H. Combinatorial Communications: Some Results on the Capacity of Graphs, *R. J. McEliece, R. P. Stanley, and H. Taylor*

1. Introduction

Let C be a finite discrete memoryless channel which is specified by its transition probability matrix $[p(j|i)]$, where $p(j|i)$ is the probability that input letter i will be received as output letter j $(i = 1, 2, \cdots, I;\ j = 1, 2, \cdots, J)$. In 1956, Shannon (Ref. 1) defined the *zero-error capacity* of such a channel as the least upper bound of rates at which it is possible to transmit information at zero-error probability. Shannon observed that the *adjacency graph* of C is fundamental to the study of zero-error capacity. The adjacency graph G has I vertices $v_1, v_2, \cdots, v_I$; v_i is connected to $v_{i'}$ if there is an output letter j such that $p(j|i)$ and $p(j|i')$ are both nonzero. That is, v_i and $v_{i'}$ are connected in G if the input letters i and i' can be confused by the channel. The importance of the adjacency graph may be seen as follows:

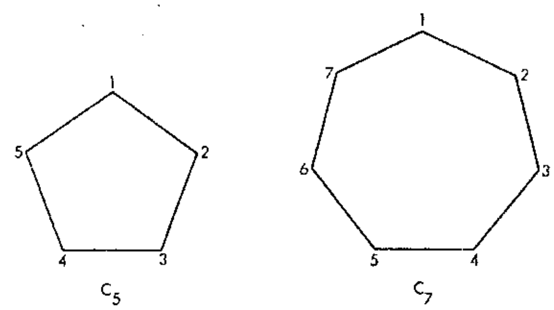
Suppose $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ and $\mathbf{y} = (y_1, y_2, \cdots, y_n)$ are two codewords in a zero-error probability block code of length n from C. Now x and y can be confused by the channel only if x_i and y_i can be confused for each i; but this is equivalent to saying that $\mathbf{x}$ and $\mathbf{y}$ are connected in the graph G^n. [If G is a graph, the direct power G^n has as vertices the set of n-tuples $(v_1, v_2, \cdots, v_n)$, where v_i are vertices of G; $(v_1, v_2, \cdots, v_n)$ and $(v'_1, v'_2, \cdots, v'_n)$ are connected if and only if for each i either $v_i = v'_i$ or v_i, and v'_i are connected in G.] Thus, the number of words in the largest error-free code of length n from C is the largest number of vertices in G^n, no two of which are adjacent. Berge (Ref. 2) defines the *coefficient of internal stability* of any graph G, $\alpha(G)$, as the largest number of vertices of G which may be chosen such that no two are adjacent. Hence, using Berge's notation, the zero-error capacity of C is

$$\sup_n \frac{1}{n} \log \alpha(G^n)$$

(And, in fact, it is not hard to see that we may replace "sup" by "lim".) This leads us to define the *capacity* of any finite undirected graph G as

$$\text{cap}(G) = \lim_{n\to\infty} \frac{1}{n} \log \alpha(G^n)$$

It turns out that for most graphs G, $\text{cap}(G) = \log \alpha(G)$; but for the few which have $\text{cap}(G) > \log \alpha(G)$, $\text{cap}(G)$ is unknown! It is the object of this article to study the functions α and cap, especially as applied to the so-called *odd cycle* graphs C_{2m+1}, which have an odd number of vertices $v_1, v_2, \cdots, v_{2m+1}$ and for which v_i is connected to v_j if and only if $i - j \equiv \pm 1 \pmod{2m+1}$. C_5 and C_7 are illustrated below:



It is very easy to show that

$$\log \frac{p-1}{2} \leq \text{cap}(C_p) \leq \log \frac{p}{2}$$

We shall be able to increase this lower bound for all odd $p \geq 5$; in particular, we shall show

$$\alpha(C_p^2) = \left\lfloor \frac{p^2 - p}{4} \right\rfloor$$

for all odd p, and that

$$\text{cap}(C_p) > \frac{1}{2} \log \alpha(C_p^2)$$

for infinitely many p, including $p = 7$ and $p = 9$.

