefficient of $a_1 b_1 \cdots a_k b_k$ on the left side of (1.4) is $(k!)^2$ (and hence, $c_\lambda = 1$), provided that the coefficient of $a_1 b_1 \cdots a_k b_k$ in $\mathcal{E}_U(p_\mu(AUBU^*))$ is zero whenever $\mu \neq (1^k)$. This can be established by elementary methods, as we shall see in §6. $\square$

## 2. The Complex Case

We first briefly review a few facts about group characters. Let $G$ be a finite group, and let $C_1, \ldots, C_t$ denote the conjugacy classes of $G$. By abuse of notation, we identify each $C_i$ with its corresponding sum in the complex group algebra $\mathbf{C}G$. These class sums form a basis for the center of $\mathbf{C}G$. If $\chi^{(1)}, \ldots, \chi^{(t)}$ are the irreducible (complex) characters of $G$ and $\deg(\chi^{(i)}) = d_i$, then the elements

$$E_i = \frac{d_i}{|G|}\sum_{j=1}^{t} \bar{\chi}_j^{(i)} C_j \qquad (1 \leq i \leq t)$$

form a complete set of orthogonal idempotents for the center of $\mathbf{C}G$, where we use $\chi_j^{(i)}$ to denote the common value of $\chi^{(i)}$ at any $w \in C_j$. Inverting this yields

$$C_j = |C_j| \sum_{i=1}^{t} \frac{\chi_j^{(i)}}{d_i} E_i,$$

by the orthogonality properties of characters [**B**, §236]. Since the $E_j$'s are orthogonal idempotents (i.e., $E_iE_j = \delta_{i,j}E_i$), it follows that

$$
\begin{aligned}
(2.1)\qquad C_iC_j &= |C_i|\cdot|C_j|\sum_{r=1}^{t}\frac{\chi_i^{(r)}\chi_j^{(r)}}{d_r^2}E_r \\
&= \frac{|C_i|\cdot|C_j|}{|G|}\sum_{r=1}^{t}\frac{\chi_i^{(r)}\chi_j^{(r)}}{d_r}\sum_{k=1}^{t}\bar{\chi}_k^{(r)}C_k \\
&= \frac{|C_i|\cdot|C_j|}{|G|}\sum_{k=1}^{t}C_k\sum_{r=1}^{t}\frac{1}{d_r}\chi_i^{(r)}\chi_j^{(r)}\bar{\chi}_k^{(r)}.
\end{aligned}
$$

If we specialize to the symmetric group $S_k$, then the characters and conjugacy classes are indexed by partitions $\lambda$ of $k$; say $\chi^\lambda$ and $C_\lambda$, respectively. We will write $\chi^\lambda(\mu)$ for the value of $\chi^\lambda$ at any $w \in C_\mu$ (i.e., any $w \in S_k$ of cycle-type $\mu$).

Let $\tau(w)$ denote the cycle-type of a permutation $w \in S_k$, and for each partition $\lambda$ of $k$, choose a fixed element $w_\lambda$ of $C_\lambda$, so that $\tau(w_\lambda) = \lambda$. It is easy to see that

$$C_\mu C_\nu = \sum_\lambda a_{\mu,\nu}^\lambda C_\lambda,$$

where $a_{\mu,\nu}^\lambda$ is the number of pairs $u, v \in S_k$ such that $\tau(u) = \mu$, $\tau(v) = \nu$ and $uv = w_\lambda$. The quantity $|C_\lambda|\,a_{\mu,\nu}^\lambda$ has a more symmetric interpretation as the number of triples $u, v, w \in S_k$ such that $uvw = \mathrm{id}$, $\tau(u) = \mu$, $\tau(v) = \nu$, and $\tau(w) = \lambda$.

Let $H_\lambda$ denote the product of the hook-lengths of $\lambda$ [**M1**, p.9], so that the degree of $\chi^\lambda$ is given by $f^\lambda = k!/H_\lambda$, and let $z_\lambda = k!/|C_\lambda|$ denote the size of the centralizer of $w_\lambda$.

2.1 Lemma. *We have* $a_{\mu,\nu}^\lambda = z_\mu^{-1}z_\nu^{-1}\sum_\beta H_\beta\,\chi^\beta(\mu)\chi^\beta(\nu)\chi^\beta(\lambda)$.

Proof. The characters of $S_k$ are all real-valued, so (2.1) implies

$$(2.2)\qquad C_\mu C_\nu = z_\mu^{-1}z_\nu^{-1}\sum_\lambda C_\lambda\sum_\beta H_\beta\,\chi^\beta(\mu)\chi^\beta(\nu)\chi^\beta(\lambda).$$

Extract the coefficient of $C_\lambda$. □

For each partition $\lambda$ of $k$, let $s_\lambda$ denote the Schur function indexed by $\lambda$ [**M1**], and recall that the power-sum expansion of $s_\lambda$ due to Frobenius [**M1**, (7.10)] is given by

$$(2.3)\qquad s_\lambda = \sum_{\mu\vdash k} z_\mu^{-1}\chi^\lambda(\mu)p_\mu.$$

Using this, it is easy to reformulate Lemma 2.1 as a symmetric function identity. In fact, the following is a slightly more general result.

2.2 PROPOSITION. *If $x^{(1)}, \ldots, x^{(l)}$ are disjoint sets of variables, then*

$$\sum_{\lambda \vdash k} H_\lambda^{l-2} s_\lambda(x^{(1)}) \cdots s_\lambda(x^{(l)}) = \frac{1}{k!} \sum_{\substack{w_1 \cdots w_l = \mathrm{id} \\ \text{in } S_k}} p_{\tau(w_1)}(x^{(1)}) \cdots p_{\tau(w_l)}(x^{(l)}).$$

PROOF. Fix an $l$-tuple of partitions $\mu^{(1)}, \ldots, \mu^{(l)} \vdash k$, and set $\mu = \mu^{(l)}$. Now extract the coefficient of $p_{\mu^{(1)}}(x^{(1)}) \cdots p_{\mu^{(l)}}(x^{(l)})$ from the left side of the claimed identity, using (2.3). By the obvious extension of (2.2) to multiple products of class sums, one may identify the resulting sum as $1/z_\mu$ times the coefficient of $C_\mu$ in $C_{\mu^{(1)}} \cdots C_{\mu^{(l-1)}}$, or equivalently, $1/z_\mu$ times the number of solutions to the equation $w_1 \cdots w_{l-1} = w_\mu$ with $\tau(w_i) = \mu^{(i)}$. However, $|C_\mu| = k!/z_\mu$, so this is also $1/k!$ times the number of solutions to the equation $w_1 \cdots w_{l-1} = w_l$ with $\tau(w_i) = \mu^{(i)}$. □

Returning to the main theme of this paper, let $U$ be an $n \times n$ matrix whose entries are independent standard normal complex variables. For any pair of $n \times n$ Hermitian matrices $A$ and $B$, define

$$P_\lambda^{\mathbf{C}}(A, B) = \mathcal{E}_U(p_\lambda(AUBU^*)).$$

The main result of this section is as follows.

2.3 THEOREM. *If $\lambda \vdash k$, then*

$$P_\lambda^{\mathbf{C}}(A, B) = \sum_{\mu,\nu \vdash k} a_{\mu,\nu}^\lambda p_\mu(A) p_\nu(B) = \sum_{w \in S_k} p_{\tau(w)}(A) p_{\tau(ww_\lambda)}(B).$$

PROOF. The two expressions for $P_\lambda^{\mathbf{C}}(A, B)$ are clearly equivalent, so it suffices to prove the first one. For this, we begin with the well-known fact that the Schur functions are known to be scalar multiples of the Jack symmetric functions at $\alpha = 1$; more precisely, $J_\lambda(x; 1) = H_\lambda s_\lambda$ [**St**, Prop. 1.2]. By Theorem 1.1, it follows that

$$\mathcal{E}_U(s_\lambda(AUBU^*)) = H_\lambda s_\lambda(A) s_\lambda(B).$$

Now since $p_\lambda = \sum_\beta \chi^\beta(\lambda) s_\beta$ [**M1**, (7.8)], the linearity of expectation yields

$$\mathcal{E}_U(p_\lambda(AUBU^*)) = \sum_\beta H_\beta \chi^\beta(\lambda) s_\beta(A) s_\beta(B). \tag{2.4}$$

In view of (2.3), we therefore have

$$\begin{aligned} P_\lambda^{\mathbf{C}}(A, B) &= \sum_\beta H_\beta \chi^\beta(\lambda) \sum_{\mu,\nu} z_\mu^{-1} z_\nu^{-1} \chi^\beta(\mu) \chi^\beta(\nu) p_\mu(A) p_\nu(B) \\ &= \sum_{\mu,\nu} p_\mu(A) p_\nu(B) \, z_\mu^{-1} z_\nu^{-1} \sum_\beta H_\beta \chi^\beta(\lambda) \chi^\beta(\mu) \chi^\beta(\nu). \end{aligned}$$

Apply Lemma 2.1 to complete the proof. □

Now consider the consequences of setting $A = I_m \oplus O_{n-m}$ and $B = I_n$, as discussed in the Introduction. If $c(w)$ denotes the number of cycles of a permutation $w$ (i.e., the number of parts in $\tau(w)$), then clearly $p_{\tau(w)}(I_m \oplus O_{n-m}) = m^{c(w)}$ and $p_{\tau(w)}(I_n) = n^{c(w)}$. Therefore, if we define

$$Q^{\mathbf{C}}_{\lambda}(m,n) = \mathcal{E}_U(p_\lambda(AUBU^*)) = \mathcal{E}_V(p_\lambda(VV^*)),$$

where $V$ is a random $m \times n$ matrix with independent standard complex normal entries, we obtain the following.

