

# Robinson–Schensted Algorithms for Skew Tableaux

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We introduce several analogs of the Robinson–Schensted algorithm for skew Young tableaux. These correspondences provide combinatorial proofs of various identities involving  $f_{\lambda/\mu}$ , the number of standard skew tableaux of shape  $\lambda/\mu$ , and the skew Schur functions  $s_{\lambda/\mu}(\mathbf{x})$ . For example, we are able to show bijectively that

$$\sum_{k, n} \left( \sum_{\substack{\lambda/\mu \vdash n \\ \mu \vdash k}} f_{\lambda/\mu}^2 \right) q^k t^n / n! = \frac{1}{1-t/(1-q)} \prod_i (1-q^i)^{-1}$$

and

$$\sum_{\lambda} s_{\lambda/\alpha}(\mathbf{x}) s_{\lambda/\beta}(\mathbf{y}) = \sum_{\mu} s_{\beta/\mu}(\mathbf{x}) s_{\alpha/\mu}(\mathbf{y}) \prod_{i, j} (1-x_i y_j)^{-1}.$$

It is then shown that these new algorithms enjoy some of the same properties as the original. In particular, it is still true that replacing a permutation by its inverse exchanges the two output tableaux. This fact permits us to derive a number of other identities as well. © 1990 Academic Press, Inc.

## 1. INTRODUCTION AND DEFINITIONS

The last few years have seen a veritable explosion in the production of various analogs of the Robinson–Schensted algorithm [Rob, Sch]. After

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Knuth's generalization to semistandard tableaux [Knu] came versions for rim hook tableaux [SW], oscillating tableaux [Ber, Sun, Pro], shifted tableaux [Sag, Wor], and various others [Tho, McL]. In this paper we present several new correspondences that apply to skew shapes. They permit us to give combinatorial derivations of a plethora of identities that heretofore had only algebraic proofs. In particular, we are able to bijectively explain formulae originally proved by Towber, Lascoux, Macdonald, and Zelivinsky [Mac] (using the theory of symmetric functions) and by Stanley [Sta2] (using linear operators on partially ordered sets).

We assume familiarity in general with the language of tableaux and symmetric functions (see, e.g., [Mac]) and, in particular, with the Robinson-Schensted algorithm as presented in [Sch]. Any definitions not given below can be found in these two sources.

We will let  $\lambda = (\lambda_1, \dots, \lambda_k)$  stand for both a *partition* and the corresponding *Ferrers diagram* displayed in "English" notation with the longest part,  $\lambda_1$ , in the top row and no zero parts allowed. The *conjugate* of  $\lambda$  is the partition  $\lambda' = (\lambda'_1, \dots, \lambda'_m)$ , where  $\lambda'_j$  is the length of the  $j$ th column of  $\lambda$ . The *node* of the diagram in row  $i$  and column  $j$  will be denoted  $v = (i, j)$  so  $v \in \lambda$  iff  $1 \leq j \leq \lambda_i$ . If  $\mu \subseteq \lambda$  then the corresponding *skew shape*  $\lambda/\mu$  is the set  $\{v \mid v \in \lambda, v \notin \mu\}$ . If  $|\lambda/\mu| = n$  then we write  $\lambda/\mu \vdash n$  and say that  $\lambda/\mu$  is a (*skew*) *partition of  $n$* .

A *Young tableau  $P$  of shape  $\lambda/\mu$*  is a labeling of the nodes of  $\lambda/\mu$  with an ordered alphabet so that the rows and columns are weakly increasing. The reader may assume that our alphabet will be the positive integers until stated otherwise. The element of  $P$  labeling node  $v = (i, j)$  is denoted  $p_v$  or  $p_{ij}$  so that  $k \in P$  means  $k = p_{ij}$  for some  $i, j$ . A Young tableau is *partial* if its elements are distinct, *standard* if it is partial and the labels are 1 through  $n = |\lambda/\mu|$ , and *generalized* if the columns strictly increase. For example, when  $\lambda = (5, 4, 1)$  and  $\mu = (3, 2)$  possible tableaux of each of the three types are

$$\begin{array}{cccccc}
 \square & \square & \square & 5 & 8 & \square & \square & \square & 3 & 5 & \square & \square & \square & 1 & 3 \\
 P = \square & \square & 2 & 7 & , & \square & \square & 1 & 4 & , & \square & \square & 2 & 2 & . \\
 1 & & & & & 2 & & & & & 1 & & & & 
 \end{array}$$

