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f-vectors and h-vectors of simplicial posets

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Abstract

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A simplicial poset is a (finite) poset P with $\hat{0}$ such that every interval $[\hat{0}, x]$ is a boolean algebra. Simplicial posets are generalizations of simplicial complexes. The f-vector $f(P) = (f_0, f_1, \ldots, f_{d-1})$ of a simplicial poset P of rank d is defined by $f_i = \#\{x \in P\colon [\hat{0}, x] \cong B_{i+1}\}$, where B_{i+1} is a boolean algebra of rank $i+1$. We give a complete characterization of the f-vectors of simplicial posets and of Cohen–Macaulay simplicial posets, and an almost complete characterization for Gorenstein simplicial posets. The Cohen–Macaulay case relies on the theory of algebras with straightening laws (ASL's).

1. Introduction

A *simplicial poset* [16, p. 135] (also called a *boolean complex* [9, p. 130] and a *poset of boolean type* [3, §2.3]) is a poset P with $\hat{0}$ (i.e., $y \geq \hat{0}$ for all $y \in P$) such that every interval $[\hat{0}, y]$ is a boolean algebra. All posets in this paper will be assumed to be finite. If a simplicial poset P is in addition a meet-semilattice, then P is just the poset of faces (ordered by inclusion) of a (finite) simplicial complex. Simplicial posets are special cases of CW-posets, as defined in [3] (see Section 2.3 of [3]). This implies that a simplicial poset P is the face poset $\mathscr{F}(\Gamma)$ of a regular CW-complex $\Gamma = \Gamma(P)$. Moreover (see the bottom of p. 10 of [3]), Γ has a well-defined *barycentic subdivision* $\beta\Gamma$ which is a simplicial complex isomorphic to the order complex $\Delta(\overline{P})$ of the poset $\overline{P} = P - \{\hat{0}\}$. (By definition, $\Delta(\overline{P})$ is the simplicial complex on the vertex set $\overline{P}$ whose faces are the chains of $\overline{P}$.) We may informally regard Γ as a 'generalized simplicial complex' whose faces are still

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simplices, but we allow two faces to intersect in any subcomplex of their boundaries, rather than just in a single face. Moreover, a simplicial poset may be regarded as the face poset of a simplicial complex modulo a certain equivalence relation; see [3, §2.3] for further details.

Given a simplicial poset P, let $f_i = f_i(P)$ denote the number of $y \in P$ for which the interval $[\hat{0}, y]$ is a boolean algebra B_{i+1} of rank $i+1$. In particular, $f_{-1} = 1$. Also set

$$d = d(P) = 1 + \max\{i\colon f_i(P) \neq 0\}\ .$$

Call d the *rank* and $d-1$ the *dimension* of P. We also say that $x \in P$ has *rank* i, denoted $\rho(x) = i$, if $[\hat{0}, x] \cong B_i$. The vector $f(P) = (f_0, f_1, \ldots, f_{d-1})$ is called the *f-vector* of P. When P is the face-poset of a simplicial complex Σ, then $f(P)$ coincides with the usual notion of the f-vector $f(\Sigma)$ of Σ.

Much research has been devoted to f-vectors of simplicial complexes. In particular, for many interesting classes of simplicial complexes the f-vector has been completely characterized. (For a recent survey of this subject, see [4].) In this paper we begin a similar investigation of f-vectors of simplicial posets. We will consider the following three classes of simplicial posets:

(a) arbitrary,
(b) Cohen–Macaulay (as defined e.g. in [6]),
(c) Gorenstein (see [15, Section 6(e)]).

For the first two we obtain complete characterizations of the f-vector. For case (a) both the statement and proof are much easier than for simplicial complexes (where the corresponding result is the celebrated Kruskal–Katona theorem). For (b), the statement is again much simpler than for simplicial complexes, but the proof is of comparable difficulty. As in the case of simplicial complexes the proof is based on the theory of Cohen–Macaulay rings. We need to use the theory of ASL's (algebras with straightening laws) to prove that a certain ring is Cohen–Macaulay. Finally for (c), in the case of simplicial complexes there is a conjectured characterization which seems at present extremely difficult to decide. (The conjecture is known to hold for boundary complexes of simplicial polytopes, a special case of Gorenstein simplicial complexes.) For simplicial posets we are also unable to settle (c) completely, but we will give some necessary and some sufficient conditions which are not too far apart. Unlike the situation for simplicial complexes, it seems conceivable that a complete solution to (c) for simplicial posets may not be too difficult.

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2. The f-vector of an arbitrary simplicial poset

The main result here is the following:

Theorem 2.1. *Let $f = (f_0, f_1, \ldots, f_{d-1}) \in \mathbb{Z}^d$. The following two conditions are equivalent*:

(i) *There exists a simplicial poset P of dimension $d-1$ with f-vector $f(P) = f$.*

(ii) $f_i \geq \binom{d}{i+1}$ *for* $0 \leq i \leq d-1$.

