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A combinatorial decomposition of acyclic simplicial complexes

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Abstract

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It is proved that if Δ is a finite acyclic simplicial complex, then there is a subcomplex $\Delta' \subset \Delta$ and a bijection $\eta: \Delta' \to \Delta - \Delta'$ such that $F \subset \eta(F)$ and $|\eta(F) - F| = 1$ for all $F \in \Delta'$. This improves an earlier result of Kalai. An immediate corollary is a characterization (first due to Kalai) of the f-vector of an acyclic simplicial complex. Several generalizations, some proved and some conjectured, are discussed.

1. Introduction

Let Δ be an (abstract) simplicial complex on the vertex set V, i.e., a collection of subsets F (called *faces*) of V such that $\{x\} \in \Delta$ for all $x \in V$ and such that if $F \in \Delta$ and $G \subset F$, then $G \in \Delta$. The *dimension* of a face $F \in \Delta$ is defined by $\dim F = |F| - 1$, and the *dimension* of Δ is given by

$$d - 1 = \dim \Delta = \max\{\dim F\colon F \in \Delta\}.$$

Let $f_i = f_i(\Delta)$ denote the number of i-dimensional faces of Δ, so $f_{-1} = 1$ unless $\Delta = \emptyset$. The vector $f(\Delta) = (f_0, \ldots, f_{d-1})$ is called the *f-vector* of Δ.

Fix a ground field K. Δ is called *acyclic* (over K) if $\tilde{H}^i(\Delta) = 0$ for all i, where $\tilde{H}^i(\Delta)$ denotes the i-th reduced simplicial cohomology group of Δ over the coefficient field K. Note that the empty set $\emptyset$ is acyclic, but that the simplicial complex $\{\emptyset\}$ is not acyclic since $\tilde{H}^{-1}(\{\emptyset\}) = K$.

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A special class of acyclic simplicial complexes are *cones*. If Δ' is a simplicial complex on V', and if x is a new vertex not in V', then the cone $\Delta = x * \Delta'$ (the join of Δ' with the vertex x) is defined to be the simplicial complex

$$\Delta = x * \Delta' = \Delta' \cup \{\{x\} \cup F \colon F \in \Delta'\}.$$

Note the following property of the cone $x * \Delta'$: Define $\eta : \Delta' \to \Delta - \Delta'$ by $\eta(F) = F \cup \{x\}$. Then η is a bijection from Δ' to $\Delta - \Delta'$ satisfying:

(P1) The domain Δ' of η is a subcomplex of Δ.

(P2) For all $F \in \Delta'$, we have $\eta(F) \supset F$ and $|\eta(F) - F| = 1$.

An immediate corollary to these properties is the following:

(P3) There exists a simplicial complex Δ_0 such that

$$\sum_{i \geqslant 0} f_{i-1}(\Delta) x^i = (1+x) \sum_{i \geqslant 0} f_{i-1}(\Delta_0) x^i.$$

Of course the simplicial complex Δ_0 of (P3) is just Δ'. We can ask whether for *any* acyclic simplicial complex Δ there is a subset $\Delta' \subset \Delta$ and bijection $\eta : \Delta' \to \Delta - \Delta'$ satisfying (P1)–(P3). Kalai [8], using his technique of algebraic shifting, first proved (P3). Björner and Kalai [3, 4] went on to give a vast generalization of this result. Meanwhile Kalai (private communication) found a proof of (P2) based on algebraic shifting, and asked whether (P1) was also true. In this paper we will prove (P1) (as well as (P2) and hence (P3)) using a somewhat simplified variant of Kalai's methods. In particular, though we work in the exterior algebra as does Kalai, we do not use algebraic shifting.

Let Γ be a directed graph with vertex set $V = \{x_1, \dots, x_n\}$. Recall that a *one-factor* of Γ is a collection $\{e_1, \dots, e_m\}$ of edges which are not loops and such that every vertex is incident to exactly one of the edges. (Thus $n = 2m$.) The following lemma is closely related to but not a direct consequence of [5, Thm. 2.1] [12, Thm. 6.2].

