VARIATIONS ON DIFFERENTIAL POSETS*

RICHARD P. STANLEY†

1. Introduction. Differential posets were introduced in [Sta$_3$]. They are partially ordered sets with many remarkable algebraic and combinatorial properties. In this paper we will consider ways to modify or extend the definition of differential posets and still retain some of their basic properties. This paper is essentially a sequel to [Sta$_3$], and familiarity with [Sta$_3$] will be useful but not essential for understanding this paper. In particular, if P is a poset and K a field, then KP denotes the K-vector space with basis P, while $\hat{K}P$ denotes the K-vector space of arbitrary (i.e., infinite) linear combinations of elements of P. If P is locally finite and $x \in P$, then define

$$C^+(x) = \{y \in P : y \text{ covers } x\},$$
$$C^-(x) = \{y \in P : x \text{ covers } y\}.$$

Furthermore, define continuous (i.e., infinite linear combinations are preserved) linear transformations $U, D : \hat{K}P \to \hat{K}P$ by

$$Ux = \sum_{y \in C^+(x)} y, \quad Dx = \sum_{y \in C^-(x)} y,$$

for all $x \in P$. If $S \subseteq P$ then we write

$$\mathbf{S} = \sum_{x \in S} x \in \hat{K}P.$$

If P is r-differential then we have

$$DU - UD = rI, \tag{1}$$

$$D\mathbf{P} = (U + r)\mathbf{P}. \tag{2}$$

(See Theorems 2.2 and 2.3 of [Sta$_3$].) We will consider three main variations of differential posets, all involving either modifications of the definition of U and D or modifications of equations (1) and (2).

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†Department of Mathematics, MIT, Cambridge, MA 02139

2. Sequentially differential posets. Let $\mathbf{r} = (r_0, r_1, \dots)$ be an infinite sequence of integers.

2.1 Definition. A poset P is called $\mathbf{r}$-*differential* if it satisfies the following three conditions:

(S1) P is locally finite and graded, and has a $\hat{0}$ element.

(S2) If $x \neq y$ in P and there are exactly k elements of P which are covered by both x and y, then there are exactly k elements of P which cover both x and y.

(S3) If $x \in P_i$ and x covers exactly k elements of P, then x is covered by exactly $k + r_i$ elements of P. (Here P_i denotes the set of elements of P of rank i.)

If P is an $\mathbf{r}$-differential poset for some sequence $\mathbf{r}$, then we call P a *sequentially differential poset.* □

Properties (S1) and (S2) coincide with (D1) and (D2) of [Sta$_3$, Def. 1.1], while (S3) is a weakening of (D3). Thus Proposition 1.2 of [Sta$_3$] remains true for sequentially differential posets, i.e., the integer k of (S2) must be 0 or 1. (Thus given (S1), condition (S2) coincides with what Proctor calls *uniquely modular* [P_1, p. 270].) Moreover, the next three results are proved in exactly the same way as the corresponding results of [Sta$_3$] (viz., Proposition 1.3 and Theorems 2.2 and 2.3).

2.2 PROPOSITION. *Let L be a lattice satisfying (S1) and (S3). Then L is $\mathbf{r}$-differential if and only if L is modular.* □

As in [Sta$_3$], if $A : \hat{K}P \to \hat{K}P$ is a linear transformation then A_j denotes the restriction of A to KP_j; and we can unambiguously use notation such as AB_j, since $A(B_j)$ and $(AB)_j$ have the same meaning. In particular I_j denotes the identity transformation $I : \hat{K}P \to \hat{K}P$ restricted to KP_j. We often omit subscripts if they are clear from context, e.g., in equation (18) it is clear that U^iD^j means $U^iD^j_k$.

2.3 PROPOSITION. *Let P be a locally finite graded poset with $\hat{0}$, with finitely many elements of each rank. Let $\mathbf{r} = (r_0, r_1, \dots)$ be a sequence of integers. The following two conditions are equivalent:*

$$(3) \qquad \begin{array}{ll} (a) & P \text{ is } \mathbf{r}\text{–differential}, \\ (b) & DU_j - UD_j = r_jI_j, \quad \text{for all } j \geq 0. \end{array} \qquad \square$$

If P is finite in Proposition 2.3 and if j is greater than the rank of P (so that $P_j = \varnothing$), then (b) is regarded as vacuously true (whatever the value of r_j). For instance, a single point is $\mathbf{r}$-differential for any sequence $\mathbf{r}$ with $r_0 = 0$.

2.4 PROPOSITION. *If P is an $\mathbf{r}$-differential poset, then*

$$(4) \qquad D_{j+1}\mathbf{P}_{j+1} = U_{j-1}\mathbf{P}_{j-1} + r_j\mathbf{P}_j. \quad \square$$

Before discussing properties of $\mathbf{r}$-differential posets, let us list some finite examples.

2.5 Example. The following finite posets are $\mathbf{r}$-differential:

(a) An n-element chain ($r_0 = 1, r_i = 0$ for $1 \le i \le n-2, r_{n-1} = -1$).

(b) The boolean algebra B_n of rank n ($r_i = n - 2i$, $0 \le i \le n$).

(c) A product C_3^n of n 3-element chains C_3 ($r_i = n - i$, $0 \le i \le 2n$).

(d) The lattice $L_n(q)$ of subspaces of an n-dimensional vector space over the finite field $\mathbb{F}_q$ ($r_i = 1 + q + \cdots + q^{n-i-1} - (1 + q + \cdots + q^{i-1})$, $\quad 0 \le i \le n$).

(e) If P is $\mathbf{r}$-differential and finite of rank n (in which case it must have a $\hat{1}$) and Q is $\mathbf{s}$-differential, then the poset $P * Q$ obtained by identifying $\hat{1} \in P$ with $\hat{0} \in Q$ is $(r_0, \ldots, r_{n-1}, r_n + s_0, s_1, s_2, \ldots)$-differential.

(f) If P is $\mathbf{r}$-differential and finite of rank n, then the dual P^* is $(-r_n, -r_{n-1}, \ldots, -r_0)$-differential.

There are many other finite $\mathbf{r}$-differential posets, and it is probably hopeless to attempt to classify them all. It may be more tractable to find the finite $\mathbf{r}$-differential distributive lattices $L = J(P)$ (where $J(P)$ has the meaning of [Sta$_2$, Thm. 3.4.1]). Example 2.5(b, c, e) gives some examples. A further class of finite $\mathbf{r}$-differential distributive lattice may be constructed as follows. Suppose P and Q are finite posets of cardinalities m and n, respectively, such that $J(P)$ is $\mathbf{r}$-differential for $r = (r_0, \ldots, r_m)$ and $J(Q)$ is $\mathbf{s}$-differential for $\mathbf{s} = (s_0, \ldots, s_n)$. Suppose also that $r_m = s_0 = t$, say; and let $M = \{x_1, \ldots, x_t\}$ be the set of maximal elements of P and $N = \{y_1, \ldots, y_t\}$ the set of minimal elements of Q. Define a new poset $P\#Q$ on the disjoint union $P + Q$ by imposing the additional relations $x_i < y_j$ whenever $i \ne j$ (and all relations then implied by transitivity). Then $J(P\#Q)$ is $(r_0, \ldots, r_{m-2}, r_{m-1} + 1, 0, s_1 - 1, s_2, \ldots, s_n)$-differential. If we take P and Q to be 2-element antichains, then $J(P\#Q) = C_3 \times C_3$, where C_3 denotes a 3-element chain. If we take $P = Q = B_3$ (the boolean algebra of rank 3), then $J(P\#Q) = FD(3)$, the free distributive lattice on 3 generators [Sta$_2$, Exer. 3.24].

