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Derangements on the n-cube

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Abstract

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Let Q_n be the n-dimensional cube represented by a graph whose vertices are sequences of 0's and 1's of length n, where two vertices are adjacent if and only if they differ only at one position. A k-dimensional subcube or a k-face of Q_n is a subgraph of Q_n spanned by all the vertices $u_1u_2\ldots u_n$ with constant entries on $n-k$ positions. For a k-face G_k of Q_n and a symmetry w of Q_n, we say that w fixes G_k if w induces a symmetry of G_k; in other words, the image of any vertex of G_k is still a vertex in G_k. A symmetry w of Q_n is said to be a k-dimensional derangement if w does not fix any k-dimensional subcube of Q_n; otherwise, w is said to be a k-dimensional rearrangement. In this paper, we find a necessary and sufficient condition for a symmetry of Q_n to have a fixed k-dimensional subcube. We find a way to compute the generating function for the number of k-dimensional rearrangements on Q_n. This makes it possible to compute explicitly such generating functions for small k. Especially, for $k=0$, we have that there are $1\cdot 3\cdots(2n-1)$ symmetries of Q_n with at least one fixed vertex. A combinatorial proof of this formula is also given.

1. Introduction

Let Q_n denote the n-dimensional cube. In this paper, we shall adopt the well-known representation of Q_n as a graph $Q_n=(V_n,E_n)$, where V_n is the set of all sequences of 0's and 1's of length n and $(u_1u_2\cdots u_n, v_1v_2\cdots v_n)\in E_n$ if and only if $u_1u_2\cdots u_n$ and $v_1v_2\cdots v_n$ differ at only one position. Let B_n denote the group of symmetries of the cube Q_n, or, equivalently, the automorphism group of the graph Q_n. B_n is the *hyperoctahedral group of degree n* or (by abuse of notation) the *Weyl group of type* B_n. We may represent an element $w\in B_n$ by a *signed permutation* of $\{1,2,\ldots,n\}$, i.e., a permutation of $\{1,2,\ldots,n\}$ with a + or − sign attached to each element $1,2,\ldots,n$. For simplicity of notation, we omit the + sign in examples. Thus, $(\overset{+}{2}\,\overset{+}{4}\,\bar{5})(\bar{3})(\overset{+}{1}\,\bar{6})$ or $(2\,4\,\bar{5})(\bar{3})(1\,\bar{6})$ represents an element of B_6 with underlying permutation

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$(2\,4\,5)(3)(1\,6)$ (written in cycle notation). We call such a representation of an element of B_n a *signed-cycle decomposition*. A signed permutation w acts on a vertex $u_1 u_2 \cdots u_n$ of Q_n by the rule

$$w(u_1 u_2 \cdots u_n) = \hat{u}_{\pi(1)} \hat{u}_{\pi(2)} \cdots \hat{u}_{\pi(n)},$$

where π is the underlying permutation of w and

$$\hat{u}_{\pi(j)} = \begin{cases} u_{\pi(j)} & \text{if } j \text{ has the sign } +, \\ 1 - u_{\pi(j)} & \text{if } j \text{ has the sign } -. \end{cases} \tag{1.1}$$

If we define the *sign vector* $(s_1, s_2, \ldots, s_n)$ of a signed permutation w as

$$s_j = \begin{cases} 0 & \text{if } j \text{ has the sign } +, \\ 1 & \text{if } j \text{ has the sign } -, \end{cases}$$

then (1.1) can be rewritten as

$$\hat{u}_{\pi(j)} \equiv s_j + u_{\pi(j)} \pmod 2.$$

Let S_n denote the subgroup of B_n consisting of those w whose signs are all $+$. Thus, S_n is isomorphic to the symmetric group of degree n. An element $w \in S_n$ will be called a *permutation*. Let Z_n denote the subgroup of B_n consisting of those w whose underlying permutation is the identity. Thus, Z_n is isomorphic to the abelian group $\mathbb{Z}_2^n$. Every element $w \in B_n$ can be written uniquely as $w = uv$, where $u \in S_n$ and $v \in Z_n$ (in fact, B_n is a semidirect product of S_n and Z_n), and $|B_n| = 2^n n!$. An element of Z_n will be called a *complementation*.