Lemma 1.1. *Let Γ be a directed graph on the n-element vertex set X. Let KX be the K-vector space with basis X. Suppose there is a linear transformation $\varphi : KX \to KX$ satisfying the two properties:*

(a) *If $x \in X$ then*

$$\varphi(x) \in \operatorname{span}_K \{y \in X \colon (x, y) \text{ is an edge of } \Gamma\}.$$

(b) $\operatorname{im} \varphi = \ker \varphi$. (*Equivalently φ is nilpotent with all Jordan blocks of size two.*)

Then Γ has a one-factor Ψ. In fact, if X' is a subset of X whose image in $KX/(\operatorname{im} \varphi)$ is a basis for $KX/(\operatorname{im} \varphi)$, then Ψ can be chosen so that

$$X' = \{x \in X \colon (x, y) \in \Psi \text{ for some } y \in X\},$$

i.e., X' is the set of initial vertices of the edges in Ψ.

Proof. Let X' be a subset of X whose image in $KX/(\operatorname{im} \varphi)$ is a basis for $KX/(\operatorname{im} \varphi)$. Since for any $\varphi : KX \to KX$ we have $\dim(\ker \varphi) + \dim(\operatorname{im} \varphi) = n$, there follows

$|X'|=|X-X'|=n/2$. By the Marriage Theorem (e.g., [1, Thm. 2.2.1] [11, Ch. 5, Thm. 1.1]) it suffices to show that for any $S\subseteq X'$, say with $|S|=k$, there are (at least) k vertices $y_1,\dots,y_k\in X-X'$ such that for each $1\leqslant i\leqslant k$ there is an $x\in S$ with (x,y_i) an edge of Γ. Suppose not. Let $S=\{x_1,\dots,x_k\}$. Then $\varphi(x_1),\dots,\varphi(x_k)$ are linearly dependent in KX/KX', since they are all in the span of fewer than k vertices of Γ. Thus some linear combination $a_1x_1+\cdots+a_kx_k$ ($a_i\in K$, not all $a_i=0$) belongs to $\ker\varphi$. But $\ker\varphi=\operatorname{im}\varphi$, and $x_1,\dots,x_k$ are linearly independent modulo $\operatorname{im}\varphi$. Hence all $a_i=0$, a contradiction. □

Theorem 1.2. *Let Δ be an acyclic simplicial complex over some field K. Then there exists a subcomplex $\Delta'\subset\Delta$ and a bijection $\eta:\Delta'\to\Delta-\Delta'$ such that for all $F\in\Delta'$ we have $F\subset\eta(F)$ and $|\eta(F)-F|=1$.*

Proof. Let C^i be the space of i-cochains of Δ (i.e., the K-vector space $K\Delta^i$, where Δ^i is the set of i-dimensional faces of Δ). Let

$$0\to C^{-1}\xrightarrow{\delta^0}C^0\xrightarrow{\delta^1}\cdots\xrightarrow{\delta^{d-1}}C^{d-1}\to 0 \tag{1}$$

be the augmented oriented cochain complex used to define the reduced simplicial cohomology of Δ. Let $C=K\Delta=C^{-1}\oplus\cdots\oplus C^{d-1}$. The coboundary maps δ^i define an overall coboundary map $\delta:C\to C$. The statement that Δ is acylic over K is equivalent to (1) being exact, i.e., $\ker\delta=\operatorname{im}\delta$.