In general, one sees as in [Sta$_3$, Prop. 5.5] that for fixed $\mathbf{r}$ there is at most one $\mathbf{r}$-differential distributive lattice $L(\mathbf{r})$ (up to isomorphism). It is probably hopeless to determine for which $\mathbf{r}$ the lattice $L(\mathbf{r})$ exists; if the sequence $r_0, r_1, r_2, \ldots$ increases "sufficiently fast" then $L(\mathbf{r})$ exists, but it seems difficult to describe the necessary rate of growth precisely. Conceivably there are no other finite $\mathbf{r}$-differential distributive lattices besides those described above. A straightforward generalization of the construction of [Sta$_3$, Prop. 6.1] shows that if all $r_i \ge 0$, then there exists an $\mathbf{r}$-differential (modular) lattice.

Let us turn to the enumerative properties of sequentially differential posets. The basic principle here is that all the enumerative results of [Sta$_3$] can be extended to $\mathbf{r}$-differential posets, but their statements no longer involve generating functions and thus become more complicated. Consider, for instance, the number $\alpha(0 \to n)$ of saturated chains from $\hat{0}$ to P_n. According to [Sta$_3$, eqn. (12)], in an r-differential poset we have

$$\sum_{n \ge 0} \alpha(0 \to n) \frac{t^n}{n!} = \exp(rt + \frac{1}{2} rt^2).$$

Equivalently,

(5) $$\alpha(0 \to n) = \sum_w r^{c(w)},$$

where w ranges over all involutions in S_n and $c(w)$ is the number of cycles of w. For $\mathbf{r}$-differential posets, the term $r^{c(w)}$ in (5) is replaced by a certain monomial $r_0^{c_0}r_1^{c_1}\ldots$ (where $\sum c_i = c(w)$), as follows:

2.1 THEOREM. *Let P be an $\mathbf{r}$-differential poset. Then*

$$\alpha(0 \to n) = \sum_w \prod_m r_{\eta(w,m)}, \tag{6}$$

where (a) w ranges over all involutions $w_1w_2\cdots w_n$ in S_n, (b) m ranges over all weak excedances of w (i.e., $w_m \geq m$), and (c) $\eta(w,m)$ is the number of integers j satisfying $j < m$ and $w_j < w_m$.

For example, when $n = 3$ we have the following table, where the positions of the weak excedances are underlined:

involution w	values of $\eta(w,m)$
$\underline{1}\ \underline{2}\ \underline{3}$	$0,1,2$
$\underline{2}\ 1\ \underline{3}$	$0,2$
$\underline{1}\ \underline{3}\ 2$	$0,1$
$\underline{3}\ \underline{2}\ 1$	$0,0$

Hence $\alpha(0 \to 3) = r_0^2 + r_0r_1 + r_0r_2 + r_0r_1r_2$. Let us note that the number of weak excedances of an involution w is equal to $c(w)$, so that Theorem 2.1 reduces to (5) when each $r_i = r$.

Proof of Theorem 2.1. Consider the element

$$D^k\mathbf{P_{i+k}} = D_{i+1}D_{i+2}\cdots D_{i+k}\mathbf{P_{i+k}}$$

of KP_i. It is evident that repeated uses of Propositions 2.3 and 2.4 will express this element as a linear combination of elements of the form $U^j\mathbf{P_{i-j}}$. For instance,

$$\begin{aligned}
D^2\mathbf{P_{i+2}} &= D(U\mathbf{P_i} + r_{i+1}\mathbf{P_{i+1}})\\
&= (UD + r_i)\mathbf{P_i} + r_{i+1}(U\mathbf{P_{i-1}} + r_i\mathbf{P_i})\\
&= U(U\mathbf{P_{i-2}} + r_{i-1}\mathbf{P_{i-1}}) + r_{i+1}U\mathbf{P_{i-1}} + r_i(1 + r_{i+1})\mathbf{P_i}\\
&= U^2\mathbf{P_{i-2}} + (r_{i-1} + r_{i+1})U\mathbf{P_{i-1}} + r_i(1 + r_{i+1})\mathbf{P_i}.
\end{aligned}$$

We claim that in general,

$$D^k\mathbf{P_{i+k}} = \sum_{j=0}^{k}\left[\sum_w \prod_m r_{i+\eta(w,m)-\nu(w,m)}\right] U^j\mathbf{P_{i-j}}, \tag{7}$$

where (a) for j fixed, w runs over all involutions in S_k with j of the fixed points in w circled (so that w has $\geq j$ fixed points), (b) m ranges over all *uncircled* weak excedances of w, (c) $\eta(w,m)$ has the same meaning as above, and (d) $\nu(w,m)$ is the number of *circled* fixed points $d = w_d$ such that $d > w_m$. For example, if $w = 16③482⑦5$, then $\eta(1) = 0, \nu(1) = 2$, $\eta(2) = 1, \nu(2) = 1, \eta(4) = 2, \nu(4) = 1, \eta(5) = 4, \nu(5) = 1$, yielding $r_{i-2}r_ir_{i+1}r_{i+3}$.

We prove (7) by induction on k. For $k = 0$ it asserts that $\mathbf{P_i} = \mathbf{P_i}$, which is clear. Assume for k. Then

$$D^{k+1}\mathbf{P_{i+k+1}} = D\sum_{j=0}^{k}\left[\sum_{w}\prod_{m} r_{i+1+\eta(w,m)-\nu(w,m)}\right] U^j\mathbf{P_{i+1-j}}. \tag{8}$$

Now an easy induction argument shows that

$$DU^j_{i+1-j} = U^j D_{i+1-j} + (r_{i+1-j} + r_{i+2-j} + \cdots + r_i)U^{j-1}_{i+1-j}, \tag{9}$$

so that by Proposition 2.4,

$$DU^j\mathbf{P_{i+1-j}} = U^j(U\mathbf{P_{i-j-1}} + r_{i-j}\mathbf{P_{i-j}}) + (r_{i+1-j} + \cdots + r_i)U^{j-1}\mathbf{P_{i+1-j}}.$$

Hence (8) becomes

$$D^{k+1}\mathbf{P_{i+k+1}} = \sum_{j=0}^{k}\left[\sum_{w}\prod_{m} r_{i+1+\eta(w,m)-\nu(w,m)}\right] \cdot (U^{j+1}\mathbf{P_{i-j-1}} + r_{i-j}U^j\mathbf{P_{i-j}} + (r_{i+1-j} + \cdots + r_i)U^{j-1}\mathbf{P_{i-j+1}}).$$

Let $I(k,j)$ denote the set of involutions in S_k with j circled fixed points. Then we need to show that for $0 \le j \le k+1$,

$$\sum_{w\in I(k+1,j)}\prod_{m} r_{i+\eta(w,m)-\nu(w,m)} = \left(\sum_{w\in I(k,j-1)} + r_{i-j}\sum_{w\in I(k,j)} + (r_{i-j} + \cdots + r_i)\sum_{w\in I(k,j+1)}\right)\prod_{m} r_{i+1+\eta(w,m)-\nu(w,m)}. \tag{10}$$

Define a bijection

$$\varphi : I(k,j-1) \cup I(k,j) \cup [I(k,j+1) \times \{1,2,\cdots,j+1\}] \to I(k+1,j)$$

as follows. If $w = w_1w_2\cdots w_k \in I(k,j-1)$ (where w_i is an uncircled or circled integer) then $\varphi(w) = ①, w_1+1, w_2+1, \cdots, w_k+1$ (where here and below w_i+1 is circled if and only if w_i is circled). If $w = w_1w_2\cdots w_k \in I(k,j)$, then $\varphi(w) = 1, w_1+1, \cdots, w_k+1$. If $w = w_1w_2\cdots w_k \in I(k,j+1)$ and $1 \le t \le j+1$, then $\varphi(w,t)$ is obtained from $w_1+1, w_2+1, \ldots, w_k+1$ by replacing the t-th circled term from the left, say $w_\ell+1$, by an uncircled 1 and placing an uncircled $w_\ell+1$ at the beginning.