2.4 COROLLARY. *If $\lambda \vdash k$, then* $Q^{\mathbf{C}}_{\lambda}(m,n) = \sum_{w \in S_k} m^{c(w)} n^{c(ww_\lambda)}$.

Thus $Q^{\mathbf{C}}_{\lambda}(m,n)$ is a polynomial function of $m$ and $n$ with nonnegative integer coefficients. Note that the coefficient of $m^i n^j$ can also be interpreted as the number of pairs $u, v \in S_k$ such that $c(u) = i$, $c(v) = j$, and $uv = w_\lambda$.

It is a well-known fact that $d(u,v) := k - c(u^{-1}v)$ defines a metric on $S_k$. Indeed, $d(u,v)$ is the distance from $u$ to $v$ in the Cayley graph of $S_k$ generated by transpositions. In particular, by the triangle inequality, we have $d(\mathrm{id}, w_\lambda) \leq d(\mathrm{id}, w^{-1}) + d(w^{-1}, w_\lambda)$, so

$$c(w) + c(ww_\lambda) \leq k + c(w_\lambda) = k + \ell(\lambda), \tag{2.5}$$

where $\ell(\lambda)$ denotes the number of parts of $\lambda$. Hence, the total degree of $Q^{\mathbf{C}}_{\lambda}(m,n)$ is at most $k + \ell(\lambda)$. In fact, we claim that the total degree is *equal* to $k + \ell(\lambda)$. To see this, consider any shortest path from the identity to $w_\lambda$ (necessarily of length $k - \ell(\lambda)$) in the Cayley graph of $S_k$. The permutation $u$ at distance $j$ along this path generates a solution to the equation with $uv = w_\lambda$ with $c(u) = k - j$ and $c(v) = j + \ell(\lambda)$, so it follows that the coefficient of $m^i n^{k+\ell(\lambda)-i}$ is nonzero whenever $\ell(\lambda) \leq i \leq k$ (and this condition is obviously necessary).

The case $\lambda = (k)$ is the one of most interest to statisticians. For example, by a result of Wigner (see [OU, p. 623]), it is known that

$$[m^i n^{k+1-i}] Q^{\mathbf{C}}_{k}(m,n) = \frac{1}{k}\binom{k}{i}\binom{k}{i-1},$$

where $[\,\cdot\,]$ denotes the operation of coefficient extraction. In fact, this result holds even for more general classes of random matrices than those we are considering here.

2.5 THEOREM. *We have*

$$Q^{\mathbf{C}}_{k}(m,n) = \frac{1}{k}\sum_{i=1}^{k} (-1)^{i-1} \frac{(m+k-i)_k (n+k-i)_k}{(k-i)!(i-1)!},$$

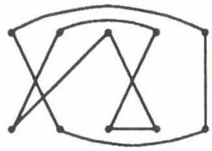
*where* $(a)_k := a(a-1)\cdots(a-k+1)$.

PROOF. Substituting $A = I_m \oplus O_{n-m}$ and $B = I_n$ into (2.4), we obtain

$$Q^{\mathbf{C}}_{\lambda}(m,n) = \sum_{\beta \vdash k} H_\beta\, \chi^\beta(\lambda) s_\beta(I_m) s_\beta(I_n).$$

$\delta_1$ $\delta_2$

FIGURE 1



$\delta_1 \cup \delta_2$

FIGURE 2

By the Murnaghan-Nakayama rule for $\chi^\beta(\lambda)$ [**M1**, Ex. I.7.5], it is known that

$$\chi^\beta(k) = \begin{cases} (-1)^{i-1}, & \text{if } \beta = (k-i+1, 1^{i-1}) \\ 0, & \text{otherwise.} \end{cases} \tag{2.6}$$

When $\beta = (k-i+1, 1^{i-1})$, it is easy to show that $H_\beta = k(k-i)!(i-1)!$ and $s_\beta(I_m) = (m+k-i)_k/H_\beta$, using [**M1**, Ex. I.3.4]. This yields the claimed identity. $\square$

## 3. The Real Case

Let $\mathcal{F}_k$ denote the set of 1-factors (or perfect matchings, or 1-regular graphs) on the vertices $\{1, 2, \dots, 2k\}$. Let $\varepsilon \in \mathcal{F}_k$ be the "identity 1-factor" in which $i$ is adjacent to $k+i$ for $1 \le i \le k$.

For each $w \in S_{2k}$, define $\delta(w)$ to be the 1-factor in which $i$ is adjacent to $j$ if and only if $|w(i) - w(j)| = k$. Note that $\delta(\text{id}) = \varepsilon$. The union $\delta_1 \cup \delta_2$ of two 1-factors is a 2-regular graph, and therefore a disjoint union of even-length cycles. Let $\Lambda(\delta_1, \delta_2)$ denote the partition of $k$ whose parts are half the cycle-lengths of $\delta_1 \cup \delta_2$. For example, in Figure 1 are two 1-factors $\delta_1$ and $\delta_2$ whose union (Figure 2) consists of a 6-cycle and a 4-cycle; thus, $\Lambda(\delta_1, \delta_2) = (3, 2)$.

Let $B_k$ denote the hyperoctahedral group, embedded in $S_{2k}$ as the centralizer of the involution $(1, k+1)(2, k+2)\cdots(k, 2k)$. Note that $B_k$ is the automorphism group of $\varepsilon$. The following well-known result shows that the right cosets of $B_k$ in $S_{2k}$ are indexed by 1-factors $\delta \in \mathcal{F}_k$, and that the double cosets $B_k \backslash S_{2k} / B_k$ are indexed by partitions of $k$.

3.1 LEMMA. *Let $w_1, w_2 \in S_{2k}$.*

(a) *$B_k w_1 = B_k w_2$ if and only if $\delta(w_1) = \delta(w_2)$.*

(b) *$B_k w_1 B_k = B_k w_2 B_k$ if and only if $\Lambda(\varepsilon, \delta(w_1)) = \Lambda(\varepsilon, \delta(w_2))$.*

PROOF. (a) Clearly, left multiplication by $B_k$ preserves $\delta(w)$. On the other hand, the number of $w \in S_{2k}$ that fix any given 1-factor is clearly $|B_k|$, so the 1-factors must completely separate the right cosets of $B_k$ in $S_{2k}$.

(b) (See also [**M2**], [**BG**], [**Ste**].) Right multiplication by $B_k$ amounts to simultaneous permutation of the vertices $\{1, \dots, k\}$ and $\{k+1, \dots, 2k\}$ of $\delta(w)$, along with interchanging the vertices $i$ and $k+i$. In particular, these operations preserve the isomorphism class of $\varepsilon \cup \delta(w)$ (or equivalently, preserve $\Lambda(\varepsilon, \delta(w))$). Conversely, the cycles of $\varepsilon \cup \delta(w)$ must alternate between edges of $\varepsilon$ and $\delta(w)$, so if $\varepsilon \cup \delta(w_1) \cong \varepsilon \cup \delta(w_2)$, then there is an isomorphism that preserves $\varepsilon$; i.e., there exists $x \in B_k$ such that $\delta(w_1 x) = \delta(w_2)$. By (a), it follows that $B_k w_1 B_k = B_k w_2 B_k$. $\square$

For each partition $\lambda$ of $k$, let $K_\lambda$ denote the double coset consisting of all $w \in S_{2k}$ with $\Lambda(\varepsilon, \delta(w)) = \lambda$, and choose a fixed representative $\hat{w}_\lambda$ from $K_\lambda$. Define $\delta_\lambda := \delta(\hat{w}_\lambda)$. By analogy with the conventions of §2, we will identify $K_\lambda$ with its sum in $\mathbf{C}S_{2k}$. These sums form a basis for a certain commutative subalgebra of $\mathbf{C}S_{2k}$; namely, the Hecke algebra of the Gelfand pair $(S_{2k}, B_k)$ (see [**BG**], [**Ste**]). In particular, it is easy to see that

$$K_\mu K_\nu = \sum_\lambda b^\lambda_{\mu,\nu} K_\lambda, \tag{3.1}$$

where $b^\lambda_{\mu,\nu}$ is the number of $u, v \in S_{2k}$ such that $u \in K_\mu$, $v \in K_\nu$ and $uv = \hat{w}_\lambda$.