Proof. (i)$\Rightarrow$(ii) Since $\dim P = d-1$, there exists $y \in P$ of rank d, i.e., $[\hat{0}, y] \cong B_d$. Since the boolean algebra B_d contains $\binom{d}{i+1}$ elements z for which $[\hat{0}, z] \cong B_{i+1}$, condition (ii) follows.

(ii)$\Rightarrow$(i) The boolean algebra B_d is a simplicial poset of dimension $d-1$ satisfying $f_i(B_d) = \binom{d}{i+1}$. Hence it suffices to show that if $f = (f_0, f_1, \ldots, f_{d-1})$ satisfies condition (i) and if $0 \leq j \leq d-1$, then the vector $f' = (f_0, f_1, \ldots, f_j + 1, \ldots, f_{d-1})$ also satisfies (i). Let P be simplicial with $f(P) = f$. Given $0 \leq j \leq d-1$, let $y \in P$ have rank $j+1$. Let $y_1, \ldots, y_{j+1}$ be the elements of P covered by y. Adjoin a new element z to P which also covers $y_1, \ldots, y_{j+1}$. Then $P \cup \{z\}$ is simplicial with f-vector equal to f', as desired. $\square$

3. Cohen–Macaulay simplicial posets

Fix a field K. We assume familiarity with the notion of a *Cohen–Macaulay* (*simplicial*) *complex* Σ (over K), as defined e.g. in [13, 15]. A poset P is called Cohen–Macaulay (over K) if its order complex $\Delta(P)$ is a Cohen–Macaulay simplicial complex (over K). From now on we omit reference to the ground field K. The question of whether Σ is Cohen–Macaulay depends only on the geometric realization $|\Sigma|$ of Σ. If P is a simplicial poset so that $P = \mathscr{F}(\Gamma)$ for some regular CW-complex Γ, then by the remarks in Section 1 the geometric realizations $|\Gamma|$ and $|\Delta(\overline{P})|$ are homeomorphic. Hence the question of whether P is Cohen–Macaulay depends only on $|\Gamma|$.

We are interested in characterizing the f-vector of a Cohen–Macaulay simplicial poset. As is the case for Cohen–Macaulay simplicial complexes, it is easier to work not with the f-vector $f(P)$ itself, but with a related vector. Given $f(P) = (f_0, f_1, \ldots, f_{d-1})$ (with $f_{-1} = 1$), define integers $h_0, h_1, \ldots, h_d$ by

$$\sum_{i=0}^{d} f_{i-1}(x-1)^{d-i} = \sum_{i=0}^{d} h_i x^{d-i} . \tag{1}$$

Call $h(P) = (h_0, h_1, \ldots, h_d)$ the *h-vector* of P. We always have $h_0 = 1$, $h_1 = f_0 - d$, and $h_0 + h_1 + \cdots + h_d = f_{d-1}$.

In [13, Theorem 6] the h-vector of a Cohen–Macaulay simplicial complex was characterized. (The details of the proof of the sufficiency of the condition appear in [5] in a more general context.) One of the conditions on the h-vector is that $h_i \geq 0$ for all i. We will show below that this condition (together with the trivial condition $h_0 = 1$) is necessary and sufficient for Cohen–Macaulay simplicial posets

P. The proof is based on showing that a certain commutative ring A_P associated with P is a Cohen–Macaulay ring, so first we will review some commutative algebra.

Let $R = R_0 \oplus R_1 \oplus \cdots$ be a finitely-generated graded (commutative) algebra over the field $K = R_0$. The *Hilbert function* of R is defined by $H(R, n) = \dim_K R_n$ for $n \geq 0$, while the *Hilbert series* is

$$F(R, \lambda) = \sum_{n \geq 0} H(R, n)\lambda^n .$$

Suppose R has the structure of a finitely-generated graded module over the polynomial ring $A = K[x_1, \ldots, x_n]$, where A is graded by setting $\deg x_i = e_i > 0$. Then $F(R, \lambda)$ is a rational function of λ of the form

$$F(R, \lambda) = \frac{P(\lambda)}{\prod_{i=1}^{n} (1 - \lambda^{e_i})} ,$$

where $P(\lambda) \in \mathbb{Z}[\lambda]$. The *Krull dimension* $\dim R$ may be defined as the order of the pole at $\lambda = 1$ of $F(R, \lambda)$ (see e.g., [14, p. 58]).

Suppose $\dim R = d$. Then there exist homogeneous elements $\theta_1, \ldots, \theta_d \in R$ of positive degree, necessarily algebraically independent over K, such that R is a finitely-generated $K[\theta_1, \ldots, \theta_d]$-module. We call $\theta_1, \ldots, \theta_d$ a *homogeneous system of parameters* (h.s.o.p.) for R. If R is generated as a K-algebra by R_1 and if $|K| = \infty$, then we can choose $\theta_1, \ldots, \theta_d$ to have degree one.