Example. Let $k=7$, $j=2$. Then

$$\begin{aligned}
\varphi(1\,5\,3\,7\,2\,⑥\,4) &= ①\,2\,6\,4\,8\,3\,⑦\,5\\
\varphi(①\,5\,3\,7\,2\,⑥\,4) &= 1\,②\,6\,4\,8\,3\,⑦\,5\\
\varphi(①\,5\,③\,7\,2\,⑥\,4,\ 1) &= 2\,1\,6\,④\,8\,3\,⑦\,5\\
\varphi(①\,5\,③\,7\,2\,⑥\,4,\ 2) &= 4\,②\,6\,1\,8\,3\,⑦\,5\\
\varphi(①\,5\,③\,7\,2\,⑥\,4,\ 3) &= 7\,②\,6\,④\,8\,3\,1\,5.
\end{aligned}$$

It is easily seen that φ is a bijection. Moreover, if $w \in I(k, j-1)$ then m is an uncircled weak excedance of $\varphi(m)$ if and only if $m-1$ is an uncircled weak excedance of w; and $\eta(\varphi(w), m) = 1 + \eta(w, m-1)$, $\nu(\varphi(w), m) = \nu(\varphi(w), m-1)$. If $w \in I(k,j)$ then m is an uncircled weak excedance of $\varphi(w)$ if and only if $m=1$ or $m-1$ is an uncircled weak excedance of w; and $\eta(\varphi(w), 1) = 0$, $\nu(\varphi(w), 1) = j$, $\eta(\varphi(w), m) = 1 + \eta(w, m-1)$ for $m > 1$, $\nu(\varphi(w), m) = \eta(w, m-1)$ for $m > 1$. Finally if $w \in I(k, j+1)$ and $1 \le t \le j+1$, then m is an uncircled weak excedance of $\varphi(w,t)$ if and only if $m=1$ or $m-1$ is an uncircled weak excedance of w; and $\eta(\varphi(w,t), 1) = 0, \nu(\varphi(w,t), 1) = t-1$, while for $m > 1$ it is not difficult to check that

$$\eta(\varphi(w,t), m) - \nu(\varphi(w,t), m) = 1 + \eta(w, m-1) - \nu(w, m-1).$$

From these observations (10) follows, and hence also (7) for $k+1$ by induction.

Now let $i=0$ in (7). The left-hand side becomes $\alpha(0 \to k)\hat{0}$, while the right-hand side becomes (since $\mathbf{P}_{\mathbf{i-j}} = 0$ for $j > 0$)

$$\Big(\sum_w \prod_m r_{\eta(w,m)}\Big)\hat{0},$$

where w and m are as in (6). This completes the proof. □

Consider now the problem of evaluating $\alpha(n \to n+k)$, i.e., the number of saturated chains from P_n to P_{n+k}. In [Sta$_3$, Thm. 3.2] a certain polynomial $A_k(q)$ (whose coefficients are polynomials in r) was defined for which

$$\sum_{n \ge 0} \alpha(n \to n+k) q^n = A_k(q) F(P, q),$$

where $F(P,q) := \sum (\#P_n) q^n$ is the rank-generating function of P. An analogous but more complicated result holds for $\mathbf{r}$-differential posets. By (7) there is a polynomial $T_{jk}(y_0, y_{\pm 1}, \dots)$ in the variables $y_0, y_{-1}, y_1, \dots$ such that

$$D^k \mathbf{P}_{\mathbf{i+k}} = \sum_{j=0}^{k} T_{jk}(r_i, r_{i\pm 1}, \dots) U^j \mathbf{P}_{\mathbf{i-j}}. \tag{11}$$

2.2 Theorem. *Let P be an $\mathbf{r}$-differential poset. Then the numbers $\alpha(n \to n+k)$ are given recursively (in k) by the formula*

$$\alpha(n \to n+k) = \sum_{\substack{0 \le i \le n \\ i \equiv n \ (\mathrm{mod}\ k)}} \sum_{j=0}^{k-1} T_{jk}(r_i, r_{i\pm 1}, \dots) \alpha(i-j \to i). \tag{12}$$

Proof. Put $i = n$ in (11) and apply the linear transformation $\sigma : KP \to K$ defined by $\sigma(x) = 1$ for all $x \in P$. We obtain

$$\begin{aligned}\alpha(n \to n+k) &= \sum_{j=0}^{k} T_{jk}(r_n, r_{n\pm1}, \dots)\alpha(n-j \to n) \\ &= \alpha(n-k \to n) + \sum_{j=0}^{k-1} T_{jk}(r_n, r_{n\pm1}, \dots)\alpha(n-j \to n),\end{aligned}$$

since $T_{kk} = 1$. The solution to this recurrence (with the initial condition $\alpha(i \to i+k) = 0$ if $i < 0$) is clearly given by (12). □

By repeated applications of (12) we will eventually express $\alpha(n \to n+k)$ in the form

$$\alpha(n \to n+k) = \sum_{i=0}^{n} R_{ik}(r_{n-i}, r_{n-i+1}, \dots, r_{n+k-1})p_{n-i},$$

where $p_{n-i} = \#P_{n-i}$ and $R_{ik}(y_1, y_2, \dots, y_{k+i})$ is a polynomial in $y_1, y_2, \dots, y_{k+i}$ (independent of n). For instance,

$$\begin{aligned}\alpha(n \to n) &= p_n \\ \alpha(n \to n+1) &= \sum_{i=0}^{n} r_i p_i \\ \alpha(n \to n+2) &= \sum_{\substack{i \le n \\ i \equiv n \pmod 2}} [(r_{i-1} + r_{i+1}) \sum_{j=0}^{i-1} r_j p_j + r_i(r_{i+1}+1)p_i].\end{aligned}$$

It would be interesting to find a more explicit formula for $\alpha(n \to n+k)$, along the lines of (6) (the case $n = 0$).

Consider now a word $w = w(U, D) = w_1 w_2 \cdots w_\ell$ in the letters U and D. Let $x \in P$. We wish to compute the quantity $\langle w\hat{0}, x\rangle$ (using the scalar product defined in [Sta$_3$, §2]), i.e., the number of Hasse walks $\hat{0} = x_0, x_1, \dots, x_\ell = x$ with the cover relation $x_{i-1} < x_i$ or $x_{i-1} > x_i$ specified by w. For instance, $\langle UDDUDUU\hat{0}, x\rangle$ is the number of Hasse walks $\hat{0} = x_0 < x_1 < x_2 > x_3 < x_4 > x_5 > x_6 < x_7 = x$. Clearly $\langle w\hat{0}, x\rangle = 0$ unless (a) for all $1 \le i \le \ell$, the number of D's among $w_i, w_{i+1}, \dots, w_\ell$ does not exceed the number of U's, and (b) the difference between the number of U's and number of D's in w is the rank $\rho(x)$ of x. Let us call such a word w a ***valid x-word.*** As in [Sta$_3$], denote by $e(x)$ the number of saturated chains from $\hat{0}$ to x.

2.3 THEOREM. *Let P be an* r*-differential poset, and let $x \in P$. Let $w = w_1 w_2 \cdots w_\ell$ be a valid x-word. Let $S = \{i : w_i = D\}$. For each $i \in S$, let a_i be the number of D's in w to the right of or including w_i, and let b_i be the number of U's in w to the right of w_i. Set $f_i = b_i - a_i$. Then*

$$\langle w\hat{0}, x\rangle = e(x) \prod_{i \in S} (r_0 + r_1 + \cdots + r_{f_i}). \tag{13}$$

Example. Let $w = DUDDUUUU$. Then $S = \{1,3,4\}$, $a_1 = 3, b_1 = 5, f_1 = 2, a_2 = 2, b_2 = 4, f_2 = 2, a_3 = 1, b_3 = 4, f_3 = 3$, so

$$(w\hat{0}, x) = e(x)(r_0 + r_1 + r_2)^2(r_0 + r_1 + r_2 + r_3).$$

Proof of Theorem 2.3. Fix $n \geq 0$. Let $w_{(n)}$ denote the linear transformation w restricted to KP_n. By successive uses of Proposition 2.3(b) we can put $w_{(n)}$ in the form

$$w_{(n)} = \sum_{i,j} c_{ij}(w) U^i D^j, \tag{14}$$

where $c_{ij}(w)$ is a polynomial in $r_0, r_1, \ldots$ (depending on n), and where if $c_{ij} \neq 0$ then $i - j = \rho(x)$. Moreover, this representation is easily seen to be unique. Now

$$\begin{aligned} Uw_{(n)} &= \sum_{i,j} c_{ij}(w) U^{i+1} D^j \\ &\Rightarrow c_{ij}(Uw) = c_{i-1,j}(w). \end{aligned}$$