3.2 LEMMA. *If $\lambda, \mu, \nu \vdash k$, then $|B_k|^{-1} b^\lambda_{\mu,\nu}$ is the number of 1-factors $\delta$ such that $\Lambda(\delta, \delta_\lambda) = \mu$ and $\Lambda(\varepsilon, \delta) = \nu$.*

PROOF. For each 1-factor $\delta$, choose a representative $v_\delta \in S_{2k}$ of the right coset of $B_k$ indexed by $\delta$. Since $uv_\delta = (ux^{-1})(xv_\delta)$ as $x$ varies through $B_k$, it follows that $|B_k|^{-1} b^\lambda_{\mu,\nu}$ is the number of 1-factors $\delta$ such that the equation $uv_\delta = \hat{w}_\lambda$ has a solution with $u \in K_\mu$ and $v_\delta \in K_\nu$. In other words, since $u$ must be $\hat{w}_\lambda v_\delta^{-1}$, $|B_k|^{-1} b^\lambda_{\mu,\nu}$ is the number of 1-factors $\delta$ such that $\Lambda(\varepsilon, \delta(\hat{w}_\lambda v_\delta^{-1})) = \mu$ and $\Lambda(\varepsilon, \delta) = \nu$. To complete the proof, note that $\Lambda(\varepsilon, \delta(\hat{w}_\lambda v_\delta^{-1})) = \Lambda(\delta, \delta(\hat{w}_\lambda))$, since for any $w \in S_{2k}$, the operation $\delta(u) \cup \delta(v) \mapsto \delta(uw) \cup \delta(vw)$ preserves the isomorphism class of $\delta(u) \cup \delta(v)$. $\square$

We now need to review a few facts about the Hecke algebra $\mathcal{H}_k$ spanned by the $K_\lambda$'s. For further details, see [**BG**] or [**Ste**]. Since left and right multiplication by the idempotent $e = |B_k|^{-1} \sum_{x \in B_k} x \in \mathbf{C}S_{2k}$ corresponds to averaging over the double cosets $B_k \backslash S_{2k} / B_k$, it follows that

$$\mathcal{H}_k = e\mathbf{C}S_{2k}e.$$

Furthermore, since $\mathcal{H}_k$ is commutative and semisimple, it has a basis of orthogonal idempotents. In fact, since the induction of the trivial character of $B_k$ to $S_{2k}$ is the multiplicity-free sum of the characters $\chi^{2\lambda}$ as $\lambda$ varies over the partitions of $k$,[2] it follows that the orthogonal idempotents of $\mathcal{H}_k$ are

$$E_\lambda := e e_{2\lambda} e = e_{2\lambda} e, \tag{3.2}$$

where $\lambda$ ranges over partitions of $k$, and $e_{2\lambda} = H_{2\lambda}^{-1} \sum_{\mu \vdash 2k} \chi^{2\lambda}(\mu) C_\mu$ denotes the primitive central idempotent of $\mathbf{C}S_{2k}$ indexed by $2\lambda$.

The following result is analogous to Lemma 2.1.

3.3 LEMMA. *We have*

$$b_{\mu,\nu}^\lambda = \frac{1}{|K_\lambda|} \sum_{\beta \vdash k} \frac{1}{H_{2\beta}} \varphi^\beta(\lambda) \varphi^\beta(\mu) \varphi^\beta(\nu),$$

*where* $\varphi^\beta(\mu) = \sum_{w \in K_\mu} \chi^{2\beta}(w)$.

PROOF. Extend the characters of $S_{2k}$ linearly so that $\chi^{2\beta}$ is the trace of an irreducible representation of $\mathbf{C}S_{2k}$. Since $E_\mu$ acts as a rank one idempotent in the representation of $\mathbf{C}S_{2k}$ indexed by $2\beta$, it follows that $\chi^{2\beta}(E_\mu) = \delta_{\beta,\mu}$. We may therefore deduce that

$$K_\mu = \sum_{\beta \vdash k} \varphi^\beta(\mu) E_\beta, \tag{3.3}$$

by applying $\chi^{2\beta}$ to both sides.

Since the $E_\beta$'s are orthogonal idempotents, it follows that

$$K_\mu K_\nu = \sum_{\beta \vdash k} \varphi^\beta(\mu) \varphi^\beta(\nu) E_\beta. \tag{3.4}$$

However, (3.2) implies

$$E_\lambda = \frac{1}{|B_k|^2} \sum_{x_1, x_2 \in B_k} \frac{1}{H_{2\lambda}} \sum_{w \in S_{2k}} \chi^{2\lambda}(w) x_1 w x_2,$$

so the coefficient of $\hat{w}_\lambda$ in $E_\beta$ is

$$\frac{1}{H_{2\beta}} \cdot \frac{1}{|B_k|^2} \sum_{x_1, x_2 \in B_k} \chi^{2\beta}(x_1 \hat{w}_\lambda x_2) = \frac{1}{H_{2\beta}} \cdot \frac{1}{|K_\lambda|} \varphi^\beta(\lambda).$$

In other words,

$$E_\beta = \frac{1}{H_{2\beta}} \sum_{\lambda \vdash k} \frac{1}{|K_\lambda|} \varphi^\beta(\lambda) K_\lambda. \tag{3.5}$$

Using this to extract the coefficient of $K_\lambda$ from (3.4) yields the claimed result. $\square$

For each partition $\lambda$, let $Z_\lambda(x) = J_\lambda(x; 2)$ denote the Jack symmetric function at $\alpha = 2$; these are the classical zonal polynomials first studied by James and Hua. The power-sum expansion of $Z_\lambda$, due to James [**J**, Thm. 4] (cf. also [**M2**], [**BG**], and [**Ste**]), can be described as follows.

[2] This can be deduced from [**M1**, Ex. I.5.5] and [**M1**, Remark 2, p.66], for example.

3.4 THEOREM. *For any $\lambda \vdash k$, we have*

$$Z_\lambda = \frac{1}{|B_k|}\sum_{\mu \vdash k} \varphi^\lambda(\mu) p_\mu = \frac{1}{|B_k|}\sum_{w \in S_{2k}} \chi^{2\lambda}(w) p_{\Lambda(\varepsilon,\delta(w))}.$$

Now let $U$ be a random $n \times n$ matrix whose entries are independent standard normal real variables, and for any $n \times n$ symmetric matrices $A$ and $B$, define

$$P_\lambda^{\mathbf{R}}(A,B) = \mathcal{E}_U(p_\lambda(AUBU^t)).$$

The main result of this section is as follows.

3.5 THEOREM. *If $\lambda \vdash k$, then*

$$P_\lambda^{\mathbf{R}}(A,B) = \frac{1}{|B_k|}\sum_{\mu,\nu \vdash k} b_{\mu,\nu}^\lambda p_\mu(A) p_\nu(B) = \sum_{\delta \in \mathcal{F}_k} p_{\Lambda(\varepsilon,\delta)}(A) p_{\Lambda(\delta,\delta_\lambda)}(B).$$

PROOF. Lemma 3.2 implies that the two expressions for $P_\lambda^{\mathbf{R}}(A,B)$ are equivalent, so it suffices to prove the first one.

If we define a linear transformation from $\mathcal{H}_k$ to symmetric functions of degree $k$ by setting $K_\nu \mapsto |K_\nu| p_\nu$, then (3.5) and Theorem 3.4 imply $E_\beta \mapsto |B_k| H_{2\beta}^{-1} Z_\beta$. Therefore by (3.3), we have

$$p_\lambda = \frac{1}{|K_\lambda|}\sum_{\beta \vdash k} \frac{|B_k|}{H_{2\beta}} \varphi^\beta(\lambda) Z_\beta. \tag{3.6}$$

Since expectation is linear, this identity and Theorem 1.1 imply

$$P_\lambda^{\mathbf{R}}(A,B) = \frac{1}{|K_\lambda|}\sum_{\beta \vdash k} \frac{|B_k|}{H_{2\beta}} \varphi^\beta(\lambda) Z_\beta(A) Z_\beta(B).$$

Using Theorem 3.4 to expand $Z_\beta(A)$ and $Z_\beta(B)$ therefore yields

$$P_\lambda^{\mathbf{R}}(A,B) = \frac{1}{|K_\lambda|}\sum_{\beta \vdash k} \frac{1}{H_{2\beta}} \cdot \frac{\varphi^\beta(\lambda)}{|B_k|} \sum_{\mu,\nu \vdash k} \varphi^\beta(\mu)\varphi^\beta(\nu) p_\mu(A) p_\nu(B).$$

Lemma 3.3 can now be used to extract the coefficient of $p_\mu(A)p_\nu(B)$. □

Now consider setting $A = I_m \oplus O_{n-m}$ and $B = I_n$, as discussed in the Introduction. By analogy with the notation of §2, let $c(\delta_1 \cup \delta_2)$ denote the number of cycles in the graph $\delta_1 \cup \delta_2$ (i.e., the number of parts in $\Lambda(\delta_1,\delta_2)$), and define

$$Q_\lambda^{\mathbf{R}}(m,n) = \mathcal{E}_U(p_\lambda(AUBU^t)) = \mathcal{E}_V(p_\lambda(VV^t)),$$

where $V$ is a random $m \times n$ matrix with independent standard real normal entries. The counterpart of Corollary 2.4 is as follows.

3.6 COROLLARY. *If* $\lambda \vdash k$, *then* $Q^{\mathbf{R}}_\lambda(m,n) = \sum_{\delta \in \mathcal{F}_k} m^{c(\varepsilon \cup \delta)} n^{c(\delta \cup \delta_\lambda)}$.