A k-dimensional subcube or a k-face of Q_n is a subgraph of Q_n spanned by all the vertices $u_1 u_2 \cdots u_n$ with constant entries on some $n-k$ positions. In particular, any vertex of Q_n is a 0-dimensional subcube of Q_n. Henceforth, we shall use a sequence of k $*$'s and $n-k$ 0's or 1's to denote a k-dimensional subcube of Q_n. For example, $*0*1$ denotes a 2-dimensional subcube of Q_4 induced by four vertices $0001, 0011, 1001, 1011$. We say that $w \in B_n$ has a *fixed k-dimensional subcube* or an *invariant k-dimensional subcube* if there exists a k-dimensional subcube G_k of Q_n such that the image of every vertex of G_k under w is still a vertex of G_k; in other words, the set of vertices of G_k is invariant under w. We shall call w a *k-dimensional rearrangement* if it has some fixed k-dimensional subcube. On the other hand, if w does not have any fixed k-dimensional subcube, we call it a *k-dimensional derangement*. In this paper, we find a necessary and sufficient condition for a symmetry w of Q_n to be a k-dimensional rearrangement. In general, we find a way to compute the generating function for the number of k-dimensional rearrangements. Especially, for $k = 0, 1, 2$ and 3, we obtain explicitly the corresponding generating functions. For $k = 0$, a 0-dimensional rearrangement is a symmetry with some fixed vertices, while for $k = 1$, a 1-dimensional rearrangement is a symmetry with some fixed edges. We also give a combinatorial proof of the formula for the number of vertex rearrangements.

For simplicity, we shall use the following notation of double factorials for non-negative integers:

$$(2n)!! = 2 \cdot 4 \cdot 6 \cdots (2n),$$

$$(2n-1)!! = 1 \cdot 3 \cdot 5 \cdots (2n-1).$$

It is clear that $(2n)!! = 2^n n!$, which is the total number of symmetries of Q_n. Moreover, we shall adopt the convention that $(-1)!! = 1$ and $(-3)!! = 0$.

2. Signed cycle decomposition

A signed cycle is said to be *balanced* if it contains an even number of minus signs. Call an element w of B_n *balanced* if every signed cycle in its signed cycle decomposition is balanced. Although we do not need this fact, let us note that $w \in B_n$ is balanced if and only if w is conjugate to an element of S_n. For instance, $(3\ \bar{1}\ \bar{4}\ 6)(5)(\bar{2}\ \bar{7})$ is balanced. We need the following definition in order to characterize elements $w \in B_n$ with a fixed k-dimensional subcube.

Definition 2.1 (*k-separable and strongly k-separable permutations*). Let $\{C_1, C_2, \ldots, C_m\}$ be a signed cycle decomposition of a symmetry w of Q_n. We say that w is k-dimensional separable (or simply k-separable) if we can partition the cycles $\{C_1, C_2, \ldots, C_m\}$ into two parts, say A and B, such that every cycle in A is balanced and B contains exactly k underlying elements (i.e., the sum of cycle lengths of B is k). Moreover, if w is both balanced and k-separable, then we say that w is strongly k-separable.

In the above definition k is allowed to be zero, in which case part B reduces to the empty set. The following proposition gives a characterization of a k-dimensional rearrangement in terms of k-separable signed permutations.

Proposition 2.2. *Let w be a symmetry of Q_n. Then w has a fixed k-dimensional subcube if and only if w is a k-separable signed permutation.*