2. A Useful Result About α

We begin with some definitions; throughout G is a finite undirected graph. A *clique* in G is a set of vertices of G such that every pair is connected in G. A *brouhaha* is a set of vertices, no two of which are connected. The *dual graph* $\overline{G}$ of G has the same vertices as G, but v and v' are connected in $\overline{G}$ if and only if they are not connected in G. Thus, a clique in G is a brouhaha in $\overline{G}$, and conversely. Finally, if H is another graph, and if each vertex of G is part of exactly r subgraphs of G isomorphic to H for some fixed r, we say that G is *H-regular*. (Thus, ordinary regularity is a special case of H-regularity, where H is the graph consisting of two connected points.)

THEOREM 1. *If G is H-regular, then $\alpha(G)/|G| \leq \alpha(H)/|H|$. ($|G|$ is the number of vertices in G.)*

Proof. Suppose X is the largest brouhaha in G; $|X| = \alpha(G)$. Then for each subgraph H' of G which is isomorphic to H, the vertices of H' which are members of X form a brouhaha in H', and so $|X \cap H'| \leqq \alpha(H') = \alpha(H)$. Since each vertex of G is part of r copies of H and each copy of H involves $|H|$ vertices of G, there are $|G| \cdot r/|H|$ copies of H altogether; thus,

$$\alpha(G) \cdot r = \sum_{\text{copies of } H} |X \cap H'| \leqq \alpha(H) \cdot \left(\frac{|G| \cdot r}{|H|}\right)$$

and the theorem follows.

Corollary 1 (the sphere-packing bound). If G is H-regular and H is a *clique*, then $\alpha(G) \leqq |G|/|H|$ and

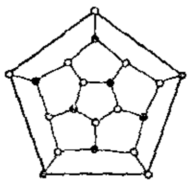
$$\text{cap}(G) \leqq \log\left(\frac{|G|}{|H|}\right)$$

Proof. The first part follows, since $\alpha(H) = 1$ if H is a clique. The second part follows from the fact that G^n is H^n-regular for all n and H^n is a clique.

Corollary 2. More generally, if G is H-regular,

$$\text{cap}(G) \leqq \text{cap}(H) + \log\left(\frac{|G|}{|H|}\right)$$

Let us give one example of the use of Theorem 1. Suppose G is the graph of the regular dodecahedron:



The eight distinguished vertices show that $\alpha(G) \geqq 8$; on the other hand, G is C_5-regular (C_5 is the pentagon). Thus, by Theorem 1 $\alpha(G) \leqq (20/5) \cdot \alpha(C_5) = 8$, and so $\alpha(G) = 8$.

3. The Graphs C_p

In this subsection, we shall restrict our attention to the graphs C_p for odd $p \geqq 5$ mentioned in *Subsection 1*. If the vertices are numbered $1, 2, \cdots, p$, the vertices $1, 3, 5, \cdots, p-2$ are a brouhaha, so that

$$\alpha(C_p) \geqq \frac{p-1}{2}$$

On the other hand, C_p is C_2-regular, so that by Theorem 1 $\alpha(C_p) \leqq p/2$. Thus, $\alpha(C_p) = (p-1)/2$, and so

$$\text{cap}(C_p) \geqq \log\frac{p-1}{2}$$

Also, by Corollary 2 to Theorem 1, $\text{cap}(C_p) \leqq \log p/2$. We shall not be able to decrease this upper bound for any p, but Theorem 2 increases the lower bound for all odd $p \geqq 5$.

THEOREM 2. *$\alpha(C_p^2) = \lfloor (p^2 - p)/4 \rfloor$, and so*

$$\text{cap}(C_p) \geqq \frac{1}{2}\log\left\lfloor \frac{1}{4}(p^2 - p)\right\rfloor$$

Proof. We first show that $\alpha(C_p^2) \leqq \lfloor (p^2 - p)/4 \rfloor$. This follows directly from Theorem 1, since C_p^2 is $C_p \times C_2$-regular, and

$$\alpha(C_p \times C_2) = \alpha(C_p) = \frac{p-1}{2}$$

To show that $\alpha(C_p^2) \geqq \lfloor (p^2 - p)/4 \rfloor$ as well, we must explicitly exhibit a brouhaha in C_p^2 of that size:

For $p \equiv 1 \pmod 4$, say $p = 4a + 1$:

$$(t, 2t + 4s)\begin{cases} t = 0, 1, \cdots, p-1 \\ s = 0, 1, \cdots, a-1 \end{cases}$$