Suppose Δ has vertex set $V=\{v_1,\dots,v_r\}$. Let $\Lambda(KV)$ denote the exterior algebra of the vector space KV. For simplicity we denote the product in $\Lambda(KV)$ by juxtaposition, rather than the more customary $\wedge$. Let I_Δ denote the two-sided ideal of $\Lambda(KV)$ generated by all monomials $v_{i_1}\cdots v_{i_s}$ where $\{v_{i_1},\dots,v_{i_s}\}\notin\Delta$. The quotient algebra $\Lambda[\Delta]:=\Lambda(KV)/I_\Delta$ is called the *exterior face ring* of Δ (over K). It has a K-basis consisting of all *face monomials* $v^F:=v_{i_1}\cdots v_{i_s}$, where $i_1<\cdots<i_s$ and $F=\{v_{i_1},\dots,v_{i_s}\}\in\Delta$. (In particular, $v^\emptyset=1$.) Thus we may identify as a vector space $\Lambda[\Delta]$ with C, by identifying $x^F\in\Lambda[\Delta]$ with $F\in C$. The coboundary map $\delta:C\to C$ then becomes right multiplication by the element $v=v_1+v_2+\cdots+v_r$ of $\Lambda[\Delta]$.

Consider the quotient space $Q=\Lambda[\Delta]/\Lambda[\Delta]v\cong C/(\operatorname{im}\delta)$, where $\Lambda[\Delta]v$ denotes the right ideal $\{xv\colon x\in\Lambda[\Delta]\}$. (Actually, $\Lambda[\Delta]v$ is a two-sided ideal and the space Q has the structure of a ring, but we do not need this fact here.) Let L be the lexicographically least basis (analogous to the definition of the basis M in the proof of [14, Thm. 2.1]) of Q consisting of face monomials x^F, with respect to the ordering $v_1<\cdots<v_r$ of the vertices of Δ. Thus if $G\in\Delta$ and $x^G\notin L$, then we have a linear relationship

$$x^G=a_1x^{F_1}+\cdots+a_tx^{F_t}+yv \tag{2}$$

in $\Lambda[\Delta]$, where $y\in\Lambda[\Delta]$, $a_i\in K$, and each $F_i<G$ (in the lexicographic order just defined). Let $\Delta'=\{F\in\Delta\colon x^F\in L\}$. We claim that Δ' is a subcomplex of Δ. This is a standard argument, first done in the context of quotients of polynomial rings

by Macaulay (see [14, Thm. 2.1]). Namely, suppose $x^G \notin L$ and $F \supset G$. We need to show $x^F \notin L$. Multiply (2) on the left by x^{F-G}. Since in $\Lambda[\Delta]$ we have $x^{F-G}x^{F_i} = \pm x^{(F-G)\cup F_i}$, we obtain an expression for x^F as a linear combination of earlier (in lexicographic order) monomials $x^{(F-G)\cup F_i}$ modulo the space $K[\Delta]v$. Hence $x^F \notin \Delta'$ as desired, so Δ' is a simplicial complex.

Now let Γ be the directed graph whose vertex set is Δ, and whose edges are the pairs (F,G) with $F \subset G \in \Delta$ and $|G-F|=1$. Define $\varphi: K\Delta \to K\Delta$ by $\varphi=\delta$. By definition of the coboundary operator δ, we have that φ satisfies condition (a) of Lemma 1.1, while (b) holds since Δ is acyclic. We can take $X'=\Delta'$ in Lemma 1.1 by the definition of Δ'. Thus by Lemma 1.1 Δ has a one-factor $\Psi=\{(F,G)\colon F\in\Delta'\}$, so we can define $\eta: \Delta' \to \Delta - \Delta'$ by $\eta(F)=G$, where $(F,G)\in\Psi$. □

Corollary 1.3 ([8]). *Fix a field K. Let $(f_0,\dots,f_{d-1})\in\mathbb{Z}^d$. The following conditions are equivalent:*

(a) *There exists a $(d-1)$-dimensional acyclic (over K) simplicial complex Δ with $f(\Delta)=(f_0,\dots,f_{d-1})$.*

(b) *There exists a $(d-2)$-dimensional simplicial complex Δ' whose f-vector $f(\Delta')=(f'_0,\dots,f'_{d-2})$ is given by*

$$(1+x)\sum_{i=0}^{d-2} f'_i x^i = \sum_{i=0}^{d-1} f_i x^i.$$

The Kruskal–Katona theorem gives an explicit characterization of the f-vectors $(f'_0,\dots,f'_{d-2})$ arising in (b) above. See e.g. [1, Ch. 7.3], [7, Thm. 8.5].