Proof. Let $\{C_1, C_2, \ldots, C_m\}$ be the signed cycle decomposition of the symmetry w, and $(s_1, s_2, \ldots, s_n)$ be the sign vector of w. First we suppose that w has a fixed k-dimensional subcube; without loss of generality, say the subcube $G_k = a_1 a_2 \cdots a_{n-k} * * \cdots *$, where $a_1 a_2 \cdots a_{n-k}$ is a given sequence of 0's and 1's. We would like to show that any two elements i and j satisfying $i \leqslant n-k$ and $j > n-k$ cannot be in the same cycle in the signed cycle decomposition of w. Otherwise, there must exist two elements l and r with $l \leqslant n-k$ and $r > n-k$ appearing in the same cycle C. Let L be the set of all elements i in C such that $i \leqslant n-k$, and R be the set of all elements j in C such that $j > n-k$. Since $l \in L$ and $r \in R$, we know that $L \neq \emptyset$ and $R \neq \emptyset$. Because the elements of L and R are arranged on a cycle, there must exist a pair of elements (i, j) such that $i \in L$ and $j \in R$ and i and j are adjacent on the cycle C. Moreover, we may assume that j follows i in C, namely C can be written in the form of $C=(\dots ij \dots)$, regardless of signs. Given a vertex $b_1 b_2 \cdots b_n$ of G_k, let $c_1 c_2 \cdots c_n = w(b_1 b_2 \cdots b_n)$. Since j follows i in C, we have

$$c_i \equiv s_i + b_j (\bmod 2). \tag{2.1}$$

Then it is easy to see that w fixes the ith position of G_k (i.e., $c_i = b_i$ for any vertex $b_1 b_2 \dots b_n \in G_k$) if and only if $b_i \equiv s_i + b_j (\bmod 2)$. Consider the two vertices in the subcube G_k: $u = a_1 a_2 \cdots a_{n-k} 00 \cdots 0$ and $v = a_1 a_2 \cdots a_{n-k} 00 \cdots 1 \cdots 0$ (where the 1 appears in the jth position). Let $c_1 c_2 \cdots c_n = w(u)$ and $d_1 d_2 \cdots d_n = w(v)$. From (2.1) it follows that

$$c_i \equiv s_i (\bmod 2) \quad \text{and} \quad d_i \equiv s_i + 1 (\bmod 2). \tag{2.2}$$

Since G_k is a fixed k-dimensional subcube of Q_n, w must fix the ith position for both u and v. Hence, we must have $c_i = d_i = a_i$, which is a contradiction to (2.2). It follows that i and j cannot be in the same cycle in the signed cycle decomposition of w. Therefore, $\{C_1, C_2, \dots, C_k\}$ can be partitioned into two parts A and B such that the underlying set for A is $\{1, 2, \dots, n-k\}$ (note that B reduces to the empty set if $k=0$.)

What we still need to show is that every cycle in A is balanced. Let w' be the signed permutation on $\{1, 2, \dots, n-k\}$ with signed cycle decomposition A. Then w' fixes all the positions of $a_1, a_2, \dots, a_{n-k}$ for any vertex $a_1 a_2 \cdots a_{n-k} b_1 b_2 \cdots b_k$ of G_k. Therefore, we may assume, without loss of generality, that $k=0$, namely $a_1 a_2 \dots a_n$ is a vertex fixed by w. Let C be a signed cycle of w. Without loss of generality, we may assume that the underlying permutation of C is $(1, 2 \cdots r)$. Let $c_1 c_2 \cdots c_n = w(a_1 a_2 \cdots a_n)$. Since w fixes all the positions of $a_1, a_2, \dots, a_r$, i.e., $c_i = a_i$ for $1 \leqslant i \leqslant r$, we have

$$\begin{cases} a_1 \equiv s_1 + a_2 (\bmod 2), \\ a_2 \equiv s_2 + a_3 (\bmod 2), \\ \quad \cdots\cdots \\ a_r \equiv s_r + a_1 (\bmod 2). \end{cases} \tag{2.3}$$

It follows that

$$s_1 + s_2 + \cdots + s_r \equiv 0 (\bmod 2).$$

Thus, C must contain an even number of minus signs. This proves the first part of the proposition. Because equation (2.3) always has a solution if $s_1 + s_2 + \cdots + s_r \equiv 0 (\bmod 2)$, the converse of the proposition can be proved by reversing the steps of the above argument. □

Corollary 2.3. *Let $w \in B_n$. Then w has some fixed vertex if and only if w is balanced.*

Corollary 2.4. *Let V_n be the number of vertex rearrangements on Q_n. Then we have $V_n = (2n-1)!!$.*