Moreover, by (9) we have

$$\begin{aligned} Dw_{(n)} &= \sum_{i,j} c_{ij}(w) D U^i_{n-j} D^j_n \\ &= \sum_{i,j} c_{ij}(w)(U^i D + (r_{n-j} + r_{n-j+1} + \cdots + r_{n-j+i-1}) U^{i-1}) D^j \\ &= \sum_{i,j} c_{ij}(w) U^i D^{j+1} \\ &\qquad + \sum c_{ij}(w)(r_{n-j} + \cdots + r_{n-j+i-1}) U^{i-1} D^j. \end{aligned}$$

It follows that

$$c_{ij}(Dw) = c_{i,j-1}(w) + c_{i+1,j}(w)(r_{n-j} + \cdots + r_{n-j+i}).$$

In particular, when $j = n = 0$ we have

$$c_{i0}(Uw) = c_{i-1,0}(w) \tag{15}$$

$$c_{i0}(Dw) = c_{i+1,0}(w)(r_0 + \cdots + r_i). \tag{16}$$

Now put $n = 0$ in (14) and operate on $\hat{0}$. Since $D^j\hat{0} = 0$ for $j > 0$, we get (setting $\rho = \rho(x)$)

$$w\hat{0} = c_{\rho 0}(w) U^\rho \hat{0}.$$

Thus

$$(w\hat{0}, x) = c_{\rho 0}(w) e(x).$$

It is easy to see from (15) and (16) that

$$c_{\rho 0}(w) = \prod_{i \in S} (r_0 + r_1 + \cdots + r_{f_i}),$$

so the proof follows. □

The previous theorem generalizes [Sta$_3$, Thm. 3.7]. When we put $w = D^n U^n$ in Theorem 2.3 (so $x = \hat{0}$), we obtain the following generalization of [Sta$_3$, Cor. 3.9]:

2.4 COROLLARY. *Let P be an* r*-differential poset. Then*

$$\alpha(0 \to n \to 0) = \sum_{x \in P_n} e(x)^2 = \prod_{i=0}^{n-1} (r_0 + r_1 + \cdots + r_i). \qquad \square$$

As a variation of Theorem 2.3, let us replace the word w by $(D+U)^n$. Thus $\langle (D+U)^n \hat{0}, x\rangle$ is the number of Hasse walks $\hat{0} = x_0, x_1, \ldots, x_n = x$ of length n from $\hat{0}$ to x.

2.5 THEOREM. *Let P be an* r*-differential poset, and let $x \in P_i$. Then*

$$\langle (D+U)^n \hat{0}, x\rangle = e(x) \sum_w \prod_s r_{\gamma(w,s)+\delta(w,s)} \tag{17}$$

where (a) $w = w_1 w_2 \cdots w_n$ ranges over all involutions in S_n with exactly i fixed points, (b) s ranges over the excedance set $\{s : w_s > s\}$, (c) $\gamma(w,s)$ is the number of fixed points $t = w_t$ such that $s < t < w_s$, and (d) $\delta(w,s) = \#\{t : s < t < w_s < w_t\}$.

Example. Let $n = 4$ and $i = 0$. For each involution $w \in S_4$ with no fixed points, let a and b denote the two indices s for which $w_s > s$. Then we have:

w	a	$\gamma(w,a)$	$\delta(w,a)$	b	$\gamma(w,b)$	$\delta(w,b)$
2143	1	0	0	3	0	0
3412	1	0	1	2	0	0
4321	1	0	0	2	0	0

Hence

$$\langle (D+U)^4 \hat{0}, \hat{0}\rangle = 2r_0^2 + r_0 r_1.$$

Proof of Theorem 2.5 (sketch). The proof is analogous to that of Theorem 2.1. Instead of (7), one proves by induction on n that

$$(D+U)_k^n = \sum_{i,j} b_{ij}(n) U^i D^j, \tag{18}$$

where

$$b_{ij}(n) = \sum_w \prod_s r_{k+\gamma(w,s)+\delta(w,s)-\epsilon(w,s)} \tag{19}$$

where (a) $w = w_1 w_2 \cdots w_n$ ranges over all involutions in S_n with i uncircled fixed points and j circled fixed points, (b) s ranges over the set $\{s : w_s > s\}$ (the number of such s is $\frac{1}{2}(n-i-j)$), (c) $\gamma(w,s)$ is the number of *uncircled* fixed points $t = w_t$ such that $s < t < w_s$ (d) $\delta(s) = \#\{t : s < t < w_s < w_t\}$, and (e) $\epsilon(s)$ is the number of *circled* fixed points $t = w_t$ such that $t < w_s$. The proof of (17) then follows by applying (18) to $\hat{0}$ (so $k = 0$) and taking the coefficient of x. □

When all $r_i = r$ (i.e, P is r-differential) then (17) simplifies considerably. For each w, the product over s is just $r^{\frac{1}{2}(n-i)}$, so that

$$\begin{aligned}\langle (D+U)^n\hat{0}, x\rangle &= e(x) r^{\frac{1}{2}(n-i)} I_i(n)\\ &= e(x) r^{\frac{1}{2}(n-i)} \binom{n}{i} (1\cdot 3\cdot 5\cdots (n-i-1)),\end{aligned}$$

where $I_i(n)$ is the number of involutions in S_n with i fixed points. This is essentially the result appearing after Proposition 3.17 in [Sta$_3$].

In the special case when P is the infinite chain $0 < 1 < 2 < \cdots$ (so $\mathbf{r} = (1,0,0,\cdots)$), $\langle (D+U)^{2n}\hat{0},\hat{0}\rangle$ is well-known to equal the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. On the other hand, (17) yields that $\langle (D+U)^{2n}\hat{0},\hat{0}\rangle$ is the number of fixed-point free involutions $w = w_1 w_2 \cdots w_{2n}$ in S_{2n} such that we never have $i < j < w_i < w_j$. This is another well-known combinatorial interpretation of the Catalan number C_n.

A second major class of results in [Sta$_3$] dealt with the evaluation of eigenvalues and eigenvectors of certain linear transformations associated with differential posets. Analogous results hold for sequentially differential posets, and moreover most of the proofs are exactly the same. We therefore will simply state most results without proof. As in [Sta$_3$], Ch A or Ch(A,λ) denotes the characteristic polynomial $\det(\lambda I - A)$ (normalized to be monic) of the linear transformation $A : V \to V$ on a finite-dimensional vector space V. Moreover, we write $p_j = \#P_j$ and $\Delta p_j = p_j - p_{j-1}$.

2.6 Theorem (see [Sta$_3$, Thm. 4.1]). *Let P be an $\mathbf{r}$-differential poset, and let $j \geq 0$. Then*

$$\text{Ch}(UD_j) = \prod_{i=0}^{j} (\lambda - (r_i + r_{i+1} + \cdots + r_{j-1}))^{\Delta p_i}. \qquad (20) \qquad \square$$

2.7 Corollary (see [Sta$_3$, Cor. 4.2-4.4]). *Let $j \geq 0$. Suppose that for all $0 \leq i \leq j$, we have $r_i + r_{i+1} + \cdots + r_j \neq 0$. Then U_j is injective and D_{j+1} is surjective. Hence $p_j \leq p_{j+1}$, and there is an order-matching $\mu : P_j \to P_{j+1}$ (i.e., μ is injective and $\mu(x) > x$ for all $x \in P_j$).* □

We will omit here the (easy) extension to sequentially differential posets of the discussion of balanced endomorphisms in [Sta$_3$]. However, there is a special case which is worthy of mention here.