2. Variations and generalizations

There are many possible directions for extending Theorem 1.2. We discuss in this section some of the possibilities. For the most interesting ones we can only offer conjectures and not proofs. I am grateful to A. Björner, G. Kalai, and J. Munkres for helpful discussions related to the work in this section.

First we can ask whether there is a generalization of Theorem 1.2 valid for *any* simplicial complex. We regard the field K as fixed throughout. Given any simplicial complex Δ, a *Betti set* is a subset $B \subseteq \Delta$ such that for all i,

$$\#\{F\in B\colon \dim F = i\} = \tilde{\beta}_i(\Delta).$$

Here

$$\tilde{\beta}_i(\Delta) = \dim \tilde{H}^i(\Delta; K),$$

the i-th reduced Betti number of Δ (over K).

Proposition 2.1. *Any (finite) simplicial complex Δ can be written as a disjoint union $\Delta=\Delta'\cup B\cup\Omega$, where:*

(a) *Δ' is a subcomplex of Δ,*

(b) *B is a Betti set,*

(c) *$\Delta'\cup B$ is a subcomplex of Δ,*

(d) *there exists a bijection $\eta:\Delta'\to\Omega$ such that for all $F\in\Delta'$ we have $F\subset\eta(F)$ and $|\eta(F)-F|=1$.*

Proof (sketch). The proof is a simple extension of the proof of Theorem 1.2. Using notation from that proof, let L be the lexicographically least basis of $\Lambda[\Delta]/(\ker\delta)$ consisting of face monomials x^F. Let M be the lexicographically least set of face monomials for which $L\cup M$ is a basis of $\Lambda[\Delta]/(\mathrm{im}\,\delta)$. Choose $\Delta'=\{F\in\Delta\colon x^F\in L\}$, $B=\{F\in\Delta\colon x^F\in M\}$. By definition of $\tilde{H}^i(\Delta;K)$ it follows that B is a Betti set, while assertions (a) and (c) follow by reasoning similar to that used in the proof of Theorem 1.2. □

Unfortunately Proposition 2.1 is not a satisfactory generalization of Theorem 1.2, because it is not strong enough to imply the Björner–Kalai characterization [3, 4] of pairs (f,β) such that f is the f-vector and β the sequence of Betti numbers of some simplicial complex. The result actually needed (as first observed by Kalai) is the following[1].

Conjecture 2.2. The Betti set B in Proposition 2.1 can be chosen to be an *antichain*, i.e., rif $F,G\in B$ and $F\subseteq G$, then $F=G$.

Another conjectured generalization of Theorem 1.2 is motivated by the following result.

Proposition 2.3. *Fix an integer $k\geqslant 0$. The following conditions are equivalent on a vector $f=(f_0,f_1,\ldots,f_{d-1})$.*

(a) *There exists a simplicial complex Λ with f-vector f such that if $F\in\Delta$ and $|F|\leqslant k$, then $\mathrm{lk}(F)$ is acyclic. Here $\mathrm{lk}(F)$ denotes the link of F in Δ.*

(b) *There exists a simplicial complex on the vertex set V with f-vector f such that if $W\subseteq V$ and $|V-W|\leqslant k$, then the restriction Δ^W of Δ to W is acyclic.*

(c) *There exists a simplicial complex Δ' with f-vector $(f'_0,f'_1,\ldots,f'_{d-k-2})$ such that*

$$\sum_{i=0}^{d} f_{i-1}x^i=(1+x)^{k+1}\sum_{i=0}^{d-k-1} f'_{i-1}x^i.$$

Proof. (a) $\Leftrightarrow$ (b) Straightforward topological argument.