The sets of partial, standard, and generalized tableaux of shape  $\lambda/\mu$  will be denoted  $PT(\lambda/\mu)$ ,  $ST(\lambda/\mu)$ , and  $GT(\lambda/\mu)$ , respectively. Also let  $f_{\lambda/\mu} = |ST(\lambda/\mu)|$  with the corresponding *Schur function* in the infinite set of variables  $x = \{x_1, x_2, \dots\}$  being given by  $s_{\lambda/\mu}(\mathbf{x}) = \sum_{P \in GT(\lambda/\mu)} (x_1^{n_1(P)} x_2^{n_2(P)} \dots)$ , where  $n_k(P)$  is the number of elements equal to  $k$  in  $P$ .

A *biword*,  $\pi$ , is a sequence of vertical pairs of positive integers  $\pi = \begin{matrix} i_1 & i_2 & \dots & i_k \\ j_1 & j_2 & \dots & j_k \end{matrix}$  with  $i_1 \leq i_2 \leq \dots \leq i_k$ . We denote the top and bottom lines of  $\pi$  by

$\hat{\pi} = i_1 i_2 \cdots i_k$  and  $\tilde{\pi} = j_1 j_2 \cdots j_k$ . The three types of biwords that we will consider are *partial permutations of  $n$* , where in each line the elements are distinct and of size at most  $n$ , *permutations of  $n$*  which are partial permutations with top line  $1, 2, \dots, n$ , and  *$n$  by  $n$  matrix words*, where all elements are at most  $n$  with the pairs arranged lexicographically:  $i_r = i_{r+1}$  implies  $j_r \leq j_{r+1}$ . Examples of these definitions when  $n = 5$  are

$$\pi = \begin{array}{ccccc} 1 & 2 & 4 & & \\ 4 & 2 & 3' & & \end{array} \quad \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 5 & 3 & 1' \end{array} \quad \begin{array}{cccccc} 1 & 1 & 1 & 2 & 2 & 4 & 4 & 4 \\ 1 & 3 & 3 & 2 & 5 & 1 & 5 & 5' \end{array}$$

Note that there is a bijective correspondence between matrix words and  $n$  by  $n$  matrices  $M = (m_{ij})$  with non-negative integral entries, where  $m_{ij}$  is the number of times the pair  $\binom{i}{j}$  appears in  $\pi$ . The matrix corresponding to the matrix word above is

$$M = \begin{pmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The sets of partial permutations, permutations, and  $n$  by  $n$  matrix words will be denoted  $PS_n$ ,  $S_n$ , and  $MAT_n$ , respectively.

In the next section, we present the skew Robinson–Schensted algorithm in its simplest form. Next we investigate the effect that inverting the partial permutation has on the output tableaux. In Sections 4 and 5 we present two different generalizations of the basic correspondence: one obtained by iteration and the other by using tableaux of different shapes. Section 6 extends everything in the first five sections to tableaux and words with repetitions; it contains the most general formulae yet presented (Corollaries 6.2, 6.5, and 6.12) from which all previous identities can be derived. The skew dual and skew shifted maps appear in the next two sections. Finally, we close with some comments and open questions.

## 2. THE FUNDAMENTAL ALGORITHM

The heart of any Robinson–Schensted algorithm is the process of *insertion*. For skew tableaux there are two types of row insertion: external and internal.