Proof. Let $V_{n,k}$ be the number of symmetries w such that w has some fixed vertices and w has k cycles in its cycle decomposition. Given any unsigned cycle C of length l, it is clear that there are 2^{l-1} balanced cycles based on C. Therefore, for any permutation π on $\{1, 2, \ldots, n\}$ with k cycles, there are 2^{n-k} signed permutations based on π with each cycle balanced. Since we know that there are $|s(n, k)|$ permutations on n elements with k cycles, where $s(n, k)$ is the Stirling number of the first kind, satisfying

$$x(x+1)(x+2)\cdots(x+n-1)=\sum_{k=1}^{n}|s(n,k)|x^k.$$

We have $V_{n,k}=|s(n,k)|2^{n-k}$, and the total number of vertex rearrangements equals

$$\sum_{k=0}^{n}|s(n,k)|2^{n-k}=2^n\cdot\frac{1}{2}\cdot\frac{3}{2}\cdots\frac{2n-1}{2}=(2n-1)!!. \qquad \square$$

Let $V(x)$ be the exponential generating function for V_n. From the well-known generating function

$$\sum_{n\geqslant 0}\binom{2n}{n}x^n=\frac{1}{\sqrt{1-4x}},$$

we obtain that

$$V(x)=\sum_{n\geqslant 0}V_n\frac{x^n}{n!}=\frac{1}{\sqrt{1-2x}}. \tag{2.4}$$

We can also give a combinatorial proof of Corollary 2.4 based on Corollary 2.3. Define a signed-cycle decomposition of $w\in B_n$ to be standard if in each cycle the minimum element appears at the beginning. For instance, $w=(\bar{2}\,8\,3\,\bar{5})(1\,6)(4\,\bar{9}\,\bar{7})\in B_9$ is standard. We now describe a way of inserting $n+1$ into the standard cycle notation for a balanced standard element $w\in B_n$ to create balanced standard elements $w'\in B_{n+1}$. Either put $n+1$ into a cycle of its own (with a + sign), or else insert $n+1$ into a cycle $(i_1, i_2, \ldots, i_k)$ of w. We can place $n+1$ immediately after i_j for $1\leqslant j\leqslant k$ (we cannot put $n+1$ before i_1 because the new cycle would no longer be standard). Choose arbitrarily the sign of the largest element among $i_1, i_2 \ldots, i_k$ and keep all other signs the same. The sign of $n+1$ is then uniquely determined in order for the new cycle to be balanced. Thus, there are a total of $2n+1$ ways to insert $n+1$ into w, as described above. Given w', we can uniquely recover w by removing $n+1$ and adjusting the sign of the largest element (if it exists) of the cycle containing $n+1$ to insure that it is balanced. From this it follows that we obtain every balanced element w' of B_{n+1} exactly once by the above procedure; so, $V_{n+1}=(2n+1)V_n$. Since $V_1=1$ is trivial, we have obtained a combinatorial proof of Corollary 2.4.

The referee of this paper suggested the following combinatorial proof of Corollary 2.4 based on the 'greedy method'. We shall denote a signed permutation on $\{1, 2, \ldots, n\}$ in the following form:

$$w = \begin{pmatrix} 1 & 2 & \cdots & n \\ w_1 & w_2 & \cdots & w_n \end{pmatrix},$$

where to each w_i is attached a sign $+$ or $-$. In order to construct all the balanced permutations w, we can use the following greedy algorithm:

(1) Choose w_1 as any signed element except $\bar{1}$; otherwise, w would contain an unbalanced cycle $(\bar{1})$. So, there are $2n-1$ possibilities for w_1.

(2) Now suppose $w_1, w_2, \ldots, w_{i-1}$ have been so chosen that every completed cycle is balanced. Ignoring the balanced cycle condition, there are $2n-2i+2$ possibilities for w_i. However, among these $2n-2i+2$ choices for w_i, exactly one choice would create a complete unbalanced cycle (containing w_i), because such a w_i must be chosen as the element j with proper sign such that $j \leqslant i$ and j is the first element in the uncompleted cycle containing i: in other words, i is in an uncompleted cycle $(j \cdots i$ regardless of signs. Therefore, there are $2n-2i+1$ choices for w_i such that no unbalanced cycle would occur.

