$\lambda = t - 1$ difference sets, and equivalently cyclic Hadamard matrices, can, to a greater extent than would be possible without the multiplier theorem, be approached by performing an exhaustive search on a digital computer.

In employing the above technique, for each different assignment of 1's and -1's to the individual cosets, the first test is to determine if the number of -1's that were accordingly assigned to the individual members of the sequence of length $N - 1$ is equal to $N/2$. Only those sequences which satisfy this condition require further testing; the others are no longer candidates. Next, the autocorrelation is computed, beginning with a phase difference of one element and increasing this difference until either the autocorrelation for a particular phase shift is other than -1 or the phase difference becomes $(N - 1)/2$. In either case, the disposition of the test sequence is determined: failure to generate the proper Hadamard matrix if the autocorrelation is anywhere other than -1 (except when in phase); success if the phase difference reaches $(N - 1)/2$ without the -1 condition being violated.

Table 1. Existence of cyclic Hadamard matrices[a]

b \ a	0	100	200	300	400	500	600	700	800	900
4	E	E	U	S	S	E	U	U	U	U
8	E	E	S	E	S	S	E	S	S	E
12	E	S	E	E	S	E	S	U	E	E
16	E	S	S	U	S	S	U	U	S	U
20	E	S	S	S	E	S	E	E	U	E
24	E	S	E	E	S	E	U	U	E	S
28	S	E	E	S	S	S	U	E	E	S
32	E	E	U	E	E	U	E	S	U	U
36	E	U	S	S	U	S	S	U	S	U
40	S	E	E	S	E	S	U	E	E	S
44	E	E	S	S	E	S	E	E	U	U
48	E	S	U	E	S	E	E	S	S	E
52	S	E	E	S	U	S	U	E	S	S
56	S	S	E	S	S	U	S	S	U	S
60	E	S	S	E	S	U	E	U	E	S
64	E	E	E	S	E	E	U	U	E	S
68	E	E	U	E	E	U	S	S	S	E
72	E	U	E	S	S	E	U	U	S	E
76	S	U	U	S	U	S	S	S	S	U
80	E	E	S	E	E	S	S	U	S	S
84	E	S	E	E	S	S	E	U	E	E
88	S	S	U	U	E	E	U	E	E	S
92	S	E	S	S	E	S	E	S	U	E
96	S	S	S	S	U	U	S	U	S	S
100	S	E	U	U	E	E	S	S	E	S

[a]Key: E ~ Cyclic Hadamard matrix known to exist from previously published results.
S ~ Cyclic Hadamard matrix derivable from difference set found not to exist by this study.
U ~ Unsettled.
$N = a + b$, order of the Hadamard matrix.

3. Results

Although many irregularities occur, the number of cosets which are generated for a particular value of N generally increases as N grows in size. This fact and the increased length of the generated sequences for larger values of N combine to increase, in the extreme, computer time required to settle the issue for larger matrices. The cases considered were confined to $N \leq 1000$ (Table 1). Even then, 54 of these 250 cases could not be tested, due to the time each would have required on the USC Honeywell 800 computer. Of the 196 remaining cases, 95 had already been settled by virtue of $N - 1$ being prime or a twin prime product, or N being a power of 2. The 101 cases which could be tested in what was deemed a reasonable length of computer time (the longest required on the order of four hours of machine time, the shortest only seconds, and the others, with few exceptions, less than 30 min) revealed that no cyclic Hadamard matrix derivable from a difference set exists for these values.

B. Combinatorial Communications: Enumeration of a Special Class of Permutations

R. Stanley

1. Summary

Let π be a permutation $a_1, a_2, \cdots, a_n$ of the symbols $1, 2, \cdots, n$. We say that π contains a *run* of length t, if for some i and k, $1 \leq i \leq n$, $1 \leq k \leq n$, we have $a_i = k$, $a_{i+1} = k + 1, \cdots, a_{i+t-1} = k + t - 1$. All numbers are regarded as taken mod n; for instance, the permutation 2 3 5 4 6 1 contains the run 6 1 2 3 of length 4. Let $F(n, t)$ denote the number of permutations of $1, 2, \cdots, n$ which contain no runs of length t (or greater). Our object is to estimate the function $F(n, t)$ for fixed t.