*External insertion* is very much like Schensted’s original procedure. Initially one has a Young tableau  $P$  of shape  $\lambda/\mu$  and an element  $m$  to be

added to  $P$ . To start,  $m$  replaces the smallest  $p_{1j}$  such that  $p_{1j} > m$ ; in the case where  $m \geq p_{1j}$  for all  $j$ , it is placed at the row's right end. If a  $p_{1j}$  was displaced from the first row then it is inserted into the second using the same rules as above. This process continues until some element comes to rest at the end of a row. The only need for caution in the skew case is when something is to be inserted into row  $i$  which is empty (this can only happen at the beginning or the end of an insertion). If this happens, we must have  $\lambda_i = \mu_i$  and we put the element in cell  $(i, \lambda_i + 1)$ . If external row insertion of  $m$  into  $P$  yields  $P'$  we will write  $R_m(P) = P'$ . For example, the reader can verify that

$$R_2 \begin{pmatrix} \square & \square & \square & 3 \\ \square & 1 & 5 & 6 \\ \square & & & \end{pmatrix} = \begin{pmatrix} \square & \square & \square & 2 \\ \square & 1 & 3 & 6 \\ \square & 5 & & \end{pmatrix}.$$

With *internal row insertion*, no new element is added to the tableau  $P$ . Instead we start with an *inner corner* of the shape  $\lambda/\mu$  of  $P$ : a cell  $(i, j)$  with  $(i, j) \in \lambda/\mu$  but  $(i - 1, j), (i, j - 1) \notin \lambda/\mu$ . To start the process  $p_{ij}$  is removed from row  $i$  and inserted, using the usual rules, into row  $i + 1$ . The insertion then continues in a normal fashion, ending with an element settling at the end of some row. The internal row insertion operator will be denoted  $R_{i,j}$ . This should cause no confusion with the external notation as the latter has only a single subscript. Note, however, that in the external case the subscript is the *element* while in the internal one we use the *coordinates*. If we wish to hedge our bets as to which operator is being used we will merely write  $R$ . As an example, we have

$$R_{2,3} \begin{pmatrix} \square & \square & \square & 2 \\ \square & \square & 3 & 5 \\ \square & 1 & 4 & \\ \square & & & \end{pmatrix} = \begin{pmatrix} \square & \square & \square & 2 \\ \square & \square & \square & 5 \\ \square & 1 & 3 & \\ \square & 4 & & \end{pmatrix}.$$

The original Robinson–Schensted algorithm gave a proof of the formula

$$\sum_{\lambda \vdash n} f_\lambda^2 = n! \tag{2.1}$$

using a bijection  $\pi \leftrightarrow (P, Q)$ , where  $\pi \in S_n$  and  $P, Q \in \text{ST}(\lambda)$ . Using external and internal insertion we can give a skew analog of this correspondence and derive an equation similar to (2.1). In the sequel  $\cup$  denotes disjoint union.

**THEOREM 2.1.** *Let  $n$  be a fixed positive integer and  $\alpha$  a fixed partition (not necessarily of  $n$ ). Then there is a map*

$$(\pi, T, U) \leftrightarrow (P, Q)$$

*defined below which is a bijection between  $\pi \in PS_n$  with  $T, U \in PT(\alpha/\mu)$  such that  $\tilde{\pi} \cup T = \tilde{\pi} \cup U = \{1, 2, \dots, n\}$ , on the one hand, and  $P, Q \in ST(\lambda/\alpha)$  such that  $\lambda/\alpha \vdash n$ , on the other.*

*Proof.* First, given  $(\pi, T, U)$  we construct  $(P, Q)$  as follows. Let  $(P_0, Q_0) = (T, \emptyset_\alpha)$ , where  $\emptyset_\alpha$  is the unique empty tableau of shape  $\alpha/\alpha$ , i.e.,  $Q_0$  is just the Ferrers diagram for  $\alpha$ . Iteratively construct  $(P_k, Q_k)$  for  $k = 1, 2, \dots, n$  as follows. At the  $k$ th step, determine whether  $k \in \tilde{\pi}$  or  $k \in U$ .

In the former case, let  $m$  be the corresponding element of  $\tilde{\pi}$  and set  $P_k = R_m(P_{k-1})$ . In the latter, suppose  $k = u_{ij}$ , whence  $(i, j)$  must be an inner corner of  $P_{k-1}$  (all elements above and to the left have been removed by previous insertions). Thus we can put  $P_k = R_{i,j}(P_{k-1})$ . In either case  $Q_k$  is  $Q_{k-1}$  with  $k$  placed in the cell where the corresponding insertion terminates. Finally let  $(P, Q) = (P_n, Q_n)$ . An example of this procedure can be found following the proof.