(c) $\Rightarrow$ (a) Let σ be a k-simplex and take $\Delta=\sigma*\Delta'$ (the join of σ and Δ').

[1] Conjecture 2.2 was proved by Art Duval after this paper was completed.

(a)$\Rightarrow$(c) Let Γ be the simplicial complex obtained from Δ by algebraic shifting (over the field K), as described in [3, Thm. 3.1] (but with a different notation). I am grateful to G. Kalai for informing me that algebraic shifting preserves the property that $\mathrm{lk}(F)$ is acyclic for all $F\in\Delta$ satisfying $|F|\leqslant k$, so Γ also has this property. It is easy to see that any shifted complex (as defined in [3]) with this property is the join of the simplex $\sigma=2^{\{1,2,\ldots,k+1\}}$ with another simplicial complex Δ'. Since Γ is shifted and $f(\Delta)=f(\Gamma)$ [3, Thm. 3.1], we have

$$\sum_{i=0}^{d} f_{i-1}(\Delta)x^i=\Sigma f_{i-1}(\sigma*\Delta')x^i$$

$$=(1+x)^{k+1}\Sigma f_{i-1}(\Delta')x^i,$$

as desired. □

Note. It is easy to see, without using any algebraic machinery, that if Δ satisfies the condition of Proposition 2.3(a) then the polynomial $F(x)=\Sigma f_{i-1}x^i$ is divisible by $(1+x)^{k+1}$. But we do not see how to show without algebraic shifting even that the quotient $F(x)/(1+x)^{k+1}$ has nonnegative coefficients.

The 'simplest' example of a simplicial complex Δ satisfying the condition of Proposition 2.3(a) is just the join $\sigma*\Delta'$ used to prove (c)$\Rightarrow$(a). In this case Δ (regarded as a poset under inclusion) can be partitioned into intervals $[F,G]$, $F\in\Delta'$, each of length $k+1$. Moreover, the bottom elements F of these intervals form a simplicial complex (namely, Δ'). This motivates the following conjecture.

Conjecture 2.4. Let k and Δ be as in Proposition 2.3(a). Then Δ can be partitioned into disjoint intervals $[F,G]$, all of length $k+1$. Moreover, the bottoms F of these intervals form a subcomplex Δ' of Δ.

There is a further and well known conjecture concerning partitioning of simplicial complexes Δ [15, p. 149] [6, Rmk. 5.2]. Namely, if Δ is Cohen–Macaulay then it can be partitioned into disjoint intervals $[F,G]$ such that the tops G of these intervals are all facets of Δ. We may extend this conjecture as follows: Suppose depth $K[\Delta]=\delta$, where $K[\Delta]$ denotes the face ring (or Stanley–Reisner ring) of Δ. (A topological description of depth $K[\Delta]$ is given by Munkres [10, Thm. 3.1].) Then (conjecturally) Δ can be partitioned into disjoint intervals $[F,G]$ such that the tops G of these intervals satisfy $\dim G\geqslant\delta$. We can combine this conjecture with Conjecture 2.4 as follows.

Conjecture 2.5. Suppose Δ satisfies the condition of Proposition 2.3(a), and let depth $K[\Delta]=\delta$. Then Δ can be partitioned into disjoint intervals $[F,G]$ satisfying:

$$\dim G\geqslant\delta,\qquad \dim G-\dim F=k+1.$$

Let us now consider some generalizations of the concept of simplicial complex itself. There are three successively more general classes of complexes with which we will be concerned. All these complexes are always assumed to be finite.

(A) CW-semilattices, or equivalently, regular CW-complexes for which the intersection of two closed cells is a closed cell.

(B) CW-posets, or equivalently, regular CW-complexes. These are essentially posets P with $\hat{0}$ such that the order complex of every open interval $(\hat{0}, x)$ is a triangulation of a sphere. If P is in addition a semilattice, then we get a CW-semilattice. For further information, see [2].