This gives that the number of balanced permutations on n elements is $(2n-1)!!$.

From the proof of Proposition 2.2, we may obtain the structure of the set of all fixed vertices of a symmetry of Q_n.

Proposition 2.5. *Let F_w be the set of all vertices of Q_n fixed by an element $w \in B_n$. Suppose $F_w \neq \emptyset$. Then there exists a partition $\pi = \{D_1, \ldots, D_k\}$ of the set $\{1, \ldots, n\}$ with the following property: If $u_1 u_2 \ldots u_n$ is any given element of F_w, then all the elements of F_w are obtained by choosing a subset $\{D_{i_1}, \ldots, D_{i_j}\}$ of the blocks of π and complementing those u_r for which $r \in D_{i_s}$ for some $1 \leqslant s \leqslant j$. In particular, if w contains k signed cycles, then $|F_w| = 2^k$.*

Proof. Let $\{C_1, C_2, \ldots, C_k\}$ be the signed-cycle decomposition of w and $(s_1, s_2, \ldots, s_n)$ the sign vector of w. Suppose C is any signed cycle of w. Without loss of generality, we may assume that the underlying permutation of C is $(1\,2 \cdots r)$. By Proposition 2.2, it follows that C is a balanced cycle. Therefore, $s_1 + s_2 + \cdots + s_r$ is even. Let $a_1 a_2 \cdots a_n$ be any vertex fixed by w. Then $(a_1, a_2, \ldots, a_r)$ is a solution to the system of equations (2.3). It is easy to see that we can arbitrarily choose a_1; then the other a_i's $(2 \leqslant i \leqslant r)$ are uniquely determined by the value of a_1. Moreover, if $(a_1, a_2, \ldots, a_r)$ is a solution to (2.3), so is the complementary sequence $(1-a_1, 1-a_2, \ldots, 1-a_r)$. Clearly, these two sequences are the only solutions to (2.3). This completes the proof. $\square$

It should be noted that the set of fixed vertices of an automorphism of Q_n is not necessarily a face of Q_n. In fact, F_w is a face of Q_n if and only if $F_w = \emptyset$ or w is the identity. Thus, the problem of counting derangements of Q_n is not a Möbius inversion problem on the face lattice of Q_n, as it may first look like.

3. k-Dimensional rearrangements on Q_n

Let $S_{n,k}$ be the number of strongly k-separable (balanced and k-separable) permutations on n elements and $S_k(x)$ be the exponential generating function for the sequence $\{S_{n,k}\}_{n\geq 0}$:

$$S_k(x)=\sum_{n\geq 0} S_{n,k}\frac{x^n}{n!}.$$

Let $R_{n,k}$ be the number of k-dimensional rearrangements on Q_n and $R_k(x)$ be the exponential generating function

$$R_k(x)=\sum_{n\geq 0} R_{n,k}\frac{x^n}{n!}.$$

Proposition 3.1. *We have*

$$R_{n,k}=\sum_{0\leq i\leq k}\binom{n}{i}(2i-1)!!\,S_{n-i,k-i}, \tag{3.1}$$

$$R_k(x)=\sum_{0\leq i\leq k}(2i-1)!!\,S_{k-i}(x)\frac{x^i}{i!}. \tag{3.2}$$

Proof. From Proposition 2.2, we know that a symmetry w of Q_n is a k-dimensional rearrangement if and only if it is k-separable. Thus, w may have some unbalanced cycles on an underlying set with no more than k elements. Since we can always change the sign of the maximum element in an unbalanced cycle to make it into a balanced cycle, we see, by Corollary 2.4, that there are $(2i-1)!!$ signed permutations on i elements with every cycle unbalanced. If w contains some unbalanced cycles with underlying set of i elements, the remaining cycles of w must correspond to a strongly $(k-i)$-separable permutation on $n-i$ elements. This proves (3.1). Thus, we have

$$\begin{aligned}R_k(x)&=\sum_{n\geq 0} R_{n,k}\frac{x^n}{n!}\\&=\sum_{n\geq 0}\sum_{0\leq i\leq k}\binom{n}{i}(2i-1)!!\,S_{n-i,k-i}\frac{x^n}{n!}\\&=\sum_{0\leq i\leq k}(2i-1)!!\frac{x^i}{i!}\sum_{n\geq i} S_{n-i,k-i}\frac{x^{n-i}}{(n-i)!}\\&=\sum_{0\leq i\leq k}(2i-1)!!\,S_{k-i}(x)\frac{x^i}{i!}. \qquad \square\end{aligned}$$