It is easy to see that at all times  $P_k$  and  $Q_k$  are partial Young tableaux with the shape of  $Q_k$  being the same as the shape of the portion of  $P_k$  outside of  $\alpha$ . Since after the  $n$ th step we have removed all the elements inside  $\alpha$ , it follows that  $P, Q \in SYT(\lambda/\alpha)$  for some  $\lambda$  with  $\lambda/\alpha \vdash n$ .

To show that this map is a bijection, we construct its inverse. The deletion process is exactly like that for the normal Robinson–Schensted algorithm except that at some point the element removed from row  $i + 1$  may be smaller than every element in row  $i$ . In this case the deletion terminates by placing the element at the *left* end of the  $i$ th row since the corresponding insertion was internal.

It should now be clear how the inverse works: start with  $(P_n, Q_n) = (P, Q)$  and  $\pi = \emptyset$ . To obtain  $(P_k, Q_k)$  from  $(P_{k+1}, Q_{k+1})$  locate the cell of  $k + 1$  in  $Q_{k+1}$ ; say it is  $(i, j)$ . Then  $P_k$  is  $P_{k+1}$  with its  $(i, j)$  entry removed by deletion and  $Q_k$  is  $Q_{k+1}$  with its  $(i, j)$  entry erased. If the deletion was external then  $\binom{k}{m}$  is added to the left end of  $\pi$ , where  $m$  is the element removed from the first row of  $P_{k+1}$ , otherwise  $\pi$  stays the same. ■

As an example of this construction let  $n = 5, \alpha = (2, 2, 1), \pi = \frac{1\ 2\ 4}{4\ 2\ 3}$ ,

$$T = \begin{array}{cc} \square & \square \\ \square & 5 \\ 1 & \end{array}, \quad \text{and} \quad U = \begin{array}{cc} \square & \square \\ \square & 3 \\ 5 & \end{array}.$$

The following list of the pairs  $(P_k, Q_k)$  is in order of increasing  $k$  from left to right:

$$\begin{array}{cccccccc}
 \square & \square & \square & \square & 4 & \square & \square & 2 & \square & \square & 2 & \square & \square & 2 & 3 & \square & \square & 2 & 3 \\
 \square & 5 & \square & 5 & & \square & 4 & & \square & \square & & \square & \square & & & \square & \square & & \\
 P_k: & 1 & , & 1 & , & 1 & 5 & , & 1 & 4 & , & 1 & 4 & , & 1 & 4 & , & \square & 4 & = P \\
 & & & & & & & & & 5 & & 5 & & & & & & & 1 \\
 & & & & & & & & & & & & & & & & & & 5 \\
 & & & & & & & & & & & & & & & & & & & \\
 \square & \square & \square & \square & 1 & 4 & \square & \square & 1 & 4 \\
 \square & \square & \square & \square & & & \square & \square & & \\
 Q_k: & \square & , & \square & , & \square & 2 & = Q \\
 & & & & & & & & & 3 & & 3 & & & & & & & 3 \\
 & & & & & & & & & & & & & & & & & & & 5
 \end{array} \tag{2.2}$$

The analog of Eq. (2.1) can now be read off from Theorem 2.1.

**COROLLARY 2.2.** *Let  $n$  be a fixed positive integer and  $\alpha$  be a fixed partition. Then*

$$\sum_{\lambda/\alpha \vdash n} f_{\lambda/\alpha}^2 = \sum_{k=0}^n \binom{n}{k}^2 k! \sum_{\alpha/\mu \vdash n-k} f_{\alpha/\mu}^2.$$

Of course, both Theorem 2.1 and Corollary 2.2 reduce to the original Robinson–Schensted results if  $\alpha = \emptyset$ .