Suppose now that R is an arbitrary commutative noetherian ring (with identity). We assume familiarity with the notion of R being *Cohen–Macaulay* [11, p. 84; 12, §17]. The following result appears in [11, Theorem 141]:

Lemma 3.1 *If R is Cohen-Macaulay and $y \in R$ is neither a zero-divisor nor a unit, then R/yR is Cohen–Macaulay.* $\square$

For graded algebras $R = R_0 \oplus R_1 \oplus \cdots$ there is a well-known characterization of the Cohen-Macaulay property involving Hilbert series [14, p. 63]. Let $\theta_1, \ldots, \theta_d$ be an h.s.o.p. for R with $e_i = \deg \theta_i$, and set $S = R/(\theta_1 R + \cdots + \theta_d R)$. Then R is Cohen–Macaulay if and only if

$$F(R, \lambda) = \frac{F(S, \lambda)}{\prod_{i=1}^{d} (1 - \lambda^{e_i})} . \tag{2}$$

Suppose P is a finite poset. A *monomial* $\mathcal{M}$ on P is a function $\mathcal{M} : P \to \mathbb{N}$. The *support* of $\mathcal{M}$ is the set

$$\operatorname{supp}(\mathcal{M}) = \{y \in P : \mathcal{M}(y) \neq 0\} .$$

A monomial $\mathcal{M}$ is *standard* if $\operatorname{supp}(\mathcal{M})$ is a chain of P.

If R is a commutative ring with an injection $\phi : P \to R$ given, then to each monomial $\mathcal{M}$ on P we may associate

$$\phi(\mathcal{M}) := \prod_{y \in P} \phi(y)^{\mathcal{M}(y)} \in R .$$

We will identify P with $\phi(P)$ and we will also call $\phi(\mathcal{M})$ a monomial.

Now let K be a field and R a (commutative) K-algebra in the situation of the previous paragraph. We will call R (or more precisely the pair (R, ϕ)) an *algebra with straightening law* (ASL) or *Hodge algebra* on P over K if the following two conditions are satisfied:

(ASL-1) The set of standard monomials is a basis of the algebra R as a vector space over K.

(ASL-2) If x and y in P are incomparable and if

$$xy = \sum_i a_i \, x_{i1} x_{i2} \cdots x_{ir_i} , \tag{3}$$

where $0 \neq a_i \in K$ and $x_{i1} \leq x_{i2} \cdots$, is the unique expression for $xy \in R$ as a linear combination of distinct standard monomials guaranteed by (ASL-1), then $x_{i1} \leq x$ and $x_{i1} \leq y$ for every i.

Note that the right-hand side of (3) is allowed to be the empty sum ($=0$); but that, though 1 is a standard monomial (whose support is the empty chain), no $x_{i1} x_{i2} \cdots x_{ir_i}$ can be 1.

For the basic theory of ASL's, see [7] or [8]. A somewhat different approach appears in [2]. The main result which we need about ASL's is the following [7, Corollary 7.2]:

Theorem 3.2. *Let P be a finite Cohen-Macaulay poset and R an ASL on P over K. Then the local ring $R_\mathfrak{p}$ is Cohen-Macaulay for any prime ideal $\mathfrak{p}$ of R with $P \subset \mathfrak{p}$. In particular, if $R = R_0 \oplus R_1 \oplus \cdots$ is a graded ring such that $R_0 = K$ and $P \subset \bigcup_{n \geq 1} R_n$, then R is a Cohen–Macaulay ring.* □

We now define a K-algebra A_P associated with a simplicial poset P. When P is the face-poset of a simplicial complex Δ, then A_P will coincide with the face ring (or Stanley–Reisner ring) of Δ [13, p. 52; 15, p. 62].

Definition 3.3. Let P a simplicial poset with elements $\hat{0} = y_0, y_1, \ldots, y_p$. Let $A = K[y_0, y_1, \ldots, y_p]$ be the polynomial ring over K in the variables y_i. Define I_P to be the ideal of A generated by the following elements:

(S_1) $y_i y_j$, if y_i and y_j have no common upper bound in P,

(S_2) $y_iy_j - (y_i \wedge y_j)(\sum_z z)$, where z ranges over all *minimal* upper bounds of y_i and y_j, otherwise.
Finally set $\tilde{A}_P = A/I_P$ and $A_P = \tilde{A}_P/(y_0 - 1)$.

Note. (1) It is clear that $y_i \wedge y_j$ exists whenever y_i and y_j have an upper bound z, since the interval $[\hat{0}, z]$ is a boolean algebra (and therefore a lattice).

(2) We could replace $\sum_z z$ in the above definition by $\sum \alpha_z z$, for any $0 \neq \alpha_z \in K$, without significantly altering the theory.