Thus $Q^{\mathbf{R}}_\lambda(m,n)$ is a polynomial function of $m$ and $n$ with nonnegative integer coefficients. We claim that the total degree of $Q^{\mathbf{R}}_\lambda(m,n)$ is $k + \ell(\lambda)$ (the same as $Q^{\mathbf{C}}_\lambda(m,n)$). To see this, define a graph structure on $\mathcal{F}_k$ by declaring $\delta_1$ adjacent to $\delta_2$ if $c(\delta_1 \cup \delta_2) = k-1$, or equivalently, $\Lambda(\delta_1, \delta_2) = (2, 1^{k-2})$. Note that $k - c(\delta_1 \cup \delta_2)$ is a metric on $\mathcal{F}_k$, since it is the distance from $\delta_1$ to $\delta_2$ in this graph. In particular, the triangle inequality implies

$$c(\varepsilon \cup \delta) + c(\delta \cup \delta_\lambda) \le k + c(\varepsilon \cup \delta_\lambda) = k + \ell(\lambda),$$

so the total degree of $Q^{\mathbf{R}}_\lambda(m,n)$ is at most $k + \ell(\lambda)$. Conversely, one can show that the coefficient of $m^i n^{k+\ell(\lambda)-i}$ is nonzero for $\ell(\lambda) \le i \le k$ by an argument similar to the one we used for $Q^{\mathbf{C}}_\lambda(m,n)$ in §2. In fact, one can also prove that the subgraph of shortest paths from $\varepsilon$ to $\delta_\lambda$ is isomorphic to the subgraph of shortest paths from the identity to $w_\lambda$ in the Cayley graph of $S_k$ (cf. §2), so it follows that the terms of highest total degree in $Q^{\mathbf{R}}_\lambda(m,n)$ and $Q^{\mathbf{C}}_\lambda(m,n)$ are identical.

## 4. The Quaternionic Case

Let $u$ be a standard normal quaternionic variable, as in §1. By a simple calculation, one can show that $\mathcal{E}((u\bar{u})^k) = (k+1)!/2^k$ for any nonnegative integer $k$. It follows that

$$\mathcal{E}((v\bar{v})^k) = (k+1)! \tag{4.1}$$

for the random variable $v = \sqrt{2}u$. In order to to avoid non-integral coefficients in what follows, we will therefore assume that $U = [u_{ij}]$ is an $n \times n$ matrix of random independent quaternionic variables, each distributed identically to $v$, rather than $u$.

Given this modified distribution for $U$, let us define

$$P^{\mathbf{H}}_\lambda(A,B) = \mathcal{E}_U(p_\lambda(AUBU^*)),$$

for any $n \times n$ Hermitian matrices $A$ and $B$ (over $\mathbf{H}$).

In order to use Theorem 1.1 to give an explicit formula for $P^{\mathbf{H}}_\lambda(A,B)$, we will need the power-sum expansion for the Jack symmetric functions at $\alpha = 1/2$. For this there is a dual relationship, due to Ian Macdonald [**St**, Thm. 3.3], between the Jack symmetric functions for the parameters $\alpha$ and $1/\alpha$. To describe this relationship, let $\omega_\alpha$ denote the unique automorphism of the ring of symmetric functions satisfying $\omega_\alpha(p_k) = \alpha p_k$ for $k \ge 1$, and let $\lambda'$ denote the conjugate of $\lambda$ [**M1**, (1.3)]. In these terms, we have

$$J_\lambda(x; 1/\alpha) = (-1/\alpha)^k \omega_{-\alpha} J_{\lambda'}(x;\alpha), \tag{4.2}$$

if $\lambda$ is a partition of $k$. This dual relationship will allow us to express $P^{\mathbf{H}}_\lambda(A,B)$ in terms of its real counterpart $P^{\mathbf{R}}_\lambda(A,B)$.

4.1 THEOREM. *If* $\lambda \vdash k$, *then* $P_\lambda^{\mathbf{H}}(A,B) = |B_k|^{-1}\sum_{\mu,\nu\vdash k} c_{\mu,\nu}^\lambda p_\mu(A)p_\nu(B)$, *where*

$$c_{\mu,\nu}^\lambda = (-1)^k(-2)^{\ell(\mu)+\ell(\nu)-\ell(\lambda)} b_{\mu,\nu}^\lambda,$$

*and* $b_{\mu,\nu}^\lambda$ *is defined as in* (3.1).

PROOF. Taking into account the modified distribution for $U$, Theorem 1.1 implies

$$\mathcal{E}_U(J_\lambda(AUBU^*;1/2)) = 2^k J_\lambda(A;1/2)J_\lambda(B;1/2),$$

for any partition $\lambda$ of $k$. However, if we use (4.2) to apply $\omega_{-2}$ to (3.6), we obtain

$$p_\lambda(x) = \frac{(-2)^{k-\ell(\lambda)}}{|K_\lambda|}\sum_{\beta\vdash k}\frac{|B_k|}{H_{2\beta}}\varphi^\beta(\lambda)J_{\beta'}(x;1/2),$$

and therefore

$$P_\lambda^{\mathbf{H}}(A,B) = (-1)^k\frac{(-2)^{2k-\ell(\lambda)}}{|K_\lambda|}\sum_{\beta\vdash k}\frac{|B_k|}{H_{2\beta}}\varphi^\beta(\lambda)J_{\beta'}(A;1/2)J_{\beta'}(B;1/2).$$

Now by Theorem 3.4 and a second application of (4.2), we have

$$J_{\beta'}(x;1/2) = (-1/2)^k\omega_{-2}Z_\beta(x) = \frac{1}{|B_k|}\sum_{\mu\vdash k}(-1/2)^{k-\ell(\mu)}\varphi^\beta(\mu)p_\mu(x).$$

Using this to expand $J_{\beta'}(A;1/2)J_{\beta'}(B;1/2)$, we obtain

$$\begin{aligned} P_\lambda^{\mathbf{H}}(A,B) = \frac{(-1)^k}{|K_\lambda|}&\sum_{\beta\vdash k}\frac{1}{H_{2\beta}}\cdot\frac{\varphi^\beta(\lambda)}{|B_k|} \\ &\times\sum_{\mu,\nu\vdash k}(-2)^{\ell(\mu)+\ell(\nu)-\ell(\lambda)}\varphi^\beta(\mu)\varphi^\beta(\nu)p_\mu(A)p_\nu(B). \end{aligned}$$

By Lemma 3.3, the coefficient of $p_\mu(A)p_\nu(B)$ in this expression is $|B_k|^{-1}c_{\mu,\nu}^\lambda$. □

Now consider the special case $A = I_m \oplus O_{n-m}$, $B = I_n$, and define

$$Q_\lambda^{\mathbf{H}}(m,n) = \mathcal{E}_U(p_\lambda(AUBU^*)) = \mathcal{E}_V(p_\lambda(VV^*)),$$

where $V$ is an $m\times n$ matrix of independent quaternionic random variables, each distributed identically to $v$. An immediate consequence of Theorem 4.1 is the fact that $Q_\lambda^{\mathbf{H}}(m,n)$ can be easily expressed in terms of its real counterpart $Q_\lambda^{\mathbf{R}}(m,n)$.

4.2 COROLLARY. *If* $\lambda\vdash k$, *then* $Q_\lambda^{\mathbf{H}}(m,n) = (-1)^{k+\ell(\lambda)}2^{-\ell(\lambda)}Q_\lambda^{\mathbf{R}}(-2m,-2n)$.

Since the terms of highest total degree in $Q_\lambda^{\mathbf{R}}(m,n)$ are contributed by the terms $p_\mu(A)p_\nu(B)$ with $\ell(\mu)+\ell(\nu) = k+\ell(\lambda)$, it follows that $c_{\mu,\nu}^\lambda = 2^k b_{\mu,\nu}^\lambda$ in such cases. In other words, the terms of highest total degree in $Q_\lambda^{\mathbf{H}}(m,n)$ are $2^k$ times the corresponding terms in $Q_\lambda^{\mathbf{R}}(m,n)$.

The following result shows that $|B_k|^{-1}c_{\mu,\nu}^\lambda$ is an integer divisible by $2^{\ell(\mu)}$; in particular, it follows that $Q_\lambda^{\mathbf{H}}(m,n)$ has integer coefficients.

4.3 PROPOSITION. *If $\lambda, \mu, \nu \vdash k$, then $|B_k|^{-1}2^{\ell(\nu)-\ell(\lambda)}b^{\lambda}_{\mu,\nu}$ is an integer.*

PROOF. Let $\mathcal{B}^{\lambda}_{\mu,\nu}$ be the set of 1-factors $\delta$ with $\Lambda(\delta, \delta_\lambda) = \mu$ and $\Lambda(\delta, \varepsilon) = \nu$. We know that $|\mathcal{B}^{\lambda}_{\mu,\nu}| = |B_k|^{-1}b^{\lambda}_{\mu,\nu}$, by Lemma 3.2. Now if $w \in S_{2k}$ is any permutation that preserves both $\varepsilon$ and $\delta_\lambda$, then $\mathcal{B}^{\lambda}_{\mu,\nu}$ will be stable under the action of $w$. Thus it suffices to find a subgroup $G$ of $\mathrm{Aut}(\varepsilon) \cap \mathrm{Aut}(\delta_\lambda)$ with the property that the orbit of any 1-factor $\delta$ is divisible by $2^{\ell(\lambda)-c(\varepsilon\cup\delta)}$.

For this, consider the cycles $C_1, \ldots, C_l$ of $\varepsilon \cup \delta_\lambda$ (where $l = \ell(\lambda)$), labeled so that $C_i$ is a cycle of length $2\lambda_i$. For each cycle $C_i$, let $w_i \in S_{2k}$ be the permutation that interchanges each pair of vertices in $C_i$ at distance $\lambda_i$, and fixes the remaining vertices. It is easy to see that $w_i$ preserves both $\varepsilon$ and $\delta_\lambda$. Now let $G$ be the group (isomorphic to the direct product of $l$ copies of $\mathbf{Z}_2$) generated by $w_1, \ldots, w_l$.