For $p \equiv 3 \pmod 4$, say $p = 4a + 3$:

$$(2s, 2t + s)\begin{cases} s = 0, 1, \cdots, 2a+1 \\ t = 0, 1, \cdots, a-1 \end{cases}$$

and

$$(2s + 1, 2t + s + 2a + 1)\begin{cases} s = 0, 1, \cdots, 2a \\ t = 0, 1, \cdots, a \end{cases}$$

We omit the straightforward but tedious verification that these sets of vertices do form brouhahas and regard Theorem 2 as proved.

Theorem 2 shows that for all odd $p \geqq 5$,

$$\text{cap}(C_p) > \log \alpha(C_p)$$

Indeed, combining our results, we have shown that for $p \geq 5$,

$$\frac{1}{2}\log\left[\frac{1}{4}(p^2-p)\right] \leq \operatorname{cap}(C_p) \leq \log\frac{p}{2}$$

and for $p = 5$, that is where matters have stood since Shannon's original paper. In *Subsections 4* and 5, we will show that this lower bound can be improved for infinitely many p, including $p = 7$ and $p = 9$.

Good Packings in C_p^n for $n \geq 3$

THEOREM 3. $$\alpha(C_p^n) \leq \frac{p}{2}\,\alpha(C_p^{n-1}).$$

Proof. Immediate from Theorem 1, since C_p^n is $C_p^{n-1} \times C_2$-regular, and $\alpha(C_p^{n-1} \times C_2) = \alpha(C_p^{n-1})$.

Corollary. $$\alpha(C_p^n) \leq \frac{(p^n - p^{n-1})}{2^n}.$$

Proof. From Theorem 3 (or Theorem 2),

$$\alpha(C_p^2) \leq \frac{p^2-p}{4}$$

The corollary follows from Theorem 3 by induction on n.

Although for fixed p, as n increases the upper bound of the corollary is doubtless very crude; for fixed n and large p, it is probably very good. In particular, we present the following conjecture:

For all $p \geq 2^n + 1$, $\alpha(C_p^n) = \frac{p}{2}\,\alpha(C_p^{n-1})$

And while we will not be able to prove this conjecture (except for $n = 2$), we will be able to prove Theorem 4, which is related:

THEOREM 4. If $p = 2^n + 1$,

$$\alpha(C_p^n) = \frac{p^n - p^{n-1}}{2^n} = p^{n-1}$$

Proof. We have seen that

$$\alpha(C_p^n) \leq \frac{p^n - p^{n-1}}{2^n}$$

Thus, it remains to exhibit a brouhaha of size p^{n-1}. We claim the following set will do:

$$(x_1, x_2, \cdots, x_{n-1}, x_n)$$

where $x_1, \cdots, x_{n-1}$ are arbitrary and

$$x_n = 2x_1 + 4x_2 + \cdots + 2^{n-1}x_{n-1} \pmod p$$

To see this, suppose $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$ are two vectors in the set, and that they are connected in C_p^n. Then

$$x - y = (x_1 - y_1, \cdots, x_n - y_n) = (z_1, \cdots, z_n)$$

has all coordinates congruent to 0, or $\pm 1 \pmod p$. But since

$$z_n = 2z_1 + \cdots + 2^{n-1}z_{n-1}$$

we must show that if each $z_i, i < n$ is $0, \pm 1 \pmod p$, that $2z_1 + \cdots + 2^{n-1}z_{n-1}$ cannot be. Now let P be the set of indices i for which $z_i = +1$, and let M be the set for which $z_i = -1$. Then, if

$$2z_1 + \cdots + 2^{n-1}z_{n-1} \equiv 0, \pm 1 \pmod p$$

we have a congruence of the form

$$\sum_{i \in P} 2^i \equiv \sum_{j \in M} 2^j \qquad \pmod p$$

(where either P or M has been intended to include 0, if necessary). But unless both sums are empty, they represent different integers of the range $[0, 2^n - 1] = [0, p-2]$ and so cannot be congruent $(\bmod\, p)$.

5. Miscellaneous Results

We present three miscellaneous results concerning $\alpha(C_p^n)$ and $\operatorname{cap}(C_p)$.