Lemma 3.4. *$\tilde{A}_P$ is an ASL on P.*

Proof. Axiom (ASL-2) is clear from the definition of the ideal I_P. We need to prove (ASL-1). We first show that the standard monomials are linearly independent. Suppose there is a linear relation

$$\sum \alpha_i u_i = 0 \tag{4}$$

in $\tilde{A}_P$, where each u_i is a standard monomial and $0 \neq \alpha_i \in K$. Let $C = \mathrm{supp}(u_1)$. Since C is a chain there is a maximal element z of P satisfying $z \geq y$ for all $y \in C$. Let $B = [\hat{0}, z]$ (a boolean algebra). Multiplying (4) by z and discarding all terms $\alpha_i u_i z$ equal to 0 in $\tilde{A}_P$ yields a nontrivial linear relation among standard monomials in $\tilde{A}_B$. It is known (see [10, Theorem on p. 100]), however (and easy to prove), that for boolean algebras (or more generally, distributive lattices) B the relations (S_1) and (S_2) (or just (S_2), since (S_1) never arises) define an ASL on B. This contradicts the linear relation just described in $\tilde{A}_B$, so the standard monomials are linearly independent.

It now follows from [7, Proposition 1.1, p. 15] that $\tilde{A}_P$ is indeed an ASL on P. (We do not need to verify separately that the standard monomials span $\tilde{A}_P$.) □

Corollary 3.5. *Let P be a finite Cohen–Macaulay simplicial poset. Then the ring $\tilde{A}_P$ is Cohen–Macaulay.*

Proof. Combine Theorem 3.2 and Lemma 3.4. □

Lemma 3.6. *Let P be a finite poset with $\hat{0} = y_0$. Let A_P be an ASL on P. Then $y_0 - 1$ is a non-zero-divisor of A_P.*

Proof. If u is a standard monomial then so is $y_0 u$. Hence if $0 \neq z \in A_P$ and $(y_0 - 1)z = 0$, then we get a nontrivial relation among standard monomials, contradicting (ASL-1). □

The previous lemma is also a special case of [7, Theorem 5.4] and [8, Corollary 3.6(2)].

Corollary 3.7. *Let $P = \{y_0, y_1, \ldots, y_p\}$ be a finite Cohen–Macaulay simplicial poset with $y_0 = \hat{0}$. Then the ring $A_P = \tilde{A}_P/(y_0 - 1)$ is Cohen–Macaulay.*

Proof. Combine Corollary 3.5, Lemma 3.6, and Lemma 3.1. □

Now preserve the notation of Definition 3.3. Suppose we define a 'quasi-grading' on $A = K[y_0, y_1, \ldots, y_p]$ by setting $\deg y_i = \rho(y_i)$. We do not get an actual graded algebra as we have defined it because $\deg y_0 = 0$, so $\dim_K A_0 = 2$ instead of 1. The relations (S_1), (S_2), and $y_0 - 1$ are homogeneous with respect to this grading, so A_P inherits a grading from A. Moreover, since $\deg(y_0 - 1) = 0$ it follows that $\dim_K(A_P) = 1$, so A_P is a genuine graded algebra. (The condition on a graded algebra R that $R_0 \cong K$ is sometimes called *connected*.)

Proposition 3.8. Let P be a finite simplicial poset of rank d and with h-vector $(h_0, h_1, \ldots, h_d)$. With the grading of A_P just defined, we have

$$F(A_P, \lambda) = \frac{h_0 + h_1\lambda + \cdots + h_d\lambda^d}{(1-\lambda)^d} \, . \tag{5}$$

Proof. Since a K-basis of $\tilde{A}_P$ consists of the standard monomials and since $y_0 - 1$ is a non-zero-divisor, a K-basis of A_P will consist of the standard monomials corresponding to the poset $\overline{P} = P - \{\hat{0}\}$, together with 1. For a boolean algebra B_r of rank r, it is easily seen (and follows e.g. from [10, Theorem on p. 100]) that $A_{B_r} \cong K[y_1, \ldots, y_r]$, a polynomial ring in r variables, each of degree one. Hence if $y \in P$ has rank r, then the contribution to the Hilbert series $F(A_P, \lambda)$ of all standard monomials whose support has maximum element x is given by $\lambda^r F(A_{B_r}, \lambda) = \lambda^r/(1-\lambda)^r$. Therefore

$$F(A_P, \lambda) = \sum_{x \in P} \frac{\lambda^{\rho(x)}}{(1-\lambda)^{\rho(x)}} = \sum_{i=0}^{d} f_{i-1}(P) \frac{\lambda^i}{(1-\lambda)^i} \, .$$

Comparing with (1) yields (5). □

Now let A_P^* denote the subalgebra of A_P generated by $(A_P)_1$.