In Section 2, we consider the case $t = 2$. We obtain a recurrence relation for $F(n, 2)$, compute its generating function, give a closed expression for it, and derive its complete asymptotic expansion. In particular we show that the probability that a random permutation of $1, 2, \cdots, n$ has no runs of length 2, approaches $1/e$ as $n \to \infty$.

The situation becomes more complicated when $t > 2$. We are unable to give as exact results as obtained for $t = 2$, but use of the principle of inclusion and exclusion enables us to develop a method which yields the asymptotic expansion of $F(n,t)$. Here we get the probability that a random permutation of $1, 2, \cdots, n$ has no runs of length t, $t > 2$, approaches 1 as $n \to \infty$.

2. Permutations With No Runs of Length Two

It is convenient to work with the function $G(n,t) = (1/n) F(n,t)$. If we consider two permutations as equivalent, if they are cyclic shifts of each other, then $G(n,t)$ enumerates the number of equivalence classes of permutations containing no permutation with a run of length t.

We claim that the number of classes of cyclically equivalent permutations containing precisely k runs of length 2, $1 \leq k \leq n-2$, is $\binom{n}{k} G(n-k, 2)$. For there are $\binom{n}{k}$ ways of choosing which k runs occur, namely, the $\binom{n}{k}$ ways of choosing k elements from the set $\{12, 23, 34, \cdots, n1\}$. Each choice partitions the set $\{1, 2, \cdots, n\}$ into $n-k$ subsets, each subset representing symbols which must remain adjacent. For instance, the choice 12, 23, 56 when $n = 7$ gives the partition 123, 4, 56, 7. The number of cyclically inequivalent ways of rearranging the parts of the partition without introducing any new runs is clearly $G(n-k, 2)$, and the assertion follows.

When $k = n - 1$ the argument breaks down, as the partition $123 \cdots n$ introduces the extra run $n1$. If we define $G(0,t) = 1$, then the above result is also valid for $k = n$. Hence, we obtain the recurrence relation

$$G(n,2) = (n-1)! - \left[\sum_{k=1}^{n} \binom{n}{k} G(n-k, 2) - n\right] \quad (1)$$

The $-n$ term appears to cancel the term $\binom{n}{1} G(1,2) = n$. If we put $G_k = G(k,2)$, then Eq. (1) can be written symbolically as

$$(n-1)! + n = (1 + G)^n, \quad (2)$$

where exponents are changed to subscripts after expanding the right side by the binomial theorem. Eq. (2) provides a rapid method for calculating $G(n,2)$. Table 2 gives the values for $1 \leq n \leq 15$.

We now define the generating functions

$$G(x) = \sum_{n=0}^{\infty} \frac{G(n,2)}{n!} x^n,$$

$$F(x) = \sum_{n=0}^{\infty} \frac{F(n,2)}{n!} x^n,$$

Table 2. Values of $G(n,2) = 1/n\, F(n,2)$

n	$G(n,2)$
1	1
2	0
3	1
4	1
5	8
6	36
7	229
8	1,625
9	13,208
10	120,288
11	1,214,673
12	13,496,897
13	162,744,944
14	2,128,047,988
15	29,943,053,061