(C) CW-complexes, as defined e.g. in [9, 13].

The possible (f, β)-pairs of all three classes (A)–(C) have been characterized. (A) appears in [4, Thm. 1] while (B) and (C) are in [3, Thm. 6.1]. In all three cases, the characterization is implied by (but does not directly imply) decomposition properties analogous to our previous results and conjectures. We first state these properties and then discuss their validity. We omit the easy arguments which show that these decompositions do indeed imply the (f, β)-characterizations of Björner and Kalai. As in the case of simplicial complexes, a *Betti set* (over K) of a more general complex P is a subset B of its closed cells such that $\tilde{\beta}_i(P) = \#\{F \in P\colon \dim F = i\}$, where $\tilde{\beta}_i(P)$ denotes the ith reduced Betti number of P (over the coefficient field K).

(A′) (CW-semilattices). We can write P as a disjoint union $P = P' \cup Q$, where: (a) P' is a subcomplex of P (i.e., an order of ideal of P, regarded as a poset), (b) B is a Betti set, (c) B is an antichain, (d) $P' \cup B$ is a subcomplex of P, and (e) there exists a bijection $\eta: P' \to Q$ such that for all $F \in P'$ we have $F < \eta(F)$ and $\dim \eta(F) - \dim F = 1$. (Note that these conditions are exactly analogous to those of Proposition 2.1 and Conjecture 2.2.)

(B′) (CW-posets). We can write P as a disjoint union $P = P' \cup B \cup Q$ such that (a) P' contains faces of dimensions $-1, 0, 1, \ldots, \dim P - 1$, (b) B is a Betti set, and (c) same as condition (e) of (A′).

(C′) (CW-complexes). We can write the set P of closed cells as a disjoint union $P = P' \cup B \cup Q$ such that (a) B is a Betti set, and (b) there exists a bijection $\eta: P' \to Q$ such that for all $F \in P'$ we have that F lies on the boundary of $\eta(F)$ and $\dim \eta(F) - \dim F = 1$. (We do not say $F < \eta(F)$ as in (A′) and (B′), since for general CW-complexes the closure of every cell need not be a union of open cells, so there is no nice poset structure on the cells. Moreover, even if the closure of every cell is a union of open cells, the poset of closed cells ordered by inclusion does not determine the CW-complex or even its homology.)

The status of the possible decompositions (A′)–(C′) is as follows. For an arbitrary CW-complex P, let C^k be the space of k-chains, i.e., the K-vector space with basis consisting of the k-dimensional open cells of P. Then there exists a coboundary map $\delta = \delta^k: C^k \to C^{k+1}$ such that (a) for any k-cell F, we have that $\delta(F)$ is a linear combination of $(k+1)$-cells G whose closure $\bar{G}$ contains F, (b) $\delta\delta = 0$, and (c) $H^k(|P|; K) \cong (\ker \delta^k)/(\operatorname{im} \delta^{k-1})$, where $H^k(|P|; K)$ denotes singular cohomology and $|P| = \bigcup_{F \in P} F$. These facts follow, e.g., from [9, §39] (which deals with homology instead of cohomology, but the arguments are essentially the same in either case). We then obtain (C′) by reasoning as in the proof of Theorem 1.2 and Proposition 2.1, but taking L and M to be *any* set of faces forming a basis of the appropriate vector spaces.

To prove (B′), we need only to show in addition that for $k<\dim P$ there exists a k-chain $u\in C^k$ such that $\delta(u)\neq 0$. But if F is any k-cell then we can take $u=F$. Thus (B′) and (C′) are true, so we have a combinatorial refinement of the (f,β)-characterizations in these cases.

There remains the case (A′). Here we do not see how to show properties (a), (c) and (d) so this case remains open. Even when P is acyclic (so $B=\emptyset$, (c) is trivially true, and (d) follows from (a)), we do not see how to prove (a).

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