### 3. INVERTING PERMUTATIONS AND TABLEAUX

One of the many beautiful properties of Robinson–Schensted is that if  $\pi \leftrightarrow (P, Q)$  then  $\pi^{-1} \leftrightarrow (Q, P)$ . Thus if  $\pi$  is an involution we have  $\pi = \pi^{-1}$  and so  $\pi \leftrightarrow (P, P) \leftrightarrow P$ . The corresponding identity is

$$\sum_{\lambda \vdash n} f_{\lambda} = \text{Inv}(n), \tag{3.1}$$

where  $\text{Inv}(n)$  is the number of involutions in  $S_n$ . The skew algorithm introduced above enjoys similar properties, but first we must set up some additional definitions and notation.

The *inverse* of any biword  $\pi$  is constructed by interchanging the two lines and the lexicographically ordering the pairs with the new top line taking precedence. If we compute the inverse of the three example permutations of Section 1 in turn, we get

$$\pi^{-1} = \begin{array}{ccc}
 2 & 3 & 4 \\
 2 & 4 & 1'
 \end{array} \quad \begin{array}{ccc}
 1 & 2 & 3 & 4 & 5 \\
 5 & 2 & 4 & 1 & 3'
 \end{array} \quad \begin{array}{cccccc}
 1 & 1 & 2 & 3 & 3 & 5 & 5 & 5 \\
 1 & 4 & 2 & 1 & 1 & 2 & 4 & 4'
 \end{array}$$



(iii) If  $s \in U$ , say  $s$  is in column  $j$  and  $b$ th from the bottom, then let  $t = j^{[b]}$ .

To illustrate this somewhat lengthy definition, one can verify that for the triple

$$(\pi, T, U) = \left( \begin{array}{cccc} 1 & 2 & 4 & \square & \square & \square & \square \\ 4 & 2 & 3 & \square & 5 & \square & 3 \\ & & & 1 & & 5 & \end{array} \right)$$

introduced after Theorem 2.1 we have

$$[\pi] = \begin{array}{cccccccccccc} 1^{[1]} & 1^{[2]} & 1^{[3]} & 2^{[1]} & 2^{[2]} & 1 & 2 & 3 & 4 & 5 & & \\ 1 & 1^{[3]} & 1^{[2]} & 5 & 2^{[2]} & 4 & 2 & 2^{[1]} & 3 & 1^{[1]} & & \end{array} \quad (3.2)$$

*Note.* It will be useful for future reference to observe that for any  $j$  and  $b$  the elements  $j^{[b]}, j^{[b-1]}, \dots, j^{[1]}$  form a decreasing subsequence of  $[\tilde{\pi}]$ .

The important property of the bracket operation is that it commutes with the versions of Robinson–Schensted at our disposal.

LEMMA 3.1. *With the same notation as Theorem 2.1, the following diagram commutes*

$$\begin{array}{ccc} (\pi, T, U) & \longleftrightarrow & (P, Q) \\ \downarrow [\cdot] & & \downarrow [\cdot] \times [\cdot] \\ \pi' & \longleftrightarrow & (P', Q') \end{array}$$

where the top and bottom bijections are the skew and ordinary Robinson–Schensted maps, respectively.

*Proof.* Let  $(P_0, Q_0), \dots, (P_n, Q_n)$  be the tableaux pairs constructed by applying skew Robinson–Schensted to  $(\pi, T, U)$ . Also let  $(P'_0, Q'_0), \dots, (P'_r, Q'_r)$ , where  $r = n + |\alpha|$ , be the corresponding pairs for non-skew Robinson–Schensted applied to  $\pi' = [\pi]$ . We must show that  $P'_r = [P_n]$  and  $Q'_r = [Q_n]$ . In fact, we will show that for  $k = 0$  to  $n$  we have  $P'_{k+|\alpha|} = [P_k]_\alpha$  and similarly for  $Q$ . This will complete the proof because when  $k = n$  we have  $P_n$  of shape  $\lambda/\alpha$  so that  $[P_n] = [P_n]_\alpha = P'_{n+|\alpha|} = P'_r$ .