Clearly $F(x) = 1 + xG'(x)$. We compute

$$G(x)\,e^x = \left(\sum_{n=0}^{\infty} \frac{G(n,2)}{n!} x^n\right)\left(\sum_{m=0}^{\infty} \frac{x^m}{m!}\right)$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{G(k,2)}{k!\,(n-k)!}\right) x^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{k=0}^{n} \binom{n}{k} G(k,2)\right) x^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \Big[n + (n-1)!\Big] x^n$$

$$= \sum_{n=1}^{\infty} \frac{x^n}{n} + x \sum_{n=0}^{\infty} \frac{x^n}{n!} + 1$$

$$= -\log(1-x) + xe^x + 1.$$

Hence

$$G(x) = e^{-x}[1 - \log(1-x)] + x, \quad (3)$$

and

$$F(x) = 1 + xG'(x) = xe^{-x}\left(\frac{x}{1-x} + \log(1-x)\right) + x + 1.$$

From Eq. (3) we get

$$\sum_{n=0}^{\infty} \frac{G(n,2)}{n!} x^n = \left(\sum_{i=0}^{\infty} \frac{(-1)^i x^i}{i!}\right)\left(1 + \sum_{j=1}^{\infty} \frac{x^j}{j}\right) + x$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n-1} \frac{(-1)^k}{k!\,(n-k)}\right) x^n + \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} + x$$

Equating coefficients gives

$$G(n,2) = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!\,(n-k)} + (-1)^n, \qquad n \neq 1 \tag{4}$$

and

$$F(n,2) = nG(n,2) = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} \frac{n}{n-k} + n(-1)^n, \qquad n \neq 1. \tag{5}$$

Eqs. (4) and (5) can also be obtained from the principle of inclusion and exclusion (Ref. 3, Ch. 3 or Ref. 5, Ch. 2).

We now use Eq. (5) to obtain the asymptotic expansion of $F(n,2)/n!$. We have

$$\frac{F(n,2)}{n!} = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \frac{n}{n-k} + \frac{(-1)^n}{(n-1)!}$$

$$= \sum_{k=0}^{n} \frac{(-1)^k}{k!} \left[1 + \frac{k}{n} + \left(\frac{k}{n}\right)^2 + \cdots + \left(\frac{k}{n}\right)^r + \frac{n}{n-k}\left(\frac{k}{n}\right)^{r+1} \right] + \frac{(-1)^n}{(n-1)!}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left[1 + \frac{k}{n} + \left(\frac{k}{n}\right)^2 + \cdots + \left(\frac{k}{n}\right)^r \right] + \sum_{k=0}^{n} \frac{(-1)^k}{k!} \frac{n}{n-k} \left(\frac{k}{n}\right)^{r+1} - \sum_{k=n+1}^{\infty} \frac{(-1)^k}{k!} \left[1 + \frac{k}{n} + \left(\frac{k}{n}\right)^2 + \cdots + \left(\frac{k}{n}\right)^r \right] + \frac{(-1)^n}{(n-1)!}.$$

Hence

$$n^r \left| \frac{F(n,2)}{n!} - \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left[1 + \frac{k}{n} + \left(\frac{k}{n}\right)^2 + \cdots + \left(\frac{k}{n}\right)^r \right] \right|$$

$$= \left| \sum_{k=0}^{n-1} \frac{(-1)^k k^{r+1}}{k!\,(n-k)} - n^r \sum_{k=n+1}^{\infty} \frac{(-1)^k}{k!} \left[\frac{1 - \left(\frac{k}{n}\right)^{r+1}}{1 - \frac{k}{n}} \right] + \frac{(-1)^n n^r}{(n-1)!} \right|$$

$$\leq \sum_{k=0}^{\log n} \frac{k^{r+1}}{k!\,(n-k)} + \sum_{k=\log n}^{n-1} \frac{k^{r+1}}{k!\,(n-k)} + n^r \left| \sum_{k=n+1}^{\infty} \frac{(-1)^k}{k!} \frac{1 - \left(\frac{k}{n}\right)^{r+1}}{1 - \frac{k}{n}} \right| + \frac{n^r}{(n-1)!}$$

$$< \frac{(\log n)(\log n)^{r+1}}{n - \log n} + \frac{n^{r+1} \cdot n}{(\log n)!} + n^r \frac{1}{(n+1)!} \left[\frac{1 - \left(\frac{n+1}{n}\right)^{r+1}}{1 - \left(\frac{n+1}{n}\right)} \right] + \frac{n^r}{(n-1)!}$$

The next to last term is obtained by using the fact that the error in truncating an alternating series of decreasing terms tending to zero is less than the first term omitted.