Lemma 3.9. *Let P be a finite simplicial poset. Then A_P is integral over A_P^*.*

Proof. We prove by induction on $\rho(y)$ that every $y \in P$ is integral over A_P^*. Since $1 \in A_P^*$ and $y \in A_P^*$ for all $y \in P$ of rank one, the induction hypothesis is true for $\rho(y) \le 1$. Now let $\rho(y) > 1$ and assume that if $x \in P$ satisfies $\rho(x) < \rho(y)$, then x is integral over A_P^*. In the open interval $(\hat{0}, y)$ pick y' and y'' such that $y' \wedge y'' = \hat{0}$ and $y' \vee y'' = y$, where $\vee$ is computed in $[\hat{0}, y]$. (Since $[\hat{0}, y]$ is a boolean algebra of rank ≥ 2, such y' and y'' always exist.) By (S_2) and the relation $\hat{0} = y_0 = 1$, we have

$$y'y'' = y + \sum_z z\ , \tag{6}$$

where z ranges over all minimal upper bounds of y' and y'' except y. (There may be no such z's.) Any two distinct minimal upper bounds z' and z'' of y' and y'' must satisfy $z'z''=0$; otherwise they would have an upper bound w, and hence $[\hat{0}, w]$ would not be a lattice (so also not a boolean algebra). Thus, if we multiply (6) by y we get $yy'y''=y^2$, so y is integral over $K[y', y'']$ and hence by induction over A_P^*. $\square$

We come to the main result of this section.

Theorem 3.10. *Let* $\boldsymbol{h}=(h_0, h_1, \ldots, h_d) \in \mathbb{Z}^{d+1}$. *The following two conditions are equivalent*:

(i) *There exists a Cohen–Macaulay simplicial poset* P *of rank* d *with* h*-vector* $h(P)=\boldsymbol{h}$.

(ii) $h_0=1$, *and* $h_i \geq 0$ *for all* i.

Proof. (i)$\Rightarrow$(ii) We may assume K is infinite by replacing it with an extension field if necessary. Then the subalgebra A_P^* of A_P defined above has an h.s.o.p. $\theta_1, \ldots, \theta_e$ of degree one, since by definition A_P^* is generated by $(A_P^*)_1$. Since A_P is clearly a finitely-generated A_P^*-algebra (or even a finitely-generated K-algebra), and since by the previous lemma A_P is integral over A_P^*, it follows that A_P is a finitely-generated A_P^*-module (see e.g. [1, p. 60]). Hence $\theta_1, \ldots, \theta_e$ is an h.s.o.p. for A_P, so (by (2) and (5), or by standard algebraic arguments) $e=d$. Comparing (2) (with each $e_i=1$) and (5) yields (ii).

(ii)$\Rightarrow$(i) Given $\boldsymbol{h}=(h_0, h_1, \ldots, h_d) \in \mathbb{N}^{d+1}$ with $h_0=1$, we will construct a shellable (in the sense of [3, Section 4]) regular CW-complex Γ of dimension $d-1$ whose face-poset is P and whose h-vector is $\boldsymbol{h}$. By [3, Proposition 44], the barycentric subdivision $\beta\Gamma$ of Γ is shellable. Hence the order complex $\Delta(P)$ is shellable and therefore Cohen–Macaulay [13, §5].

First note that a $(d-1)$-simplex is shellable with h-vector $(1, 0, \ldots, 0)$.Thus, it suffices to show that if $\boldsymbol{h}=(h_0, h_1, \ldots, h_d)$ is the h-vector of a shellable regular CW-complex Γ of dimension $d-1$ whose face-poset is simplicial and if $1 \leq i \leq d$, then the vector $\boldsymbol{h}\langle i\rangle := (h_0, h_1, \ldots, h_{i-1}, h_i+1, h_{i+1}, \ldots, h_d)$ is also the h-vector of a shellable regular CW-complex of dimension $d-1$ whose face-poset is simplicial. Choose a $(d-1)$-cell F of Γ. Let $F_1, F_2, \ldots, F_i$ be any i $(d-2)$-faces of F. Let G be a $(d-1)$-simplex disjoint from Γ, and let $G_1, G_2, \ldots, G_i$ be any i $(d-2)$-faces of G. Attach G to Γ by identifying (in a compatible way) each G_j with F_j, yielding a new complex Γ' whose face-poset is simplicial. By attaching G we have extended the shelling of Γ to Γ', so Γ' is shellable. It is easy to check that $h(\Gamma')=\boldsymbol{h}\langle i\rangle$, and we are done. $\square$