To prove that this group satisfies the necessary properties, choose a 1-factor $\delta$, and let $c$ denote the number of connected components of $\varepsilon \cup \delta \cup \delta_\lambda$. The $G$-stabilizer of $\delta$ must be a divisor of $2^c$, so the $G$-orbit of $\delta$ will be a multiple of $2^{l-c}$. However, the number of connected components of $\varepsilon \cup \delta$ is $c(\varepsilon \cup \delta)$, so we have $l - c \geq l - c(\varepsilon \cup \delta)$. $\square$

As a final remark, note that (4.1) implies $Q^{\mathbf{H}}_{\lambda}(1,1) = (k+1)!$, or equivalently,

$$Q^{\mathbf{R}}_{\lambda}(-2,-2) = (-1)^{k+\ell(\lambda)}2^{\ell(\lambda)}(k+1)!.$$

It would be interesting to give a combinatorial proof of this based on Corollary 3.6.

## 5. An Explicit Formula for $Q^{\mathbf{R}}_k(m,n)$

In this section, we will derive an explicit Jack symmetric function expansion for $p_k$; this will lead to an explicit formula for $Q^{\mathbf{R}}_k(m,n)$ analogous to the formula for $Q^{\mathbf{C}}_k(m,n)$ we gave in Theorem 2.5. In view of Corollary 4.2, this also yields a formula for $Q^{\mathbf{H}}_k(m,n)$. Alternatively, one could use this technique to give a uniform treatment of all three cases.

To begin, let us introduce a scalar product $\langle \cdot, \cdot \rangle_\alpha$ on the ring of symmetric functions (say with real coefficients) by setting

$$\langle p_\mu, p_\nu \rangle_\alpha = z_\mu \alpha^{\ell(\mu)} \delta_{\mu,\nu}. \tag{5.1}$$

It is known that the Jack symmetric functions $J_\lambda = J_\lambda(x;\alpha)$ are orthogonal with respect to this inner product; in fact, by Theorem 5.8 of [St], we have

$$\langle J_\lambda, J_\mu \rangle_\alpha = j_\lambda(\alpha)\delta_{\lambda,\mu},$$

where

$$j_\lambda(\alpha) = \prod_{(i,j)\in\lambda} (\lambda'_j - i + \alpha(\lambda_i - j + 1))(\lambda'_j - i + 1 + \alpha(\lambda_i - j)). \tag{5.2}$$

Here we identify $\lambda$ with its diagram $\{(i,j) \in \mathbf{Z}^2 : 1 \leq i \leq \ell(\lambda),\ 1 \leq j \leq \lambda_i\}$.

By Theorem 5.4 of [**St**], there is an explicit formula for the value of $J_\lambda$ at the identity matrix; namely,

$$J_\lambda(I_n;\alpha) = \prod_{(i,j)\in\lambda} (n-(i-1)+\alpha(j-1)). \tag{5.3}$$

5.1 LEMMA. *We have*

$$p_k(x) = k\alpha \sum_{\lambda\vdash k} \frac{1}{j_\lambda(\alpha)} J_\lambda(x;\alpha) \prod_{\substack{(i,j)\in\lambda \\ (i,j)\neq(1,1)}} (\alpha(j-1)-(i-1)).$$

PROOF. Let $\psi_\mu^\lambda(\alpha)$ denote the coefficient of $p_\mu$ in $J_\lambda$, so that $J_\lambda = \sum \psi_\mu^\lambda(\alpha)p_\mu$. In view of (5.3), we have

$$\prod_{(i,j)\in\lambda} (n-(i-1)+\alpha(j-1)) = \sum_{\mu\vdash k} \psi_\mu^\lambda(\alpha)n^{\ell(\mu)}.$$

Note that both sides of this identity are polynomials in $n$. Since the term on the left side indexed by $(i,j)=(1,1)$ is $n$, it follows that if we extract the coefficient of $n$ from both sides, we obtain

$$\psi_k^\lambda(\alpha) = \prod_{\substack{(i,j)\in\lambda \\ (i,j)\neq(1,1)}} (\alpha(j-1)-(i-1)). \tag{5.4}$$

Now let $c_\lambda(\alpha)$ denote the coefficient of $J_\lambda$ in $p_k$, so that

$$p_k(x) = \sum_{\lambda\vdash k} c_\lambda(\alpha)J_\lambda(x;\alpha). \tag{5.5}$$

Taking the scalar product of both sides with $J_\lambda$ yields

$$\langle J_\lambda, p_k\rangle_\alpha = c_\lambda(\alpha)\langle J_\lambda, J_\lambda\rangle_\alpha = c_\lambda(\alpha)j_\lambda(\alpha).$$

On the other hand, (5.1) implies

$$\langle J_\lambda, p_k\rangle_\alpha = k\alpha\psi_k^\lambda(\alpha),$$

so we have

$$c_\lambda(\alpha) = k\alpha j_\lambda(\alpha)^{-1}\psi_k^\lambda(\alpha).$$

The result is now an immediate consequence of (5.2) and (5.4). □

5.2 COROLLARY. *If $\lambda \vdash k$ and $\alpha$ is a positive integer, then $c_\lambda(\alpha)$ (the coefficient of $J_\lambda$ in $p_k$) is zero if $\lambda_{\alpha+1} \geq 2$. In other words, if $c_\lambda(\alpha) \neq 0$, then $(\alpha+1,2) \notin \lambda$.*

PROOF. If $(\alpha+1,2)\in\lambda$, then (5.4) vanishes. □

Note that when $\alpha = 1$, we have $c_\lambda(1) \neq 0$ only if $\lambda$ is of the form $(i, 1^{k-i})$ (i.e., a hook). In this case, it is easy to show that Lemma 5.1 is equivalent to (2.6).

It would be interesting to generalize Lemma 5.1 to arbitrary power sums $p_\mu$. To do so, it would suffice to give a rule for the $J_\lambda$-expansion of $p_k J_\mu$; this would amount to a generalization of the Murnaghan-Nakayama rule [**M1**, Ex. I.3.11] to

Jack symmetric functions. Although this is still an open problem, one can show that the coefficient of $J_\lambda$ in $p_k J_\mu$ is nonzero only if $\lambda \supseteq \mu$ (diagram inclusion) and the skew diagram $\lambda - \mu$ contains no $(\alpha+1) \times 2$ rectangle (assuming $\alpha$ is a positive integer).

Returning to the problem of determining a formula for $Q_k^{\mathbf{R}}(m,n)$, note that by Corollary 5.2, the partitions $\lambda \vdash k$ for which $c_\lambda(2) \neq 0$ are of the form $(a, b, 1^{k-a-b})$, where $a \geq b \geq 1$ or $a = k$, $b = 0$. The following formulas for $c_\lambda(2)$ are easily obtained by specializing Lemma 5.1; we leave the details to the reader.

5.3 LEMMA.

(a) *If* $\lambda = (a, b, 1^{k-a-b})$ *and* $a \geq b \geq 1$, *then*[3]

$$c_\lambda(2) = (-1)^k \frac{(-2)^{a-b+1} k(2a-2b+1)(a-1)!}{(k+a-b+1)_2 (k-a+b)_2 (k-a-b)!(2a-1)!(b-1)!}.$$

(b) *If* $\lambda = (k)$, *then* $c_\lambda(2) = 2^k k!/(2k)! = 1/(1 \cdot 3 \cdots (2k-1))$. □

Now if $\lambda = (a, b, 1^{k-a-b})$ (where $a \geq b \geq 1$ or $a = k$, $b = 0$), let us define

$$F_\lambda(x) = 2^{a-b}\left(\frac{x}{2} + a - 1\right)_{a-b} (x + 2b - 2)_{k-a+b}.$$

5.4 THEOREM. *We have* $Q_k^{\mathbf{R}}(m,n) = \sum_\lambda c_\lambda(2) F_\lambda(m) F_\lambda(n)$, *where* $\lambda$ *ranges over all partitions of the form* $(a, b, 1^{k-a-b})$, *and* $c_\lambda(2)$ *is given by Lemma* 5.3.

PROOF. Proceed by analogy with the proof of Theorem 2.5. If we apply (5.5) and Theorem 1.1 in the case $\alpha = 2$, we obtain

$$Q_k^{\mathbf{R}}(m,n) = \sum_{\lambda \vdash k} c_\lambda(2) J_\lambda(I_m; 2) J_\lambda(I_n; 2),$$

by the linearity of expectation. However, if $\lambda = (a, b, 1^{k-a-b})$, then we have $J_\lambda(I_n; 2) = F_\lambda(n)$, by (5.3). Since $c_\lambda(2) = 0$ unless $\lambda$ is of the form $(a, b, 1^{k-a-b})$ (Corollary 5.2), the result follows. □

This formula for $Q_k^{\mathbf{R}}(m,n)$ is rather unsightly, and it is by no means obvious that it has integer coefficients. But the number of terms is only a quadratic function of $k$, and each term is a simple product, so it is more efficient for computational purposes than Corollary 3.6, which has exponential complexity.