Clearly

$$\log_{n\to\infty} \frac{(\log n)^{r+2}}{n-\log n} = 0.$$

Moreover, using the estimate $x! > (x/e)^x$

$$\lim_{n\to\infty} \frac{n^{r+2}}{(\log n)!} \leq \lim_{n\to\infty} \frac{n^{r+2}}{(\log n/e)^{\log n}}$$

$$= \lim_{n\to\infty} e^{(r+3)\log n-(\log n)^2}$$

$$= 0.$$

Next

$$\lim_{n\to\infty} \frac{n^r}{(n+1)!}\left[\frac{1-\left(\frac{n+1}{n}\right)^{r+1}}{1-\left(\frac{n+1}{n}\right)}\right]$$

$$= \lim_{n\to\infty} \frac{(n+1)^{r+1}-n^{r+1}}{(n+1)!}$$

$$= 0.$$

Finally

$$\lim_{n\to\infty} \frac{n^r}{(n-1)!} = 0$$

If we put

$$b_r = \sum_{k=0}^{\infty} \frac{k^r(-1)^k}{k!}$$

then we have just established the asymptotic expansion

$$\frac{F(n,2)}{n!} = \sum_{r=0}^{s} \frac{b_r}{n^r} + 0\left(\frac{1}{n^{s+1}}\right).$$

The numbers b_r can be computed as follows: First

$$b_0 = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} = \frac{1}{e}.$$

We now have

$$b_{r+1} = \sum_{k=0}^{\infty} \frac{k^{r+1}(-1)^k}{k!}$$

$$= \sum_{k=1}^{\infty} \frac{k^r(-1)^k}{(k-1)!}$$

$$= \sum_{k=0}^{\infty} \frac{(k+1)^r(-1)^{k+1}}{k!}$$

$$= -\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \sum_{j=0}^{r} k^j\binom{k}{j}$$

$$= -\sum_{j=0}^{r} \binom{k}{j} b_j,$$

or symbolically

$$b^{r+1} = -(1+b)^r.$$

This recurrence relation shows that b_r is always an integer multiple of $1/e$, say $b_r = (1/e)\,a_r$.

If we write

$$b(x) = \sum_{r=0}^{\infty} \frac{b_r}{r!} x^r,$$

then similarly to the derivation of Eq. (3) we get

$$-b(x)e^x = \sum_{r=0}^{\infty} \frac{b_{r+1}}{r!} x^r = b'(x).$$

Hence

$$b(x) = e^{-e^x}.$$

If c_r denotes the number of partitions of a set of r elements, then it is well known (Ref. 4) that

$$\sum_{r=0}^{\infty} \frac{c_r}{r!} x_r = e^{e^x-1}.$$

Hence the a_r are the so-called Blissard or umbral inverses of the set partition function c_r (Ref. 3, p. 27).

The above results are summarized by the following theorem, which also gives the values of a_r for $0 \leq r \leq 20$.

Theorem 1. Let $F(n,2)$ be the number of permutations of $1, 2, \cdots, n$ with no runs of length 2 and put $G(n,2) = (1/n)\,F(n,2)$. Then

(i) $$n + (n-1)! = \sum_{k=0}^{n} \binom{n}{k} G(k,2).$$

(ii) $$\sum_{k=0}^{\infty} \frac{G(k,2)}{k!} x^k = e^{-x}[1-\log(1-x)] + x,$$

$$\sum_{k=0}^{\infty} \frac{F(k,2)}{k!} x^k = xe^{-x}\left[\frac{x}{1-x} + \log(1-x)\right] + x + 1.$$