2.8 Lemma (see [Sta$_3$, Prop. 4.7]). *Let P be an $\mathbf{r}$-differential poset. Then for all $j \geq n \geq 0$,*

$$U^n D_j^n = \prod_{i=1}^{n} (UD_j - r_{j-i+1} - r_{j-i+2} - \cdots - r_{j-1}).$$

Proof (sketch). First show by induction on n that

$$U^n D_{j-n+1} = (UD_j - r_{j-n+1} - \cdots - r_{j-1})U^{n-1}_{j-n+1}.$$

Then use the formula

$$U^n D^n_j = (U^n D_{j-n+1})D^{n-1}_j$$

to deduce the lemma by induction on n. □

2.9 Proposition (see [Sta$_3$, Ex. 4.11]). *Preserve the conditions of Lemma 2.8. Then*

$$\mathrm{Ch}(U^n D^n_j) = \lambda^{p_j - p_{j-n}} \prod_{i=0}^{j-n} (\lambda - \prod_{k=1}^{n} (r_i + r_{i+1} + \cdots + r_{j-k}))^{\Delta p_i} \tag{21}$$

$$= \lambda^{p_j - p_{j-n}} \mathrm{Ch}(D^n U^n_{j-n}). \tag{22}$$

Proof. It follows from Theorem 2.5 and Proposition 2.7 (as in the proof of [Sta$_3$, Prop 4.12]) that

(23)

$$\mathrm{Ch}(U^n D^n_j) = \prod_{i=0}^{j} (\lambda - \prod_{k=1}^{n} (r_i + r_{i+1} + \cdots + r_{j-1} - r_{j-k+1} - r_{j-k+2} - \cdots - r_{j-1}))^{\Delta p_i}.$$

If $j - n < i \le j$ and $k = j - i + 1$, then $r_i + \cdots + r_{j-1} - r_{j-k+1} - \cdots - r_{j-1} = 0$. Hence the factors in (22) for $i > j - n$ contribute $\lambda^{\Delta p_{j-n+1} + \cdots + \Delta p_j} = \lambda^{p_j - p_{j-n}}$. If $1 \le i \le j - n$ then $r_i + \cdots + r_{j-1} - r_{j-k+1} - \cdots - r_{j-1} = r_i + r_{i+1} + \cdots + r_{j-k}$, so the formula (21) for $\mathrm{Ch}(U^n D^n_j)$ follows. To obtain (22), use the fact (mentioned in the proof of [Sta$_3$, Thm. 4.1]) that if $A : V \to W$ and $B : W \to V$ are linear transformations on finite dimensional vector spaces V and W of dimensions v and w, respectively, then

$$\mathrm{Ch}(BA) = \lambda^{v-w} \mathrm{Ch}(AB). \quad \square$$

Note. It may happen that $\Delta p_i < 0$ in (20) and (21), so the corresponding factor of $\mathrm{Ch}(UD_j)$ or $\mathrm{Ch}(U^n D^n_j)$ actually appears in the denominator. Hence (since $\mathrm{Ch}(A)$ is always a polynomial) it must be cancelled by some factor in the numerator. This puts constraints on the possible values of $\mathbf{r}$ and $F(P, q)$ which may be interesting to investigate further.

Recall now from [Sta$_1$] that a finite graded poset P of rank n is *unitary Peck* if for $0 \le j \le [n/2]$ the linear transformation

$$U^{n-2j} : KP_j \to KP_{n-j}$$

is a bijection. In particular, this condition implies that P is rank-symmetric ($p_i = p_{n-i}$) and rank unimodal (which in the presence of rank-symmetry means $p_0 \le p_1 \le \cdots \le p_{[n/2]}$).

2.10 PROPOSITION. *Let P be a finite, rank-symmetric, rank-unimodal* $\mathbf{r}$*-differential poset of rank n. Then the following two conditions are equivalent:*

(i) *P is unitary Peck,*

(ii) *if $0 \le i < [n/2]$, $\Delta p_i > 0$, and $i \le k \le n-i-1$, then*

$$r_i + r_{i+1} + \cdots + r_k \neq 0.$$

Proof. Clearly P is unitary Peck if and only if for all $0 \le j < [n/2]$, the linear transformations $D^{n-2j}U_j^{n-2j}$ and $U^{n-2j}D_{n-j}^{n-2j}$ have no zero eigenvalues. By Corollary 2.9, we have

$$\begin{aligned}\mathrm{Ch}(U^{n-2j}D_{n-j}^{n-2j}) &= \lambda^{p_{n-j}-p_j}\prod_{i=0}^{j}(\lambda - \prod_{k=1}^{n-2j}(r_i + \cdots + r_{n-j-k}))^{\Delta p_i}\\ &= \lambda^{p_{n-j}-p_j}\,\mathrm{Ch}(D^{n-2j}U_j^{n-2j}).\end{aligned}$$

Since P is rank-symmetric, we have $p_{n-j} = p_j$. Since P is rank-unimodal, we have $\Delta p_i \ge 0$ for $0 \le i \le j$. Hence the eigenvalues of $U^{n-2j}D_{n-j}^{n-2j}$ and $D^{n-2j}U_j^{n-2j}$ are given by

$$\prod_{k=1}^{n-2j}(r_i + \cdots + r_{n-j-k}), \tag{24}$$

for those i with $0 \le i \le j$ and $\Delta p_i > 0$. One easily checks that the non-vanishing of (24) for $0 \le i \le j$ and $\Delta p_i > 0$ is equivalent to (ii). □

Note: We do not know whether every finite, rank-symmetric, rank-unimodal $\mathbf{r}$-differential poset is unitary Peck.

2.11 COROLLARY. *The boolean algebra B_n and subspace lattice $L_n(q)$ of Example 2.5(b,d) are unitary Peck.*

Proof. Using the values of r_i given by Example 2.5(b,d), it is easy to check that condition (ii) of the previous proposition is satisfied. □

The unitary Peckness of B_n is implicit in [K, p. 317], though it probably goes back much earlier. Simple proofs may be found in [F-H, Lemma 5.1][G-L-L, p. 13]. The unitary Peckness of $L_n(q)$ is equivalent to a result of Kantor [K] (see [Sta_1, Thm. 2(*d*)]). Our proof seems simpler, since it is based on only simple structural properties of $L_n(q)$. (On the other hand, Kantor obtains a related result for *affine* subspaces which does not seem to follow directly from the methods here.)

Our final result of this section is a generalization of [Sta_3, Thm. 4.14]. The proof is analogous to that of [Sta_3, Thm. 4.14] and will be omitted. As in [Sta_3], given a graded poset P let $\mathcal{H}(P_{[i,j]})$ denote the Hasse graph of the rank-selected subposet

$$P_{[i,j]} = \{x \in P : i \le \rho(x) \le j\}.$$

Thus the vertices of $\mathcal{H}(P_{[i,j]})$ are the elements of $P_{[i,j]}$, and vertices x and y are joined by an (undirected) edge if x covers y or y covers x. Denote by Ch $\mathcal{H}(P_{[i,j]})$ the characteristic polynomial (normalized to be monic) of the adjacency matrix of $\mathcal{H}(P_{[i,j]})$.