We shall use the common notation $\lambda \vdash n$ to denote that λ is a partition of n, and $\lambda = 1^{\lambda_1} 2^{\lambda_2} \cdots$ to denote a partition of an integer with λ_1 1's, λ_2 2's, and so on. Moreover, we define the *join* of two partitions λ and μ as follows:

$$(1^{\lambda_1} 2^{\lambda_2} \cdots) \vee (1^{\mu_1} 2^{\mu_2} \cdots) = 1^{\gamma_1} 2^{\gamma_2} \cdots,$$

where $\gamma_i = \max(\lambda_i, \mu_i)$.

As a refinement of the definition of strongly k-separable signed permutations, we give the following definition.

Definition 3.2 (*λ-separable permutations*). Let λ be a partition of an integer k. A balanced permutation T is said to be λ-separable if T has at least λ_i i-cycles in its cycle decomposition for any i.

Definition 3.3 (*Euler characteristic of a partition*). Let λ be a partition of an integer. Given an integer k, let $c_i(\lambda)$ be the number of i-sets of partitions of k such that their join equals λ. Then the Euler characteristic of λ is defined by

$$\chi_k(\lambda) = c_1 - c_2 + c_3 - c_4 + \cdots.$$

Proposition 3.4. *Let $S_{n,k}$ and $S_{n,\lambda}$ be the number of k-separable and λ-separable signed permutations on n elements, and let $S_\lambda(x)$ be the exponential generating function for $S_{n,\lambda}$. Then we have*

$$S_{n,k} = \sum_{\lambda} \chi_k(\lambda) S_{n,\lambda}, \tag{3.3}$$

$$S_k(x) = \sum_{\lambda} \chi_k(\lambda) S_\lambda(x). \tag{3.4}$$

Proof. Let w be a signed permutation on n elements. Then w is k-separable if and only if there exists a partition λ of k such that w is λ-separable. Let $p_1, p_2, \ldots$ be all the partitions of k. Then, by the principle of inclusion and exclusion, we have

$$S_{n,k} = \sum_{i \geqslant 1} S_{n,p_i} - \sum_{i<j} S_{n,p_i \vee p_j} + \sum_{i<j<l} S_{n,p_i \vee p_j \vee p_l} - \cdots$$
$$= \sum_{\lambda} \chi_k(\lambda) S_{n,\lambda}.$$

There follows the desired generating function $S_k(x)$. □

For simplicity, we shall use the convention

$$y_i = \frac{(2x)^i}{2i}.$$

For integers $i \geqslant 1$ and $j \geqslant 1$, set

$$Z_{ij} = \mathrm{e}^{-y_i} \sum_{t=0}^{j-1} \frac{y_i^t}{t!},$$

while, if $j=0$, set $Z_{ij}=0$.

Proposition 3.5. *Let $\lambda = 1^{\lambda_1} 2^{\lambda_2} \cdots m^{\lambda_m}$ be a partition of an integer m. Let y_i and Z_{ij} be as above. Then we have*

$$S_\lambda(x) = \frac{1}{\sqrt{1-2x}} \prod_{1 \leqslant i \leqslant m} (1 - Z_{i\lambda_i}).$$