(iii) $$F(n,2) = n! \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} \frac{n}{n-k} + n(-1)^n, n \neq 1.$$

(iv)

$$\frac{F(n,2)}{n!} \sim \frac{1}{e}\left(1 - \frac{1}{n^2} + \frac{1}{n^3} + \frac{1}{n^4} - \frac{2}{n^5} - \frac{9}{n^6} - \frac{9}{n^7} + \frac{50}{n^8} + \frac{267}{n^9} + \frac{413}{n^{10}} - \frac{2180}{n^{11}} - \frac{17731}{n^{12}} - \frac{50533}{n^{13}} + \frac{110176}{n^{14}} + \frac{1966797}{n^{15}} + \frac{9938669}{n^{16}} + \frac{8638718}{n^{17}} - \frac{278475061}{n^{18}} - \frac{2540956509}{n^{19}} - \frac{9816860358}{n^{20}} + \cdots + \frac{a_k}{n^k} + \cdots\right),$$

where

$$\sum_{k=0}^{\infty} \frac{a_k}{k!} x^k = e^{-e^x+1}.$$

In particular, the probability that a random permutation of $1,2,\cdots,n$ has no runs of length 2 is

$$\frac{F(n,2)}{n!} \to \frac{1}{e} = 0.36788 \cdots \text{ as } n \to \infty .$$

3. *An Asymptotic Formula for Arbitrary t*

The main result of this section is the following theorem:

Theorem 2. Let $F(n,t)$ be the number of permutations of $1,2,\cdots,n$ with no runs of length t. Then

$$\frac{F(n,3)}{n!} = 1 - \frac{1}{n} - \frac{3}{2}\frac{1}{n^2} - \frac{14}{3}\frac{1}{n^3} + 0\left(\frac{1}{n^4}\right),$$

$$\frac{F(n,4)}{n!} = 1 - \frac{1}{n^2} - \frac{5}{n^3} - \frac{29}{2}\frac{1}{n^4} + 0\left(\frac{1}{n^5}\right),$$

while for fixed $t > 4$

$$\frac{F(n,t)}{n!} = 1 - \frac{1}{n^{t-2}} - \frac{(t-2)(t+1)}{2}\frac{1}{n^{t-1}} - \frac{t(t+1)(3t^2-5t-10)}{24}\frac{1}{n^t} + 0\left(\frac{1}{n^{t+1}}\right).$$

Proof. The proof is based on the principle of inclusion and exclusion referred to earlier. Let $w(i_1, i_2, \cdots, i_r)$ be the number of permutations of $1,2,\cdots,n$ with runs of length t beginning on the symbols $i_1, i_2, \cdots, i_r$. Let $W(r) = \sum w(i_1, i_2, \cdots, i_r)$, where the sum is taken over all subsets $\{i_1, i_2, \cdots, i_r\}$ of $\{1,2,\cdots,n\}$ with r elements. It follows from the principle of inclusion and exclusion that

$$n! - W(1) + W(2) - W(3) < F(n,t) < n! - W(1) + W(2) - W(3) + W(4).$$

If we can show that

$$W(4)/n! = 0\left(\frac{1}{n^{t+1}}\right),$$

then

$$\frac{F(n,t)}{n!} = 1 - \frac{W(1)}{n!} + \frac{W(2)}{n!} - \frac{W(3)}{n!} + 0\left(\frac{1}{n^{t+1}}\right).$$

Then to complete the proof of Theorem 2 we need only calculate $W(1)$, $W(2)$, and $W(3)$.

We now show that for $t > 2$,

$$W(r)/n! = 0\left(\frac{1}{n^{t+r-3}}\right).$$

This will be seen to be false for $t = 2$, which explains why this case was handled separately. For each subset $T = \{i_1, i_2, \cdots, i_r\}$ of $S = \{1,2,\cdots,n\}$, we associate a partition of n as follows: the set T partitions S into subsets of symbols which must remain adjacent in order for a permutation to have runs of length t beginning on $i_1, i_2, \cdots, i_r$. The number of elements in each subset is taken to be a term in the partition of n.