In some unpublished work on Wishart eigenvalues, Mallows and Wachter have considered some combinatorial aspects of the polynomials $Q_k^{\mathbf{R}}(m,n)$ (see the related paper [**MW**], although it does not explicitly mention $Q_k^{\mathbf{R}}(m,n)$). In particular, Mallows has raised the question of computing the coefficient $t_k$ of $mn$ in $Q_k^{\mathbf{R}}(m,n)$; indeed, the sequence

$$(t_1, t_2, \ldots) = (1, 1, 4, 20, 148, 1348, 15104, \ldots)$$

[3] Recall that $(a)_i$ denotes the falling factorial $a(a-1)\cdots(a-i+1)$.

is number 1447 in **[Sl]**. Our work yields two expressions for $t_k$. By Corollary 3.6, it follows that $t_k$ is the number of 1-factors $\delta \in \mathcal{F}_k$ with $c(\varepsilon \cup \delta) = c(\delta \cup \delta_k) = 1$, whereas by Theorem 5.4, we have

$$t_k = \sum_\lambda c_\lambda(2) \left[ \frac{2^{a-b-1}(2b)!(a-1)!(k-a-b+1)!}{(2b-1)\, b!} \right]^2,$$

where $\lambda$ is restricted as in Theorem 5.4, and $c_\lambda(2)$ is given by Lemma 5.3.

## 6. An Alternative Combinatorial Approach

Returning to the general setting of §1, let $F$ be a finite-dimensional real division algebra, and let $U = [u_{ij}]$ be an $n \times n$ matrix of independent $F$-valued standard normal variables. As usual, let $U^*$ denote the conjugated transpose of $U$ (so that $U^* = U^t$ in the real case). Recall that if $A$ and $B$ are fixed $n \times n$ Hermitian matrices and $f$ is a symmetric function, then $\mathcal{E}_U(f(AUBU^*))$ is invariant under unitary transformations $A \mapsto XAX^*$ and $B \mapsto YBY^*$. We will therefore assume in what follows that $A$ and $B$ are (real) diagonal matrices; say $A = \mathrm{diag}(a_1, \ldots, a_n)$, $B = \mathrm{diag}(b_1, \ldots, b_n)$.

In the real and complex cases, it is possible to give a direct combinatorial interpretation of $\mathcal{E}_U(p_\lambda(AUBU^*))$ (as is apparent from **[OU**, §3**]**), which raises the possibility that a more combinatorial proof of Theorems 2.3 and 3.5 could be given. In the quaternionic case, the noncommutativity causes certain complications which make this approach more difficult, as we shall see below. To describe the combinatorial interpretation, first note that the $(i,j)$-entry of $AUBU^*$ is $\sum_k a_i b_k u_{ik} \bar{u}_{jk}$, so it follows that

$$p_k(AUBU^*) = \sum (a_{i_1} b_{j_1} \cdots a_{i_k} b_{j_k}) \cdot (u_{i_1 j_1} \bar{u}_{i_2 j_1} u_{i_2 j_2} \bar{u}_{i_3 j_2} \cdots u_{i_k j_k} \bar{u}_{i_1 j_k}), \tag{6.1}$$

where the indices $i_1, j_1, \ldots, i_k, j_k$ all range from 1 to $n$. Since conjugation is an anti-automorphism of $\mathbf{H}$, it follows that the above expression is real (regardless of $F$), and therefore consistent with the conventions of §1 (cf. (1.2)).

Now let $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_n\}$ be two $n$-element alphabets, and let $\mathcal{W}_k$ be the set of all words $w$ of length $2k$ in the elements of $X \cup Y$, starting with a letter of $X$, and alternating thereafter between $Y$ and $X$; say,

$$w = x_{i_1} y_{j_1} x_{i_2} y_{j_2} \cdots x_{i_k} y_{j_k}. \tag{6.2}$$

There is an obvious one-to-one correspondence between $\mathcal{W}_k$ and the terms of $p_k(AUBU^*)$ (identify the indices $i_1, j_1, \ldots, i_k, j_k$ in (6.1) and (6.2)). More generally, for each partition $\lambda = (\lambda_1, \ldots, \lambda_l)$, there is a one-to-one correspondence between the terms of $p_\lambda(AUBU^*)$ and the Cartesian product

$$\mathcal{W}_\lambda := \mathcal{W}_{\lambda_1} \times \cdots \times \mathcal{W}_{\lambda_l}.$$

In order to determine the amount contributed to $\mathcal{E}_U(p_\lambda(AUBU^*))$ by a term indexed by a typical member of $\mathcal{W}_\lambda$, it suffices to determine the expectation of an arbitrary monomial of independent standard normal variables and their conjugates. In the commutative cases (i.e., $F = \mathbf{R}$ or $\mathbf{C}$), this is easy to do, but in the quaternionic case, we do not know the answer in general. In spite of this, we can still prove the following result for all three cases, thereby completing our proof of Theorem 1.1.

6.1 PROPOSITION. *If $\lambda \vdash k$ and $\lambda \neq (1^k)$, then the coefficient of $a_1 b_1 \cdots a_k b_k$ in $\mathcal{E}_U(p_\lambda(AUBU^*))$ is zero.*

PROOF. Let $w = (w_1, \ldots, w_l)$ be a typical member of $\mathcal{W}_\lambda$, and assume $\lambda_1 \geq 2$. Let $x_i y_j$ be the first two letters of $w_1$. In order for the term indexed by $w$ to contribute to the coefficient of $a_1 b_1 \cdots a_k b_k$, there can be no other occurrence of $x_i$ or $y_j$ among $w_1, \ldots, w_l$. In particular, the term indexed by $w$ will have at most one occurrence of $u_{ij}$, and since $\lambda_1 \geq 2$, there cannot be any occurrence of $\bar{u}_{ij}$. (In graph-theoretic terms, this is equivalent to the fact that a closed path of length $2k$ with no repeated vertices cannot have repeated edges unless $k = 1$.) It follows that the term indexed by $w$ is homogeneous of degree one in the variable $u_{ij}$, and therefore has zero expectation. □

For the remainder of this section, we assume $F = \mathbf{R}$ or $\mathbf{C}$.

Given any $w = (w_1, \ldots, w_l) \in \mathcal{W}_\lambda$, let us define $\pi(w) = \pi(w_1) \cdots \pi(w_l)$ to be the monomial obtained by substituting $a_i$ and $b_j$ for each occurrence of $x_i$ and $y_j$. Furthermore, let us define $m_{ij}(w)$ to be the number of times $x_i y_j$ occurs consecutively as a subword in $w_1, w_2, \ldots$, and $w_l$. Similarly define $m'_{ij}(w)$ to be the number of times $y_j x_i$ occurs as a subword, using the convention that the last letter of each $w_r$ is followed by its first letter (i.e., each $w_r$ is a "circular word").

6.2 PROPOSITION. *Assuming $F = \mathbf{R}$ or $\mathbf{C}$, we have*

$$P_\lambda^F(A, B) = \mathcal{E}_U(p_\lambda(AUBU^*)) = \sum_{w \in \mathcal{W}_\lambda} \pi(w) \prod_{1 \leq i,j \leq n} \kappa_{ij}^F(w),$$

*where*

$$\kappa_{ij}^{\mathbf{R}}(w) = \begin{cases} 1 \cdot 3 \cdots (2r-1), & \text{if } m_{ij}(w) + m'_{ij}(w) = 2r \\ 0, & \text{if } m_{ij}(w) + m'_{ij}(w) \text{ is odd,} \end{cases}$$

$$\kappa_{ij}^{\mathbf{C}}(w) = \begin{cases} r!, & \text{if } m_{ij}(w) = m'_{ij}(w) = r \\ 0, & \text{if } m_{ij}(w) \neq m'_{ij}(w). \end{cases}$$

PROOF. This is an immediate consequence of the fact that

$$p_\lambda(AUBU^*) = \sum_{w \in \mathcal{W}_\lambda} \pi(w) \cdot \prod_{1 \leq i,j \leq n} u_{ij}^{m_{ij}(w)} \bar{u}_{ij}^{m'_{ij}(w)},$$

together with the fact that if $u$ is a standard normal variable, then

$$\mathcal{E}(u^{2r}) = 1 \cdot 3 \cdots (2r-1) \tag{6.3}$$

in the real case, and

$$\mathcal{E}(u^r \bar{u}^s) = r! \, \delta_{r,s} \tag{6.4}$$

in the complex case. □

Thus one could establish Theorem 2.3 by directly proving that

$$\sum_{w\in\mathcal{W}_\lambda} \pi(w) \prod_{1\le i,j\le n} \kappa_{ij}^{\mathbf{C}}(w) = \sum_{\mu,\nu\vdash k} a_{\mu,\nu}^{\lambda} p_\mu(A)p_\nu(B) \tag{6.5}$$

for all partitions $\lambda$ of $k$, and a similar approach to Theorem 3.5 could be attempted. If successful, this would also yield proofs of the real and complex cases of Theorem 1.1 that avoid any integration beyond the simple and well-known (6.3) and (6.4).