Proof. Let $W_n(\lambda)$ be the number of strongly λ-separable signed permutations w on n elements such that w contains at least λ_i i-cycles in the cycle decomposition. Recall that the number of unsigned permutations of type $\mu = 1^{\mu_1} 2^{\mu_2} \cdots$ is

$$\frac{n!}{1^{\mu_1} 2^{\mu_2} \cdots n^{\mu_n} \mu_1! \mu_2! \cdots \mu_n!}.$$

Let $Y_{n,\mu}$ be the number of balanced permutations of type μ. Since μ is a partition of n, we have $\mu_1 + 2\mu_2 + 3\mu_3 + \cdots = n$ and

$$\begin{aligned}
Y_{n,\mu} \frac{x^n}{n!} &= \frac{n! 2^{n-(\mu_1+\mu_2+\cdots)}}{1^{\mu_1} 2^{\mu_2} \cdots n^{\mu_n} \mu_1! \mu_2! \cdots \mu_n!} \frac{x^n}{n!} \\
&= \prod_{i \geqslant 1} \left(\frac{(2x)^i}{2i} \right)^{\mu_i} \frac{1}{\mu_i!} \\
&= \prod_{i \geqslant 1} y_i^{\mu_i} \frac{1}{\mu_i!}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\sum_{n \geqslant 0} W_n(\lambda) \frac{x^n}{n!} &= \sum_{n \geqslant 0} \sum_{\substack{\mu \vdash n \\ \mu_i \geqslant \lambda_i}} Y_{n,\mu} \frac{x^n}{n!} \\
&= \sum_{\mu : \mu_i \geqslant \lambda_i} \prod_{i \geqslant 1} \frac{y_i^{\mu_i}}{\mu_i!} \\
&= \prod_{i \geqslant 1} \sum_{\mu_i \geqslant \lambda_i} \frac{y_i^{\mu_i}}{\mu_i!} \\
&= \prod_{i \geqslant 1} \left(\mathrm{e}^{y_i} - \sum_{\mu_i < \lambda_i} \frac{y_i^{\mu_i}}{\mu_i!} \right)
\end{aligned}$$

$$=\prod_{i\geqslant 1}e^{y_i}\prod_{i\geqslant 1}\left(1-e^{-y_i}\sum_{\mu_i<\lambda_i}\frac{y_i^{\mu_i}}{\mu_i!}\right)$$

$$=e^{y_1+y_2+\cdots}\prod_{i\geqslant 1}(1-Z_{i\lambda_i})$$

$$=e^{-(1/2)\log(1-2x)}\prod_{i\geqslant 1}(1-Z_{i\lambda_i})$$

$$=\frac{1}{\sqrt{1-2x}}\prod_{i\geqslant 1}(1-Z_{i\lambda_i}).$$

Since $Z_{i\lambda_i}=0$ for $\lambda_i=0$, this completes the proof. □

By Propositions 3.1 and 3.5, we may explicitly give the generating functions $R_k(x)$ and $S_k(x)$ for $0\leqslant k\leqslant 3$:

$$S_0(x)=\frac{1}{\sqrt{1-2x}},$$

$$S_1(x)=\frac{1-e^{-x}}{\sqrt{1-2x}},$$

$$S_2(x)=\frac{1-(1+x)e^{-x-x^2}}{\sqrt{1-2x}},$$

$$S_3(x)=\frac{1}{\sqrt{1-2x}}\left[1-e^{-x-4x^3/3}-\left(x+\frac{x^2}{2}\right)e^{-x-x^2-4x^3/3}\right],$$

$$R_0(x)=\frac{1}{\sqrt{1-2x}},$$

$$R_1(x)=\frac{1+x-e^{-x}}{\sqrt{1-2x}},$$

$$R_2(x)=\frac{1}{\sqrt{1-2x}}\left[1+x+\frac{3x^2}{2}-xe^{-x}-(1+x)e^{-x-x^2}\right],$$

$$R_3(x)=\frac{1}{\sqrt{1-2x}}\left[1+x+\frac{3x^2}{2}+\frac{5x^3}{2}-\frac{3x^2}{2}e^{-x}\right.$$
$$\left.-(x+x^2)e^{-x-x^2}-e^{-x-4x^3/3}-(x+x^2)e^{-x-x^2-4x^3/3}\right].$$

From the generating function $R_1(x)$, we may obtain the following formula for the number E_n of edge rearrangements of Q_n:

$$E_n = (2n-1)!! + n(2n-3)!! - \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (2k-1)!!.$$

Finally, we remark that when n goes to infinity, almost all symmetries of Q_n are vertex derangements. It is also true that almost all symmetries of Q_n are edge derangements while $n \to \infty$. What about k-dimensional derangements (for fixed k)?

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