THEOREM 5. $\operatorname{cap}(C_7) \geq \frac{3}{5}\log 7 > \frac{1}{2}\log 10.$

Proof. It is easy to verify that the following set of 7^3 vertices is a brouhaha in C_7^5:

$$(x_1, x_2, x_3, 2x_1 + 2x_2 + 2x_3, 2x_1 + 4x_2 + 6x_3)$$

x_1, x_2, x_3 are arbitrary.

THEOREM 6. $31 \leq \alpha(C_7^3) \leq 35.$

Proof. The upper bound comes from Theorem 3 with $p = 7$, $n = 3$. Here is a brouhaha of size 31:

5	3	1				7
		5	3	1		
3	1			5	3	1,5
	5	3	1			
1			5	3	1,5	3
5	3	1				
		5	3	1,6	4	2

It has often been conjectured that $\alpha(C_5^n) = 5\alpha(C_5^{n-2})$ for $n \geqq 3$; this conjecture has been verified by exhaustive enumeration for $n \leqq 4$. [Notice that it is sufficient to verify the conjecture for odd n by Theorem 3, since $\alpha(C_5^n) \geqq 5\alpha(C_5^{n-2})$.]

Now, by a *systematic* brouhaha, we mean a brouhaha like the ones exhibited in Theorems 4 and 5, i.e., one of :he form

$$(x_1, x_2, \cdots, x_k, y_1, y_2, \cdots, y_{n-k})$$

where the x_i's vary freely over the integers (mod p), and he y_i's are uniquely determined by the x's. Let us denote he size of the largest *systematic* brouhaha in C_p^n by $\alpha_{sys}(C_p^n)$. Note that $\alpha_{sys}(C_5^2) = \alpha(C_5^2)$.

THEOREM 7. $\alpha_{sys}(C_5^n) = 5^{[n/2]}$ for $n \leqq 12$. (Thus, there is no systematic brouhaha which improves the bound cap $(C_5) \geqq \frac{1}{2} \log 5$ for $n \leqq 12$.)

Proof. If $k = [n/2]$, then for any n, the set

$$(x_1, x_2, \cdots, x_k, 2x_1, 2x_2, \cdots, 2x_k, 0)$$

a brouhaha of size 5^k in C_5^n. (The 0 coordinate is only resent for odd k.) From $\alpha(C_5^3) = 10$, $\alpha(C_5^4) = 25$, and sing

$$5\alpha(C_5^{n-2}) \leqq \alpha(C_5^n) \leqq \frac{5}{2}\alpha(C_5^{n-1})$$

we obtain the following sequence of upper and lower bounds on $\alpha(C_5^n)$:

n	Lower bound	Upper bound
5	50	62
6	125	155
7	250	387
8	625	967
9	1250	2417
10	3125	6042
11	6250	15105
12	15625	37762
13	31250	94405

Now, since the number of vertices in a systematic brouhaha is always a power of 5, and $5^{[n/2]}$ is the largest power of 5 less than the upper bound $n \leqq 12$, the theorem follows. Finally, notice that $5^7 = 78125 < 94405$, so that it is conceivable that there is a systematic brouhaha for $n = 13$ which would show cap $(C_5) \geqq 7/13 \log 5$. However, if it turns out that $\alpha(C_5^5) \leqq 51$, the above procedure would rule this out.

References

1. Shannon, E. C., "The Zero-Error Capacity of a Noisy Channel," *IRE Trans. Inform. Th.*, IT-2, pp. 8–19, 1956.
2. Berge, C., *The Theory of Graphs.* Methuen & Co., Ltd., London, England, 1962.

I. Combinatorial Communications: Negative Radix Conversion, *S. Zohar*

1. Introduction

In the common positional representation of numbers, negative numbers always present a special case. Thus, a machine that can add $3 + 5$ has to go through a special sequence when the problem is to add $3 + (-5)$.

It has recently been pointed out[0] that a computer mechanization which is completely indifferent to the sign of a number can be built if, instead of the standard positive radix usually adopted in number representation, a negative radix is chosen.

[0]News item in *Electronics*, Vol. 40, No. 26, pp. 40–41, Dec. 25, 1967. The idea is credited to Mauritz P. de Regt.