4. Gorenstein simplicial posets

We define finite poset P to be *Gorenstein* (over a field K) if its order complex $\Delta(P)$ is a Gorenstein complex in the sense of [15, Chapter 2.5]. Let $\mathrm{triv}(P)$ denote the set of elements of P that are related to *all* elements of P, and set $\mathrm{core}(P) = P - \mathrm{triv}(P)$. It follows from [15, Chapter 2, Theorem 5.1] that P is Gorenstein if and only if $\mathrm{core}(P)$ is nonacyclic Gorenstein. ('Nonacyclic' means that the reduced homology groups $\tilde{H}_i(\Delta(P); k)$ of $\Delta(P)$ do not all vanish.) Moreover, the property of being nonacyclic Gorenstein is *topological*, i.e., depends only on the geometric realization $|\Delta(P)|$. If P is a simplicial poset, then we say that P is *Gorenstein** if $\bar{P} = P - \{\hat{0}\}$ is nonacyclic Gorenstein. The only Gorenstein simplicial posets which are not Gorenstein* are boolean algebras, so we lose nothing by confining our attention to Gorenstein* simplicial posets. Let us remark that an arbitrary finite poset P is Gorenstein* if and only if P is Cohen–Macaulay and $P \cup \{\hat{0}, \hat{1}\}$ is Eulerian, as defined in [16, Chapter 3.14]. Moreover, a finite poset P is Gorenstein* if $|\Delta(P)|$ is homeomorphic to a $(d-1)$-sphere S^{d-1}.

Let P be a Gorenstein* simplicial poset of rank d with $h(P) = (h_0, h_1, \ldots, h_d)$. The *Dehn–Sommerville equations*, which are proved for simplicial posets P for which $P \cup \{\hat{1}\}$ is Eulerian in [16, equation (3.40)], assert that $h_i = h_{d-i}$ for all i. Moreover, since Gorenstein posets are Cohen–Macaulay it follows from Theorem 3.10 that $h_i \geq 0$. The conditions $h_i = h_{d-i}$, $h_i \geq 0$, and the trivial condition $h_0 = 1$ almost but not quite suffice to characterize h-vectors of Gorenstein* simplicial posets. We first will establish some sufficient conditions on such h-vectors, and then a necessary condition.

Lemma 4.1. *Let P and P' be Gorenstein* simplicial posets of rank d, with $h(P) = (1, h_1, h_2, \ldots, h_{d-1}, 1)$ and $h(P') = (1, h_1', h_2', \ldots, h_{d-1}', 1)$. Then there exists a Gorenstein* simplicial poset Q of rank d with*

$$h(Q) = (1, h_1 + h_1', h_2 + h_2', \ldots, h_{d-1} + h_{d-1}', 1)\,.$$

Proof. Let $\Gamma = \Gamma(P)$ and $\Gamma' = \Gamma(P')$ denote the regular CW-complexes of which P and P' are face posets. Let σ and σ' be open $(d-1)$-cells of Γ and Γ', respectively. The Gorenstein* condition implies that every $(d-2)$-cell of Γ and Γ' is contained in exactly two $(d-1)$-cells (see [15, Chapter 2, Theorem 5.1]). Hence when we remove σ and σ' from Γ and Γ', we obtain pseudomanifolds with boundary, whose boundaries are isomorphic to the boundary of a $(d-1)$-simplex. Glue these two pseudomanifolds together by choosing an isomorphism between their boundaries and using it to identify their boundaries. It is easy to check that we obtain a regular CW-complex Λ whose face poset $Q = \mathscr{F}(\Lambda)$ has the desired properties. $\square$

Lemma 4.2. *Fix $d \in \mathbb{P}$. Let δ_i denote the vector $(a_0, a_1, \ldots, a_d)$ with $a_j = \delta_{ij}$ (Kronecker delta). The following vectors $(h_0, h_1, \ldots, h_d)$ are h-vectors of Gorenstein* simplicial posets P of rank d:*

(a) $\delta_0 + \delta_d = (1, 0, 0, \ldots, 0, 1)$,

(b) $\delta_0 + \delta_i + \delta_{d-i} + \delta_d$, *for* $0 < i < d$,

(c) $\delta_0 + \delta_1 + \cdots + \delta_d = (1, 1, \ldots, 1)$.

Proof. (a) Take two $(d-1)$-simplices and glue them along their boundaries (according to some isomorphism of their boundaries), obtaining a regular CW-complex Λ. Take P to be the face-poset $\mathcal{F}(\Lambda)$. (Equivalently, P is obtained from the boolean algebra B_d by adjoining an element which covers every coatom of B_d.)

(b) Let $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ be $(d-1)$-simplices. Choose any i facets ($=(d-2)$-faces) of σ_1 and σ_2, and choose $d-i$ facets of σ_3.Glue σ_2 to σ_1 along the i chosen facets. Glue σ_3 to σ_1 by identifying the $d-i$ chosen facets of σ_3 with the $d-i$ remaining facets of σ_1 (the ones that were not identified with facets of σ_2). At this stage we have a pseudomanifold Λ' with boundary, whose boundary is isomorphic to the boundary of σ_4. Glue σ_4 to Λ' by identifying their boundaries. We obtain a regular CW-complex Λ, and we can take $P = \mathcal{F}(\Lambda)$.