For certain pairs $\mu,\nu\vdash k$, there is a known combinatorial proof that $a_{\mu,\nu}^{\lambda}$ is indeed the coefficient of $p_\mu(A)p_\nu(B)$ on the left side of (6.5). By the triangle inequality (cf. (2.5)), we have $a_{\mu,\nu}^{\lambda}=0$ unless $\ell(\mu)+\ell(\nu)\le k+\ell(\lambda)$, so the terms on the right side of (6.5) with $\ell(\mu)+\ell(\nu)=k+\ell(\lambda)$ are the terms of highest total degree with respect to $p_i(A)$ and $p_j(B)$. It follows that if we expand $p_\mu(x)p_\nu(y)$ in terms of monomial symmetric functions $m_\alpha(x)m_\beta(y)$, then the coefficient of $m_\mu(x)m_\nu(y)$ is one, while any other term $m_\alpha(x)m_\beta(y)$ appearing with nonzero coefficient must satisfy $\ell(\alpha)+\ell(\beta)<\ell(\mu)+\ell(\nu)$. Hence if we let $\mathcal{W}_\lambda^*$ be the set of $w\in\mathcal{W}_\lambda$ such that $\pi(w)$ is a term of $m_\alpha(A)m_\beta(B)$ with $\ell(\alpha)+\ell(\beta)=\ell(\lambda)+k$, then (6.5) implies

$$\sum_{w\in\mathcal{W}_\lambda^*} \pi(w) \prod_{1\le i,j\le n} \kappa_{ij}^{\mathbf{C}}(w) \quad = \sum_{\substack{\mu,\nu\vdash k\\ \ell(\mu)+\ell(\nu)=\ell(\lambda)+k}} a_{\mu,\nu}^{\lambda} m_\mu(A)m_\nu(B).$$

Goulden and Jackson have given a combinatorial proof of a result easily seen to be equivalent to the case $\lambda=(k)$ of this identity [**GJ**,Thm. 2.1], and they remark that the proof generalizes to arbitrary $\lambda$ [**GJ**, §4]. It would be interesting to try to extend this reasoning to give a proof of (6.5).

Finally, let us remark that when $\lambda=(k)$ and $\ell(\mu)+\ell(\nu)=\ell(\lambda)+k=k+1$, Bédard and Goupil [**BGo**], and later Goulden and Jackson [**GJ**,Thm. 2.2], have shown that

$$a_{\mu,\nu}^{(k)} = \frac{k(\ell(\mu)-1)!\,(\ell(\lambda)-1)!}{m_1(\mu)!\,m_1(\nu)!\,m_2(\mu)!\,m_2(\nu)!\cdots},$$

where $m_j(\lambda)$ denotes the multiplicity of $j$ in $\lambda$.

## Appendix

The following are tables for the polynomials $P_\lambda^F(A,B)=\mathcal{E}_U(p_\lambda(AUBU^*))$ and $Q_\lambda^F(m,n)$ for all partitions $\lambda$ of size at most 4, and each of the division algebras $F=\mathbf{R}$, $\mathbf{C}$ and $\mathbf{H}$. We use $p_r$ and $q_r$ as abbreviations for $p_r(A)$ and $p_r(B)$, respectively.

**The polynomials $P_\lambda^{\mathbf{R}}(A,B)$.**

| | |
|---|---|
| 2 | $p_1^2q_2+p_2q_1^2+p_2q_2$ |
| 11 | $p_1^2q_1^2+2p_2q_2$ |

| | |
|---|---|
| 3 | $p_1^3q_3 + 3p_1p_2q_1q_2 + p_3q_1^3 + 3p_1p_2q_3 + 3p_3q_1q_2 + 4p_3q_3$ |
| 21 | $p_1^3q_1q_2 + p_1p_2q_1^3 + p_1p_2q_1q_2 + 4p_1p_2q_3 + 4p_3q_1q_2 + 4p_3q_3$ |
| $1^3$ | $p_1^3q_1^3 + 6p_1p_2q_1q_2 + 8p_3q_3$ |

| | |
|---|---|
| 4 | $p_1^4q_4 + 4p_1^2p_2q_1q_3 + 2p_1^2p_2q_2^2 + 4p_1p_3q_1^2q_2 + 2p_2^2q_1^2q_2 + p_4q_1^4$ $+6p_1^2p_2q_4 + 8p_1p_3q_1q_3 + 4p_1p_3q_2^2 + 4p_2^2q_1q_3 + p_2^2q_2^2$ $+6p_4q_1^2q_2 + 16p_1p_3q_4 + 5p_2^2q_4 + 16p_4q_1q_3 + 5p_4q_2^2 + 20p_4q_4$ |
| 31 | $p_1^4q_1q_3 + 3p_1^2p_2q_1^2q_2 + p_1p_3q_1^4 + 3p_1^2p_2q_1q_3 + 3p_1p_3q_1^2q_2$ $+6p_1^2p_2q_4 + 10p_1p_3q_1q_3 + 6p_1p_3q_2^2 + 6p_2^2q_1q_3 + 6p_4q_1^2q_2$ $+12p_1p_3q_4 + 6p_2^2q_4 + 12p_4q_1q_3 + 6p_4q_2^2 + 24p_4q_4$ |
| 22 | $p_1^4q_2^2 + 2p_1^2p_2q_1^2q_2 + p_2^2q_1^4 + 2p_1^2p_2q_2^2 + 2p_2^2q_1^2q_2 + 8p_1^2p_2q_4 + 16p_1p_3q_1q_3$ $+5p_2^2q_2^2 + 8p_4q_1^2q_2 + 16p_1p_3q_4 + 4p_2^2q_4 + 16p_4q_1q_3 + 4p_4q_2^2 + 20p_4q_4$ |
| 211 | $p_1^4q_1^2q_2 + p_1^2p_2q_1^4 + p_1^2p_2q_1^2q_2 + 8p_1^2p_2q_1q_3 + 2p_1^2p_2q_2^2 + 8p_1p_3q_1^2q_2 + 2p_2^2q_1^2q_2$ $+8p_1p_3q_1q_3 + 2p_2^2q_2^2 + 16p_1p_3q_4 + 8p_2^2q_4 + 16p_4q_1q_3 + 8p_4q_2^2 + 24p_4q_4$ |
| $1^4$ | $p_1^4q_1^4 + 12p_1^2p_2q_1^2q_2 + 32p_1p_3q_1q_3 + 12p_2^2q_2^2 + 48p_4q_4$ |

**The polynomials $Q_\lambda^{\mathbf{R}}(m,n)$.**

| | |
|---|---|
| 2 | $m^2n + mn^2 + mn$ |
| 11 | $m^2n^2 + 2mn$ |

| | |
|---|---|
| 3 | $m^3n + 3m^2n^2 + mn^3 + 3m^2n + 3mn^2 + 4mn$ |
| 21 | $m^3n^2 + m^2n^3 + m^2n^2 + 4m^2n + 4mn^2 + 4mn$ |
| $1^3$ | $m^3n^3 + 6m^2n^2 + 8mn$ |

| | |
|---|---|
| 4 | $m^4n + 6m^3n^2 + 6m^2n^3 + mn^4 + 6m^3n$ $+17m^2n^2 + 6mn^3 + 21m^2n + 21mn^2 + 20mn$ |
| 31 | $m^4n^2 + 3m^3n^3 + m^2n^4 + 3m^3n^2 + 3m^2n^3 + 6m^3n$ $+22m^2n^2 + 6mn^3 + 18m^2n + 18mn^2 + 24mn$ |
| 22 | $m^4n^2 + 2m^3n^3 + m^2n^4 + 2m^3n^2 + 2m^2n^3 + 8m^3n$ $+21m^2n^2 + 8mn^3 + 20m^2n + 20mn^2 + 20mn$ |
| 211 | $m^4n^3 + m^3n^4 + m^3n^3 + 10m^3n^2 + 10m^2n^3$ $+10m^2n^2 + 24m^2n + 24mn^2 + 24mn$ |
| $1^4$ | $m^4n^4 + 12m^3n^3 + 44m^2n^2 + 48mn$ |

**The polynomials $P^{\mathbf{C}}_\lambda(A,B)$.**

| | |
|---|---|
| 2 | $p_1^2q_2+p_2q_1^2$ |
| 11 | $p_1^2q_1^2+p_2q_2$ |

| | |
|---|---|
| 3 | $p_1^3q_3+3p_1p_2q_1q_2+p_3q_1^3+p_3q_3$ |
| 21 | $p_1^3q_1q_2+p_1p_2q_1^3+2p_1p_2q_3+2p_3q_1q_2$ |
| $1^3$ | $p_1^3q_1^3+3p_1p_2q_1q_2+2p_3q_3$ |

| | |
|---|---|
| 4 | $p_1^4q_4+4p_1^2p_2q_1q_3+2p_1^2p_2q_2^2+4p_1p_3q_1^2q_2+2p_2^2q_1^2q_2$ $+p_4q_1^4+4p_1p_3q_4+p_2^2q_4+4p_4q_1q_3+p_4q_2^2$ |
| 31 | $p_1^4q_1q_3+3p_1^2p_2q_1^2q_2+p_1p_3q_1^4+3p_1^2p_2q_4+4p_1p_3q_1q_3$ $+3p_1p_3q_2^2+3p_2^2q_1q_3+3p_4q_1^2q_2+3p_4q_4$ |
| 22 | $p_1^4q_2^2+2p_1^2p_2q_1^2q_2+p_2^2q_1^4+4p_1^2p_2q_4$ $+8p_1p_3q_1q_3+2p_2^2q_2^2+4p_4q_1^2q_2+2p_4q_4$ |
| 211 | $p_1^4q_1^2q_2+p_1^2p_2q_1^4+4p_1^2p_2q_1q_3+p_1^2p_2q_2^2+4p_1p_3q_1^2q_2$ $+p_2^2q_1^2q_2+4p_1p_3q_4+2p_2^2q_4+4p_4q_1q_3+2p_4q_2^2$ |
| $1^4$ | $p_1^4q_1^4+6p_1^2p_2q_1^2q_2+8p_1p_3q_1q_3+3p_2^2q_2^2+6p_4q_4$ |