Regarding $\mathbf{r} = (r_0, r_1, \ldots)$ as fixed, define for $1 \leq a \leq b-1$ and $s \geq 0$ the $(b-a+2) \times (b-a+2)$ tridiagonal matrix

$$M_{ab}^{(s)} = \begin{bmatrix} 0 & R(s,s+a-1) & 0 & 0 & & \\ 1 & 0 & R(s,s+a) & 0 & & \\ & 1 & 0 & \ddots & & \\ & & & \ddots & & \\ & & \ddots & & & \\ & & & & 0 & R(s,s+b-1) \\ & & & & 1 & 0 \end{bmatrix},$$

where

$$R(s, s+c) = r_s + r_{s+1} + \cdots + r_{s+c}.$$

Finally set

$$C_{ab}^{(s)} = \operatorname{Ch} M_{ab}^{(s)}.$$

2.12 THEOREM. *Let P be an* $\mathbf{r}$*-differential poset, and let* $0 \leq i \leq j$. *Then*

$$\operatorname{Ch} \mathcal{H}(P_{[i,j]}) = \prod_{s=0}^{j-i} (C_{1s}^{(j-s)})^{\Delta p_{j-s}} \cdot \prod_{s=j-i+1}^{j} (C_{s-j+i+1,s}^{(j-s)})^{\Delta p_{j-s}}. \quad \square$$

For $j - i \leq 2$ Theorem 2.12 leads to the formulas

$$\operatorname{Ch} \mathcal{H}(P_{[j,j]}) = \lambda^{p_j}$$

$$\operatorname{Ch} \mathcal{H}(P_{[j-1,j]}) = \lambda^{\Delta p_j} \prod_{s=1}^{j} (\lambda^2 - (r_{j-s} + r_{j-s+1} + \cdots + r_{j-1}))^{\Delta p_{j-s}}$$

$$\operatorname{Ch} \mathcal{H}(P_{[j-2,j]}) = \lambda^{\Delta p_j} (\lambda^2 - r_{j-1})^{\Delta p_{j-1}} \cdot \prod_{s=2}^{j} (\lambda^3 - (2r_{j-s} + 2r_{j-s+1} + \cdots + 2r_{j-2} + r_{j-1})\lambda)^{\Delta p_{j-s}}.$$

3. Shifted partitions. In the previous section we generalized the definition of differential poset by modifying the formula $DU - UD = rI$. We could also ask for generalizations in which the definitions of U and D themselves are modified. There are now a vast number of possibilities, and if we wish to preserve interesting enumerative results then U and D cannot be too different from their original definition. Here we will consider only a single example which naturally arises in the theory of symmetric functions and tableaux. In the next section we will consider more significant alterations in the definitions of U and D, for which all enumerative results are lost but for which we can still deduce some structural properties of the poset P in the spirit of Proposition 2.10.

A *strict partition* λ of n, denoted $\lambda \models n$, is an integer sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ satisfying $\lambda_1 > \lambda_2 > \cdots > \lambda_\ell > \lambda_{\ell+1} = \lambda_{\ell+2} = \cdots = 0$ and $\sum \lambda_i = n$. We also write $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$. We call the integers $\lambda_i > 0$ the *parts* of λ, and call $\ell = \ell(\lambda)$ the *length* of λ. Define the *shifted Young's lattice* $\tilde{Y}$ to be the sublattice of Young's lattice, as defined in [Sta$_2$, p. 168] or [Sta$_3$], consisting of all strict partitions (including the empty partition ϕ of of 0). $\tilde{Y}$ is a locally finite distributive lattice with $\hat{0}$ (and hence graded) with rank-generating function

$$F(\tilde{Y}, q) = \prod_{n \geq 1} (1 + q^n).$$

The rank $\rho(\lambda)$ of $\lambda \in \tilde{Y}$ is just the sum $|\lambda|$ of its parts (the same as in Y). A saturated chain $\phi = \lambda^0 \subset \lambda^1 \subset \cdots \subset \lambda^n = \lambda$ in the interval $[\phi, \lambda]$ of $\tilde{Y}$ is equivalent to a *standard shifted tableau* of shape λ. The number $e(\lambda)$ of each tableaux is often denoted g^λ. For further information concerning these concepts, see e.g. [M, Ex. 8, pp. 134–136][Sa][Ste].

Now define two continuous linear transformations $D, \tilde{U} : \hat{K}\tilde{Y} \to \hat{K}\tilde{Y}$ as follows: D is the same as before, i.e., for $\lambda \in \tilde{Y}$,

$$D\lambda = \sum_{\mu \in C^-(\lambda)} \mu,$$

summed over all μ which λ covers in $\tilde{Y}$. $\tilde{U}$ is given by

$$\tilde{U}\lambda = 2 \sum_{\substack{\mu \in C^+(\lambda) \\ \ell(\mu) = \ell(\lambda)}} \mu + \sum_{\substack{\nu \in C^+(\lambda) \\ \ell(\nu) > \ell(\lambda)}} \nu.$$

Note that if $\nu \in C^+(\lambda)$ with $\ell(\nu) > \ell(\lambda)$ and $\lambda = (\lambda_1, \dots, \lambda_\ell)$, then $\lambda_\ell \geq 2$ and $\nu = (\lambda_1, \dots, \lambda_\ell, 1)$.

3.1 PROPOSITION. *We have $D\tilde{U} - \tilde{U}D = I$.*

Proof. The proof is a straightforward verification and will be omitted. □

Unfortunately there seems to be no analogue of (2) (or Proposition 2.4) for $\tilde{U}$ and D. This means that our previous results on differential posets (or on Young's lattice) have "shifted analogues" only in certain special cases. Before discussing these results, let us first point out the connection with symmetric functions, analogous to the connection between Young's lattice and symmetric functions discussed in various places in [Sta$_3$]. Let $Q_\lambda(x;t)$ denote the Hall-Littlewood symmetric function indexed by λ [M, Ch. III], and write $Q_\lambda = Q_\lambda(x) = Q_\lambda(x;-1)$. The symmetric functions $Q_\lambda(x)$ were created by Schur [Sc] in connection with his investigation of projective representations of the symmetric group. For further information, see e.g. [M, Ex. 8, pp. 134-136] [Sa][Ste].

Now let $\hat{\Omega}_K$ denote the algebra of symmetric formal power series over K in the variables $x = (x_1, x_2, \ldots)$ consisting of infinite linear combinations of all Q_λ where λ is a strict partition. This algebra is generated (as an algebra of formal power series) by the odd power sums $p_1, p_3, \ldots$, i.e., $\hat{\Omega}_K = K[[p_1, p_3, \ldots]]$. (See [M] for information on symmetric functions needed here.) Define a continuous vector space isomorphism $\sigma : \hat{K}\tilde{Y} \to \hat{\Omega}_K$ by $\sigma(\lambda) = 2^{-|\lambda|} Q_\lambda$ for $\lambda \in \tilde{Y}$. By known results concerning the Q_λ's, the following diagrams commute:

$$\begin{array}{ccc} \hat{K}\tilde{Y} & \xrightarrow{\sigma} & \hat{\Omega}_K \\ \tilde{U}\downarrow & & \downarrow p_1 \\ \hat{K}\tilde{Y} & \xrightarrow{\sigma} & \hat{\Omega}_K \end{array}$$

$$\begin{array}{ccc} \hat{K}Y & \xrightarrow{\sigma} & \hat{\Omega}_K \\ D\downarrow & & \downarrow \frac{\partial}{\partial p_1} \\ \hat{K}Y & \xrightarrow{\sigma} & \hat{\Omega}_K. \end{array}$$

Here p_1 and $\frac{\partial}{\partial p_1}$ have the same meaning as in [Sta$_3$, remark after Thm. 2.5]. Hence our results below on $\tilde{Y}$ can all be interpreted in terms of symmetric functions.

We now consider some enumerative results from [Sta$_3$](or Section 2 of this paper) which depend only on the formula $DU - UD = I$ and therefore carry over to $\tilde{Y}$ with U replaced by $\tilde{U}$. The first is [Sta$_3$, Thm. 3.7] (or Theorem 2.3 here). If w is a word in the letters $\tilde{U}$ and D, then $\langle w\phi, \lambda\rangle$ has the same value as in [Sta$_3$, Thm. 3.7] (or (13) here with each $r_i = 1$). But rather than counting Hasse walks $\phi = \lambda^0, \lambda^1, \ldots, \lambda^\ell = \lambda$ with each choice $\lambda^{i-1} < \lambda^i$ or $\lambda^{i-1} > \lambda^i$ specified, now $\langle w\phi, \lambda\rangle$ counts such walks W weighted by a factor $2^{t(W)}$, where $t(W)$ is the number of steps $\lambda^{i-1} < \lambda^i$ for which $\ell(\lambda^{i-1}) = \ell(\lambda^i)$. For instance, suppose $w = D^n\tilde{U}^n$. If $\phi = \lambda^0 < \lambda^1 < \cdots < \lambda^n > \lambda^{n+1} > \cdots > \lambda^{2n} = \phi$ is a Hasse walk W, then $t(w) = |\lambda^n| - \ell(\lambda^n)$. Hence

$$\langle D^n\tilde{U}^n\phi, \phi\rangle = \sum_{\lambda \models n} 2^{n-\ell(\lambda)} (g^\lambda)^2, \tag{25}$$

where $g^\lambda = e(\lambda)$ as discussed above. Thus by Corollary 2.4 or [Sta$_3$, Cor. 3.9], we get

$$\sum_{\lambda \models n} 2^{n-\ell(\lambda)} (g^\lambda)^2 = n!,$$

a well-known formula with many combinatorial and algebraic ramifications.