Example: Let $n = 15$, $t = 3$, $T = \{2,5,7,8,12\}$. Then S is partitioned into the subsets $\{1\}$, $\{2,3,4\}$, $\{5,6,7,8,9,10\}$, $\{11\}$, $\{12,13,14\}$, $\{15\}$. This yields the partition $15 = 1 + 1 + 1 + 3 + 3 + 6$.

Let $n = b_1 + 2b_2 + \cdots + nb_n$ be the partition of n induced by T. Observe that $b_1 \geqq n - rt$, with equality holding when each element of T belongs to a distinct subset of the partition of S. Hence, there are at most $\binom{rt}{r}$ distinct partitions of S induced by all subsets T with r elements, since we can assume that r elements of T are chosen from a specified set of size rt.

Given the partition $n = b_1 + 2b_2 + \cdots + nb_n$, there are

$$n \frac{\left(\sum_{i=1}^{n} b_i - 1\right)!}{b_1!\, b_2! \cdots b_n!}$$

ways of choosing subsets of S with this partition (with the property that each subset of order i must contain i consecutive elements of S). For each choice, there are

$$n\left(\sum_{i=1}^{n} b_i - 1\right)!$$

ways of rearranging these subsets to give different permutations of S. Hence

$$W(r) \leq \binom{rt}{r} \max \frac{n^2\left(\sum_{i=1}^{n} b_i - 1\right)!^2}{b_1!\, b_2! \cdots b_n!}, \qquad (6)$$

where the maximum is taken over all partitions of n which can arise from subsets T of order r.

Now if $b_2 + \cdots + b_n = b$, then

$$\sum_{i=1}^{n} b_i \leq n - bt - (r - b) + b,$$

since $b_2 = b_3 = \cdots = b_{t-1} = 0$ and there are r runs of length t. Hence, from Eq. (6) we get

$$\frac{W(r)}{n!} \leq \binom{rt}{r} \frac{n^2 (n - bt + 2b - r - 1)!^2}{b_1!\, n!}$$

$$= \binom{rt}{r} \frac{n^2 (n - bt + 2b - r - 1)(n - bt + 2b - r - 2) \cdots (n - bt + b - r)}{n(n-1)(n-2) \cdots (n - bt + 2b - r)}$$

$$= 0\left(\frac{1}{n^{bt - 3b + r}}\right).$$

Since $b \geq 1$, we get

$$bt - 3b + r \geq t + r - 3 \text{ when } t > 2.$$

Note that this is false for $t = 2$, whenever $b > 1$. It follows that

$$\frac{W(r)}{n!} = 0\left(\frac{1}{n^{t+r-3}}\right), t > 2,$$

as asserted.

Note that

$$\frac{F(n,t)}{n!} \geq 1 - \frac{W(1)}{n!} = 1 + 0\left(\frac{1}{n^{t-2}}\right),$$

so that this simple estimate suffices to show that the probability that a random permutation has no runs of length $t > 2$ approaches 1 as $n \to \infty$.

We now complete the proof by calculating $W(1)$, $W(2)$, and $W(3)$ for n sufficiently large to "accommodate" all runs in question.

One run of length t can begin on any one of n symbols, leaving $(n - t)!$ ways of permuting the $n - t$ remaining symbols and n ways of shifting and permuting cyclically. Hence

$$W(1) = n^2 (n - t)!, \qquad n \geq t + 1.$$

Two runs of length t can occur in one of two ways: (i) one run of length $t + 1, t + 2, \cdots, 2t - 1$, or (ii) two disjoint runs of length t. In the first case, one run of length $t + i$ can begin on any one of n symbols, leaving $(n - t - i)!$ ways of permuting the $n - t$ remaining symbols and n ways of shifting each permutation cyclically. In the second case, there are $n(n - 2t + 1)/2$ ways of choosing two disjoint runs of length t, $(n - 2t + 1)!$ ways of permuting the $n - 2t + 1$ subsets that remain when one run is fixed in place, and n ways of shifting each permutation cyclically.