**The polynomials $Q^{\mathbf{C}}_\lambda(m,n)$.**

| | |
|---|---|
| 2 | $m^2n+mn^2$ |
| 11 | $m^2n^2+mn$ |

| | |
|---|---|
| 3 | $m^3n+3m^2n^2+mn^3+mn$ |
| 21 | $m^3n^2+m^2n^3+2m^2n+2mn^2$ |
| $1^3$ | $m^3n^3+3m^2n^2+2mn$ |

| | |
|---|---|
| 4 | $m^4n+6m^3n^2+6m^2n^3+mn^4+5m^2n+5mn^2$ |
| 31 | $m^4n^2+3m^3n^3+m^2n^4+3m^3n+10m^2n^2+3mn^3+3mn$ |
| 22 | $m^4n^2+2m^3n^3+m^2n^4+4m^3n+10m^2n^2+4mn^3+2mn$ |
| 211 | $m^4n^3+m^3n^4+5m^3n^2+5m^2n^3+6m^2n+6mn^2$ |
| $1^4$ | $m^4n^4+6m^3n^3+11m^2n^2+6mn$ |

**The polynomials $P^{\mathbf{H}}_\lambda(A,B)$.**

| | |
|---|---|
| 2 | $4p_1^2q_2+4p_2q_1^2-2p_2q_2$ |
| 11 | $4p_1^2q_1^2+2p_2q_2$ |

| | |
|---|---|
| 3 | $8p_1^3q_3+24p_1p_2q_1q_2+8p_3q_1^3-12p_1p_2q_3-12p_3q_1q_2+8p_3q_3$ |
| 21 | $8p_1^3q_1q_2+8p_1p_2q_1^3-4p_1p_2q_1q_2+8p_1p_2q_3+8p_3q_1q_2-4p_3q_3$ |
| $1^3$ | $8p_1^3q_1^3+12p_1p_2q_1q_2+4p_3q_3$ |

| | |
|---|---|
| 4 | $16p_1^4q_4 + 64p_1^2p_2q_1q_3 + 32p_1^2p_2q_2^2 + 64p_1p_3q_1^2q_2 + 32p_2^2q_1^2q_2 + 16p_4q_1^4$ $-48p_1^2p_2q_4 - 64p_1p_3q_1q_3 - 32p_1p_3q_2^2 - 32p_2^2q_1q_3 - 8p_2^2q_2^2$ $-48p_4q_1^2q_2 + 64p_1p_3q_4 + 20p_2^2q_4 + 64p_4q_1q_3 + 20p_4q_2^2 - 40p_4q_4$ |
| 31 | $16p_1^4q_1q_3 + 48p_1^2p_2q_1^2q_2 + 16p_1p_3q_1^4 - 24p_1^2p_2q_1q_3 - 24p_1p_3q_1^2q_2$ $+24p_1^2p_2q_4 + 40p_1p_3q_1q_3 + 24p_1p_3q_2^2 + 24p_2^2q_1q_3 + 24p_4q_1^2q_2$ $-24p_1p_3q_4 - 12p_2^2q_4 - 24p_4q_1q_3 - 12p_4q_2^2 + 24p_4q_4$ |
| 22 | $16p_1^4q_2^2 + 32p_1^2p_2q_1^2q_2 + 16p_2^2q_1^4 - 16p_1^2p_2q_2^2$ $-16p_2^2q_1^2q_2 + 32p_1^2p_2q_4 + 64p_1p_3q_1q_3 + 20p_2^2q_2^2 + 32p_4q_1^2q_2$ $-32p_1p_3q_4 - 8p_2^2q_4 - 32p_4q_1q_3 - 8p_4q_2^2 + 20p_4q_4$ |
| 211 | $16p_1^4q_1^2q_2 + 16p_1^2p_2q_1^4 - 8p_1^2p_2q_1^2q_2 + 32p_1^2p_2q_1q_3$ $+8p_1^2p_2q_2^2 + 32p_1p_3q_1^2q_2 + 8p_2^2q_1^2q_2 - 16p_1p_3q_1q_3 - 4p_2^2q_2^2$ $+16p_1p_3q_4 + 8p_2^2q_4 + 16p_4q_1q_3 + 8p_4q_2^2 - 12p_4q_4$ |
| $1^4$ | $16p_1^4q_1^4 + 48p_1^2p_2q_1^2q_2 + 32p_1p_3q_1q_3 + 12p_2^2q_2^2 + 12p_4q_4$ |

**The polynomials $Q_\lambda^{\mathbf{H}}(m,n)$.**

| | |
|---|---|
| 2 | $4m^2n + 4mn^2 - 2mn$ |
| 11 | $4m^2n^2 + 2mn$ |

| | |
|---|---|
| 3 | $8m^3n + 24m^2n^2 + 8mn^3 - 12m^2n - 12mn^2 + 8mn$ |
| 21 | $8m^3n^2 + 8m^2n^3 - 4m^2n^2 + 8m^2n + 8mn^2 - 4mn$ |
| $1^3$ | $8m^3n^3 + 12m^2n^2 + 4mn$ |

| | |
|---|---|
| 4 | $16m^4n + 96m^3n^2 + 96m^2n^3 + 16mn^4 - 48m^3n$ $-136m^2n^2 - 48mn^3 + 84m^2n + 84mn^2 - 40mn$ |
| 31 | $16m^4n^2 + 48m^3n^3 + 16m^2n^4 - 24m^3n^2 - 24m^2n^3$ $+24m^3n + 88m^2n^2 + 24mn^3 - 36m^2n - 36mn^2 + 24mn$ |
| 22 | $16m^4n^2 + 32m^3n^3 + 16m^2n^4 - 16m^3n^2 - 16m^2n^3$ $+32m^3n + 84m^2n^2 + 32mn^3 - 40m^2n - 40mn^2 + 20mn$ |
| 211 | $16m^4n^3 + 16m^3n^4 - 8m^3n^3 + 40m^3n^2$ $+40m^2n^3 - 20m^2n^2 + 24m^2n + 24mn^2 - 12mn$ |
| $1^4$ | $16m^4n^4 + 48m^3n^3 + 44m^2n^2 + 12mn$ |

**Acknowledgment.** The second author would like to thank Colin Mallows for introducing him to the problems considered in this paper and for several helpful discussions.

## References

[BGo] F. Bédard and A. Goupil, *The poset of conjugacy classes and decomposition of products in the symmetric group*, Canad. Math. Bull. (to appear).

[BG] N. Bergeron and A. M. Garsia, *Zonal polynomials and domino tableaux*, preprint.

[B] W. Burnside, *Theory of groups of finite order*, 2nd ed., Dover, New York, 1955.

[G] I. M. Gelfand, *Spherical functions on symmetric Riemann spaces*, Dokl. Akad. Nauk SSSR **70** (1950), 5–8. (Russian)

[Go] R. Godement, *A theory of spherical functions*. I, Trans. Amer. Math. Soc. **73** (1952), 496–556.

[GJ] I. P. Goulden and D. M. Jackson, *The combinatorial relationship between trees, cacti and certain connection coefficients for the symmetric group*, preprint.

[GR] K. I. Gross and D. St. P. Richards, *Special functions of matrix argument I: Algebraic induction, zonal polynomials, and hypergeometric functions*, Trans. Amer. Math. Soc. **301** (1987), 781–811.

[J] A. T. James, *Zonal polynomials of the real positive definite symmetric matrices*, Ann. of Math. **74** (1961), 475–501.

[M1] I. G. Macdonald, *Symmetric functions and Hall polynomials*, Oxford Univ. Press, Oxford, 1979.

[M2] I. G. Macdonald, *Commuting differential operators and zonal spherical functions*, Algebraic Groups, Utrecht 1986 (A. M. Cohen, *et. al.*, eds.), Lecture Notes in Math., vol. 1271, Springer-Verlag, Berlin and New York, 1987, pp. 189–200.

[MW] C. L. Mallows and K. W. Wachter, *Valency enumeration of rooted plane trees*, J. Austral. Math. Soc. **13** (1972), 472–476.

[OU] W. H. Olson and V. R. R. Uppuluri, *Asymptotic distribution of eigenvalues of random matrices*, Sixth Berkeley Symposium on Mathematical Statistics and Probability, vol. III, Univ. of California Press, Berkeley, CA, 1972, pp. 615–644.

[Sl] N. J. A. Sloane, *A handbook of integer sequences*, Academic Press, New York, 1973.

[St] R. P. Stanley, *Some combinatorial properties of Jack symmetric functions*, Adv. in Math. **77** (1989), 76–115.

[Ste] J. R. Stembridge, *On Schur's Q-functions and the primitive idempotents of a commutative Hecke algebra*, preprint.

[T] A. Takemura, *Zonal polynomials*, Institute of Mathematical Statistics, Hayward, CA, 1984.

Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48109

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48109