(c) Take $P = \mathcal{F}(\Lambda)$, where Λ is the boundary complex of a d-simplex. (Equivalently, $P = B_{d+1} - \{\hat{1}\}$.) □

Combining Lemmas 4.1 and 4.2 yields the following result.

Theorem 4.3. *Let $\boldsymbol{h} = (h_0, h_1, \ldots, h_d) \in \mathbb{N}^{d+1}$, with $h_i = h_{d-i}$ and $h_0 = 1$. Any of the following (mutually exclusive) conditions are sufficient for the existence of a Gorenstein* simplicial poset P of rank d and h-vector $h(P) = \boldsymbol{h}$:*

(a) *d is odd,*

(b) *d is even and $h_{d/2}$ is even,*

(c) *d is even, $h_{d/2}$ is odd, and $h_i > 0$ for $0 \leq i \leq d$.* □

We now turn to a necessary condition on the h-vector of a Gorenstein* simplicial poset which shows in particular that the conditions $h_i \geq 0$, $h_i = h_{d-i}$, and $h_0 = 1$ are not sufficient. For the proof we use the following refinement of the Dehn–Sommerville equations:

Proposition 4.4. *Let P be a Gorenstein* simplicial poset of rank d, with atoms $x_1, \ldots, x_n$. Regard $x_1, \ldots, x_n$ as independent indeterminates. Given $y \in P$, let $A(y)$ denote the set of atoms $x_i \leq y$. Define a rational function*

$$L_P(x_1, \ldots, x_n) = \sum_{y \in P} \prod_{x_i \in A(y)} \frac{x_i}{1 - x_i} . \tag{7}$$

Then

$$(-1)^d L_P(1/x_1, \ldots, 1/x_n) = L_P(x_1, \ldots, x_n) .$$

First proof. Let $u = x_1^{a_1} \cdots x_n^{a_n}$ be a monomial with support $S := \{x_i : a_i > 0\}$. Then the coefficient of u in $L_P(x_1, \ldots, x_n)$, denoted $[u]L_P(x_1, \ldots, x_n)$, is given by

$$[u]L_P(x_1, \ldots, x_n) = \#\{y \in P \colon A(y) = S\} .$$

On the other hand, we have

$$(-1)^d L_P(1/x_1, \ldots, 1/x_n) = \sum_{y \in P} (-1)^{d-\rho(y)} \prod_{x_i \in A(y)} \frac{1}{1 - x_i} .$$

It follows that

$$[u](-1)^d L_P(1/x_1, \ldots, 1/x_n) = \sum_{\substack{y \in P \\ S \subseteq A(y)}} (-1)^{d-\rho(y)} .$$

Let $\{y_1, \ldots, y_r\} = \{y \in P \colon A(y) = S\}$, and define $V_i = \{y \in P \colon y \geq y_i\}$. Since every interval $[\hat{0}, y]$ of P is a boolean algebra, the sets $V_1, \ldots, V_r$ are pairwise disjoint. Hence

$$[u](-1)^d L_P(1/x_1, \ldots, 1/x_n) = \sum_{i=1}^{r} \sum_{y \geq y_i} (-1)^{d-\rho(y)} .$$

Let μ denote the Möbius function of the poset $P \cup \{\hat{1}\}$. For $y \geq y_i$ in the above sum, the interval $[y_i, y]$ is a boolean algebra; hence $\mu(y_i, y) = (-1)^{\rho(y)-\rho(y_i)}$ (see [16, Example 3.8.3]). Thus the above sum on $y \geq y_i$ becomes

$$\sum_{y \geq y_i} (-1)^{d-\rho(y)} = (-1)^{d-\rho(y_i)} \sum_{y \geq y_i} \mu(y_i, y) = -(-1)^{d-\rho(y_i)} \mu(y_i, \hat{1}) ,$$

by the fundamental recurrence [16, equation (3.14)] for Möbius functions. But, as mentioned earlier, $P \cup \{\hat{1}\}$ is Eulerian since P is Gorenstein*. Hence (by definition of Eulerian posets) $\mu(y_i, \hat{1}) = (-1)^{d+1-\rho(y_i)}$, so we get

$$\begin{aligned} [u](-1)^d L_P(1/x_1, \ldots, 1/x_n) &= \sum_{i=1}^{r} 1 = r \\ &= [u]L_P(x_1, \ldots, x_n) . \qquad \square \end{aligned}$$

Second proof (sketch). We can give the ring A_P an $\mathbb{N}^n$-grading by defining for $y \in P$,

$$\deg y = (\varepsilon_1, \ldots, \varepsilon_n) ,$$

where $\varepsilon_i = 1$ if $x_i \leq y$, and $\varepsilon_i = 0$ otherwise. Then $L_P(x_1, \ldots, x_d)$ becomes the Hilbert series of A_P with respect to this grading. Since P is Gorenstein*, A_P is a Gorenstein ring [7, Corollary 7.2]. The symmetry of Hilbert series of graded