Additional results from [Sta$_3$]which carry over to $\tilde{Y}$ by replacing U by $\tilde{U}$ are Theorem 3.11, Theorem 3.12, Corollary 3.14, Corollary 3.15, Corollary 3.16, as well as Theorem 2.5 from this paper. Let us state as an illustrative example the shifted analogue of [Sta$_3$, Cor. 3.14].

3.2 PROPOSITION. *Let*

$$\kappa_{2k}(n) = \sum_W 2^{t(W)},$$

summed over all closed Hasse walks $\lambda^0, \lambda^1, \ldots, \lambda^{2k} = \lambda^0$ of length $2k$ in $\tilde{Y}$ with $|\lambda^0| = n$, where $t(W)$ is the number of steps $\lambda^{i-1} < \lambda^i$ for which $\ell(\lambda^{i-1}) = \ell(\lambda^i)$. Then

$$\sum_{n\geq 0} \kappa_{2k}(n) q^n = \frac{(2k)!}{2^k k!} \left(\frac{1+q}{1-q}\right)^k \prod_{i\geq 1}(1+q^i). \qquad \square$$

The results in [Sta$_3$, Section 4] concerning characteristic polynomials all carry over to $\tilde{Y}$ with U replaced by $\tilde{U}$, so we will not state them explicitly here. Let us note, however, that the shifted analogue of [Sta$_3$, Cor. 4.2] is the result that D is surjective and $\tilde{U}$ is injective. However, since U is adjoint to D we also get that U is injective. Moreover, the shifted analogue of [Sta$_3$, Thm. 4.14] (or Theorem 2.12 of this paper) evaluates not the characteristic polynomial of the graph $\mathcal{H}(\tilde{Y}_{[i,j]})$, but rather the digraph $\tilde{\mathcal{H}}(\tilde{Y}_{[i,j]})$ whose vertices are the elements of $\tilde{Y}_{[i,j]}$, with one edge from λ to μ if λ covers μ or if μ covers λ and $\ell(\mu) = \ell(\lambda) + 1$, and with two edges from λ to μ if μ covers λ and $\ell(\mu) = \ell(\lambda)$.

It is natural to ask whether there is some modification of U and D for the poset $\tilde{Y}$ such that analogues of *both* (1) and (2) hold. A remarkable result of this nature was found by M. Haiman (private communication) and will now be briefly discussed. Let $\omega = (1+i)/\sqrt{2} = e^{2\pi i/8}$, $\overline{\omega} = (1-i)/\sqrt{2} = e^{-2\pi i/8}$ (where $i^2 = -1$). Define continuous linear transformations $V, E : \hat{K}\tilde{Y} \to \hat{K}\tilde{Y}$ as follows:

$$V\lambda = \sqrt{2} \sum_{\substack{\mu \in C^+(\lambda) \\ \ell(\mu)=\ell(\lambda)}} \mu + \omega \sum_{\substack{\nu \in C^+(\lambda) \\ \ell(\nu)>\ell(\lambda)}} \nu$$

$$E\lambda = \sqrt{2} \sum_{\substack{\mu \in C^-(\lambda) \\ \ell(\mu)=\ell(\lambda)}} \mu + \overline{\omega} \sum_{\substack{\nu \in C^-(\lambda) \\ \ell(\nu)>\ell(\lambda)}} \nu.$$

It is then straightforward to verify the following result.

3.3 PROPOSITION. *We have*

$$EV - VE = I$$
$$E\tilde{\mathrm{Y}} = (V + \overline{\omega})\tilde{\mathrm{Y}}. \qquad \square$$

Reasoning as in [Sta$_3$, Cor. 2.6] leads to such results as

$$e^{(V+E)t} = e^{\frac{1}{2}t^2 + Vt} e^{Et}$$
$$e^{Et} e^{Vt} = e^{t^2 + Vt} e^{Et}$$
$$e^{Et}\tilde{\mathrm{Y}} = e^{\omega t + \frac{1}{2}t^2 + Vt}\tilde{\mathrm{Y}}.$$

From this all the enumerative results in [Sta₃] will have shifted analogues involving V and E instead of U and D. To obtain combinatorially meaningful results one must take real and imaginary parts. As an example, consider

$$\begin{aligned}(26)\qquad \beta(0\to n) &:= \langle V^n\hat{0}, \tilde{Y}\rangle \\ &= \sum_{\lambda\models n} \omega^{\ell(\lambda)} 2^{\frac{1}{2}(n-\ell(\lambda))} g^\lambda.\end{aligned}$$

We get from the techniques of [Sta₃] that

$$\sum_{n\geq 0} \beta(0\to n)\frac{t^n}{n!} = e^{\omega t+\frac{1}{2}t^2}.$$

Thus

$$\begin{aligned}(27)\qquad \sum_{n\geq 0} \operatorname{Re}\,\beta(0\to n)\frac{t^n}{n!} &= e^{\frac{t}{\sqrt{2}}+\frac{1}{2}t^2}\cos\frac{t}{\sqrt{2}} \\ \sum_{n\geq 0} \operatorname{Im}\,\beta(0\to n)\frac{t^n}{n!} &= e^{\frac{t}{\sqrt{2}}+\frac{1}{2}t^2}\sin\frac{t}{\sqrt{2}}.\end{aligned}$$

Moreover, taking the real part of (26) yields

$$(28)\qquad \operatorname{Re}\,\beta(0\to n) = \sum_{\lambda\models n} c_\lambda g^\lambda,$$

where

$$(29)\qquad c_\lambda = \begin{cases} 2^{[\frac{1}{2}(n-\ell(\lambda))]}, & \ell(\lambda)\equiv 0,1,7 \pmod 8 \\ -2^{[\frac{1}{2}(n-\ell(\lambda))]}, & \ell(\lambda)\equiv 3,4,5 \pmod 8 \\ 0, & \ell(\lambda)\equiv 2,6 \pmod 8). \end{cases}$$

Combining (27), (28) and (29) yields a curious combinatorial result. Similarly we have

$$\operatorname{Im}\,\beta(0\to n) = \sum_{\lambda\models n} d_\lambda g^\lambda,$$

where

$$d_\lambda = \begin{cases} 2^{[\frac{1}{2}(n-\ell(\lambda))]}, & \ell(\lambda)\equiv 1,2,3 \pmod 8 \\ -2^{[\frac{1}{2}(n-\ell(\lambda))]}, & \ell(\lambda)\equiv 5,6,7 \pmod 8 \\ 0, & \ell(\lambda)\equiv 0,4 \pmod 8). \end{cases}$$

4. Non-enumerative variations. Our previous variations were close enough to differential posets to retain many enumerative features of them. In this section we will be concerned only with questions of injectivity and surjectivity of certain linear transformations and their applications to structural properties of P; no explicit formulas will be obtained.

Let P be a poset satisfying axioms (S1) and (S2) of Definition 2.1 (or (D1) and (D2) of [Sta₃, Def. 1.1]). For each $i\geq 0$, define an axiom E_i as follows:

(E_i) If $x\in P_i$, then x is covered by more elements than x covers, i.e., $\#C^+(x) > \#C^-(x)$.