Hence

$$W(2) = \sum_{i=1}^{t-1} n^2 (n-t-i)! + \frac{n^2(n-2t+1)}{2}(n-2t+1)!,$$

$$n \geqq 2t+1.$$

Three runs of length t can occur in one of three ways: (i) one run of length $t+2, t+3, \cdots, 3t-2$, (ii) one run of length t and one of length $t+1, t+2, \cdots, 2t-1$, or (iii) three disjoint runs of length t. In the first case, as before, one run of length $t+i$ can begin on any one of n symbols, leaving $(n-t-i)!$ ways of permuting the $n-t$ remaining symbols and n ways of shifting each permutation cyclically. Now, however, we get an additional factor of $i-1$, as there are $i-1$ places on which the middle run can begin. In the second case, there are $n(n-2t+1-i)$ ways of choosing two disjoint runs of length t and $t+i$, $(n-2t+1-i)!$ ways of permuting the $n-2t+1-i$ subsets that remain when one run is fixed in place, and n ways of shifting each permutation cyclically. In the third case, there are $n(n-3t+2)(n-3t+1)/6$ ways of choosing three disjoint runs of length t, $(n-3t+2)!$ ways of permuting the $n-3t+2$ subsets that remain when one run is fixed in place, and n ways of shifting each permutation cyclically.

Hence

$$W(3) = \sum_{i=2}^{2t-2} (i-1) n^2 (n-t-i)!$$

$$+ \sum_{i=1}^{t-1} n^2 (n-2t+1-i)(n-2t+1-i)!$$

$$+ \frac{n^2(n-3t+2)(n-3t+1)}{6}(n-3t+2)!,$$

$$n \geqq 3t+1.$$

We leave it to the intrepid reader to expand $1 - W(1)/n! + W(2)/n! - W(3)/n!$ in a power series in $1/n$ and verify that the terms given in the statement of the theorem are correct. With this the proof of Theorem 2 is complete.

It is evident that the above procedure can be continued to give the asymptotic expansion of $F(n,t)/n!$ to any desired accuracy.

C. Coding Theory: Moments of Weight Distributions

R. Stanley

1. Introduction

In Section 2 of this report, a general combinatorial formula is developed which allows calculation of sums of the type

$$\sum_{v \in S} [\sigma(v)]^t, \quad 0 \leqq t \leqq r$$

where S is a set of vectors v, $\sigma(v)$ is the sum of the coordinates of v, and r is an integer depending on a special property of the set S. In Section 3, this formula is applied to (n,k) binary codes and yields explicit formulas for the sums

$$\sum_{i=0}^{n} i^t a_i, \quad 0 \leqq t < d$$

where a_i words of the code have weight i, and d is the minimum weight of the dual code. When enough information about a code is known, these equations may suffice to determine its weight distribution. As an example, we calculate the weight distribution of the dual Golay (23, 11) code without using J. MacWilliams' formula.

2. A Combinatorial Formula

Let $P = \{x_1, \cdots, x_p\}$ be a subset of a commutative ring R, and let S be a subset of order s of the direct product $P^n = P \times P \times \cdots \times P$ (n times). Assume that S has the following property for some integer $r \leqq n$:

(1) The restriction of S to any r coordinates contains all p^r r-tuples of elements of P the same number of times. (This necessitates $p^r | s$.)

Let $\sigma(v)$ denote the sum (in R) of the coordinates of the element v of S. We then have:

Theorem 1. For $0 \leqq t \leqq r$, the sum

$$\sum_{v \in S} [\sigma(v)]^t$$

depends only on P, n, s (not on S), and we have

$$\frac{p^n}{s} \sum_{v \in S} (\sigma(v))^t = \sum_{v \in P^n} (\sigma(v))^t. \quad (2)$$