Gorenstein algebras [13, Theorem 6.1] gives the desired result up to a monomial factor, and it is an easy matter to compute this factor (e.g., by comparing constant terms and using $\mu(\hat{0}, \hat{1}) = (-1)^{d+1}$). $\square$

Theorem 4.5. *Let P be a Gorenstein* simplicial poset of even rank $d = 2e$. If $h_1(P) = 0$ (equivalently, P has $n = d$ atoms), then $h_e(P)$ is even.*

Proof. For the present, we do not assume that $h_1(P) = 0$. Let $L_P(x_1, \ldots, x_n)$ be defined by (7). Write

$$L_P(x_1, \ldots, x_n) = J_P(x_1, \ldots, x_n) / \prod (1 - x_i) , \tag{8}$$

where the product is over all atoms of P, so that $J_P(x_1, \ldots, x_n) \in \mathbb{Z}[x_1, \ldots, x_n]$. Proposition 4.4 implies that

$$(-1)^{n-d} x_1 \cdots x_n \, J_P(1/x_1, \ldots, 1/x_n) = J_P(x_1, \ldots, x_n) . \tag{9}$$

Now let $h_1(P) = 0$ (i.e., $n - d = 0$). Let $S \subseteq \{1, 2, \ldots, n\}$ and write $x^S := \prod_{i \in S} x_i$. Let $\bar{S} = \{1, 2, \ldots, n\} - S$. It follows from (9) that the coefficients of x^S and $x^{\bar{S}}$ in $J_P(x_1, \ldots, x_n)$ are equal. In particular, the homogeneous component of $J_P(x_1, \ldots, x_n)$ of degree e may be written as an integer linear combination of distinct terms $x^S + x^{\bar{S}}$. Thus the coefficient of x^e in $J_P(x, \ldots, x)$ is even. But comparing (1) with (7) shows that $J_P(x, \ldots, x) = \sum_{i=0}^{d} h_i x^i$, and the proof follows. $\square$

Remark 1. The proofs of Proposition 4.4 and Theorem 4.5 work just as well if we assume only that P is a simplicial poset for which $P \cup \{\hat{1}\}$ is Eulerian of rank $d + 1$.

Remark 2. Let P be a Gorenstein* simplicial poset of any rank d, with exactly d atoms $x_1, \ldots, x_d$ (so $h_1(P) = 0$). Then $x_1, \ldots, x_d$ are an h.s.o.p. for the ring A_P defined in the previous section. From this it follows from standard arguments (using the fact that A_P is Gorenstein and therefore Cohen–Macaulay) that the coefficients of the polynomial $J_P(x_1, \ldots, x_n)$ (8) are *nonnegative*.

Remark 3. The proof technique of Theorem 4.5 shows that if P is a Gorenstein* simplicial poset of rank d (or even just if P is a simplicial poset of rank d such that $P \cup \{\hat{1}\}$ is Eulerian) with an even number $n = 2m$ of atoms, then the coefficient of x^m in $(h_0 + h_1 x + \cdots + h_d x^d)(1 - x)^{n-d}$ is even. However, it is easy to show this fact for *any* symmetric (i.e., $h_i = h_{d-i}$) integer polynomial $h_0 + h_1 x + \cdots + h_d x^d$ when $n - d > 0$, so we obtain nothing new when $n - d > 0$.

Remark 4. Lemma 4.1, Lemma 4.2, and Theorem 4.3 remain valid if for every poset T appearing in these results, we replace the condition that T is Gorenstein* with the stronger condition that $|\Delta(T-\{\hat{0}\})|$ is homeomorphic to a $(d-1)$-sphere. It seems likely that any h-vector of a Gorenstein* simplicial poset is also the h-vector of a simplicial poset P for which $|\Delta(P-\{\hat{0}\})|$ is a sphere. (The corresponding result for simplicial complexes is also open and probably very difficult.)

Remark 5. What vectors are h-vectors of Gorenstein* simplicial posets? It is conceivable that the sufficient conditions of Theorem 4.3 are necessary. In this direction we state without proof some results of A. Duval.

Proposition 4.6. (a) *Suppose that P is a simplicial poset for which $P \cup \{\hat{1}\}$ is Eulerian of rank $d+1$, where d is even. Suppose also that the h-vector of P satisfies either $h_0 = h_1 = \cdots = h_i = 1$ and $h_{i+1} = 0$ for some $i < d/2$, or else $h_1 = 2$, $h_2 = 0$. Then $h_{d/2}$ is even.*

(b) *Suppose P is a Gorenstein* simplicial poset of even rank d, and that $h_1 = 3$, $h_2 = 0$. Then $h_{d/2}$ is even.* □

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