4.1 THEOREM. *Let P satisfy (S1), (S2), and (E_i) for some i. Let U and D have their usual meanings. Then $DU_i : KP_i \to KP_i$ is a bijection. Hence U_i is injective and D_{i+1} is surjective, so (as in Corollary 2.7) $p_i \leq p_{i+1}$ and there is an order-matching $\mu : P_i \to P_{i+1}$.*

Proof. Given $x \in P$, let

$$d_x = \#C^+(x) - \#C^-(x).$$

The axioms (S1), (S2), and (E_i) imply that

$$(DU - UD)_i x = d_x x,$$

for all $x \in P_i$. Hence

$$DU_i = UD_i + A_i,$$

where A is a diagonal matrix (with respect to the basis P_i of KP_i) with positive entries d_x. Since U_{i-1} and D_i are adjoints (see [Sta$_3$, Sect. 2]), UD_i is semidefinite. Since A is positive definite, the sum $UD_i + A$ is positive definite and hence invertible. □

We now give an application of Theorem 4.1. Let p be a prime and $k, n \geq 1$. Define $L_{kn}(p)$ to be the lattice of subgroups of the abelian p-group $(\mathbf{Z}/p^k\mathbf{Z})^n$. It has been conjectured (though I cannot recall by whom) that $L_{kn}(p)$ has the Sperner property. (In general, the lattice of subgroups of a finite abelian p-group need not have the Sperner property, e.g., the group $(\mathbf{Z}/p\mathbf{Z}) \oplus (\mathbf{Z}/p^2\mathbf{Z})$.) One can also ask whether $L_{kn}(p)$ has stronger properties, such as being Peck, unitary Peck, or having a symmetric chain decomposition. (It is well-known that $L_{kn}(p)$ is rank-symmetric, and by a recent result of Butler [B] is rank-unimodal.) The case $k = 1$ is well-understood (see e.g. Corollary 2.11 and [G]); here we consider $k = 2$.

4.2 PROPOSITION. *The lattice $L = L_{2n}(p)$ has order-matchings $\mu : L_i \to L_{i+1}$ for $i < n$ and $\mu : L_{i+1} \to L_i$ for $i > n$. Hence L has the Sperner property.*

Proof. Clearly L satisfies (S1), while (S2) follows since L is modular. Now let $G = (\mathbf{Z}/p^2\mathbf{Z})^n$, and let H be a subgroup of G. It is not hard to see (see [M, (4.3), p. 93] for a much stronger result) that

$$H \cong (\mathbf{Z}/p\mathbf{Z})^j \oplus (\mathbf{Z}/p^2\mathbf{Z})^k,$$
$$G/H \cong (\mathbf{Z}/p\mathbf{Z})^j \oplus (\mathbf{Z}/p^2\mathbf{Z})^{n-j-k},$$

where $j + 2k \leq 2n$. Hence in the lattice L, H covers $(p^{j+k} - 1)/(p-1)$ elements and is covered by $(p^{n-k} - 1)/(p-1)$ elements. If $j + 2k < n$ (i.e., the rank $\rho(H)$ of H in L is $< n$), then $j + k < n - k$, so $\#C^+(H) > \#C^-(H)$. Thus (E_i) holds for $i < n$, so by the previous theorem there is an order-matching $\mu : L_i \to L_{i+1}$ for $i < n$. Since L is self-dual [*M*, (1.5), p. 87] (or by an argument dual to the preceding) we get an order-matching $\mu : L_{i+1} \to L_i$ for $i > n$. It is now a standard argument (viz., the matching μ partitions L into saturated chains all passing through the middle rank L_n) to deduce the Sperner property. □

Perhaps some refinement of the preceding argument can be used to show that $D^{2n-2i}U^{2n-2i} : L_i \to L_{2n-i}$ is a bijection for $0 \le i \le n$, and therefore establish that L is unitary Peck.

We now consider a variation of Theorem 4.1 where U and D are replaced by other operators. If P is a finite poset, then as in Section 2 let $J(P)$ denote the lattice of order ideals of P [Sta$_2$, Ch. 3.4]. $J(P)$ is a graded distributive lattice, with the rank $\rho(Q)$ of an order ideal Q of P given by its cardinality $\#Q$. If $Q \in J(P)$, then define

$$M(Q) = \{x \in P : x \text{ is a maximal element of } Q\},$$
$$m(Q) = \{x \in P : x \text{ is a minimal element of } P - Q\}.$$

4.3 Proposition. *Fix an integer $0 \le i < \#P$. Suppose there is a function $\phi : P \to \mathbf{R}$ satisfying the following property: for all $Q \in J(P)_i$, we have*

$$\sum_{x \in M(Q)} \phi(x) < \sum_{x \in m(Q)} \phi(x). \tag{30}$$

Then there is an order-matching $\mu : J(P)_i \to J(P)_{i+1}$.

Proof. Define linear transformations $U(\phi), D(\phi) : K{\cdot}J(P) \to K{\cdot}J(P)$ as follows: if $Q \in J(P)$ then

$$U(\phi)Q = \sum_{Q' \in C^+(Q)} \phi(Q' - Q)Q',$$
$$D(\phi)Q = \sum_{Q' \in C^-(Q)} \phi(Q - Q')Q'.$$

Here if $Q' - Q = \{x\}$ then $\phi(Q' - Q) := \phi(x)$, and similarly for $\phi(Q - Q')$.

It is not difficult to check that distributivity of $J(P)$ insures that

$$(D(\phi)U(\phi) - U(\phi)D(\phi))Q = \left[\sum_{x \in m(Q)} \phi(x) - \sum_{x \in M(Q)} \phi(x)\right] Q. \tag{31}$$

Thus by (30) the expression in brackets in the right-hand side of (31) is positive for all $Q \in J(P)_i$. It now follows just as in the proof of Theorem 4.1 that $D(\phi)U(\phi)$ is positive definite. Thus $U(\phi)_i$ is injective, so by the usual arguments an order-matching $\mu : J(P)_i \to J(P)_{i+1}$ exists. □

Unfortunately we have been unable thus far to find any interesting applications of Proposition 4.3 that were not previously known.

The results and techniques of this section are closely related to work of Proctor [P$_1$] [P$_2$] [P$_4$]. Given a finite ranked poset P of rank n, call a linear transformation

$Y : KP \to KP$ a *lowering operator* if $Yx \in KP_{i-1}$ when $x \in P_i$. Also call $X : KP \to KP$ an *order-raising operator* if for all $x \in P$, $Xx = \sum c_{xy}y, c_{xy} \in K$, where y ranges over $C^+(x)$. If there exist X, Y as above satisfying for $0 \le i \le n$,

$$(XY - YX)_i = (2i - n)I_i, \tag{32}$$

then P is called an *sl(2)-poset*. By methods similar to those here Proctor showed that sl(2)-posets are Peck (and conversely) [P_1, Thm. 1]. In some interesting cases one can choose $X = U$ (though usually not $Y = D$) [P_1, Thm. 3]. This situation combines features of Sections 2 and 3, since (32) is analogous to (3), while U and Y replace $\tilde{U}$ and D in Section 3.

Let us also point out the similarity between our Proposition 4.3 and Proctor's concepts of *edge-labelable* posets [P_1, p. 279] and *vertex-labelable* posets [P_4, p. 105]. Indeed, a poset P is vertex-labelable if there exists a function $\phi : P \to \mathbb{Q}$ such that the equation

$$\sum_{x \in M(Q)} \phi(x) - |Q| = \sum_{y \in m(Q)} \phi(y) - |P - Q|$$

is satisfied for every order ideal $Q \in J(P)$. Hence the quantity in brackets in equation (31) is just $|P| - 2 \cdot |Q|$. From this Proctor deduces that $J(P)$ is actually an sl(2)-poset and hence Peck. Furthermore, Proctor [P_4, Thm. 1] gives an elegant classification of all vertex-labelable posets using Dynkin diagrams. If we regard the function ϕ of Proposition 4.3 as labeling the edge (I, I') of $J(P)$ by $\phi(I' - I)$,then we obtain an exact analogue of Proctor's definition of edge-labeling. One could also, as in Proctor, extend the concept to uniquely modular posets.

Finally let us mention that Proctor [P_3] contains results closely related to Proposition 2.9 for some particular sl(2)-posets.

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