The proof will proceed by induction on  $k$ . When  $k = 0$  we have inserted the first  $|\alpha|$  elements of  $[\tilde{\pi}]$ . But by definition this is just the column word of  $[T]_\alpha$ , where  $T = P_0$ ; hence  $P'_{|\alpha|} = [P_0]_\alpha$  as desired. Also these elements enter column by column, each column being filled by the corresponding entries of  $[\tilde{\pi}]$ , so we also have  $Q'_{|\alpha|} = [Q_0]_\alpha$ , where  $Q_0$  is the empty skew tableau of shape  $\alpha/\alpha$ .

To show that the equations are still valid after the  $k$ th step, we must consider whether  $k \in \hat{\pi}$  or  $k \in U$ . If  $k \in \hat{\pi}$  then the pair  $\binom{k}{m}$  must appear in both  $\pi$  and  $[\pi]$ . Thus the skew insertion  $R_m(P_{k-1})$  is external and hence exactly like ordinary insertion of  $m$  into  $P'_{k-1+|\alpha|}$  (elements of the form  $j^{[i]}$  will not be disturbed because  $m$  is positive). It follows that  $k$  will be added to both  $Q$  tableaux in the same place so the induction holds in this case.

Now if  $k \in U$ , say  $k = u_{ij}$ , then in  $[\hat{\pi}]$  we have  $m = j^{[b]}$ . By induction and the note after the definition of  $[\pi]$  we see that the insertion of  $m$  into  $P'_{k-1+|\alpha|}$  will just push down the  $j$ th column until the first positive entry is displaced, which must be the one in row  $i$  by definition. From this point on, the insertion process is exactly like the internal insertion  $R_{i,j}(P_{k-1})$  which completes the proof of the induction step. ■

To continue our running example, consider again the triple

$$(\pi, T, U) = \left( \begin{array}{cccc} 1 & 2 & 4 & \square \\ 4 & 2 & 3 & \square \end{array}, \begin{array}{cccc} \square & \square & \square & \square \\ \square & 5 & \square & 3 \\ 1 & & 5 & \end{array} \right).$$

Applying normal Robinson–Schensted to  $[\pi]$  as computed in (3.2), we obtain

$$\begin{aligned} (P'_k, Q'_k): & \left( \begin{array}{cc} 1 & 1^{[1]} \\ , & \end{array} \right), \left( \begin{array}{cc} 1^{[3]} & 1^{[1]} \\ 1 & , & 1^{[2]} \end{array} \right), \left( \begin{array}{cc} 1^{[2]} & 1^{[1]} \\ 1^{[3]} & , & 1^{[2]} \\ 1 & & 1^{[3]} \end{array} \right), \\ & \left( \begin{array}{cccc} 1^{[2]} & 5 & 1^{[1]} & 2^{[1]} \\ 1^{[3]} & , & 1^{[2]} & \\ 1 & & 1^{[3]} & \end{array} \right), \left( \begin{array}{cccc} 1^{[2]} & 2^{[2]} & 1^{[1]} & 2^{[1]} \\ 1^{[3]} & 5 & , & 1^{[2]} & 2^{[2]} \\ 1 & & & 1^{[3]} \end{array} \right), \\ & \left( \begin{array}{cccccc} 1^{[2]} & 2^{[2]} & 4 & 1^{[1]} & 2^{[1]} & 1 \\ 1^{[3]} & 5 & , & 1^{[2]} & 2^{[2]} & \\ 1 & & & 1^{[3]} & & \end{array} \right), \left( \begin{array}{cccccc} 1^{[2]} & 2^{[2]} & 2 & 1^{[1]} & 2^{[1]} & 1 \\ 1^{[3]} & 4 & , & 1^{[2]} & 2^{[2]} & \\ 1 & 5 & & 1^{[3]} & 2 & \end{array} \right), \\ & \left( \begin{array}{cccccc} 1^{[2]} & 2^{[1]} & 2 & 1^{[1]} & 2^{[1]} & 1 \\ 1^{[3]} & 2^{[2]} & , & 1^{[2]} & 2^{[2]} & \\ 1 & 4 & , & 1^{[3]} & 2 & \\ 5 & & & 3 & & \end{array} \right), \left( \begin{array}{cccccc} 1^{[2]} & 2^{[1]} & 2 & 3 & 1^{[1]} & 2^{[1]} & 1 & 4 \\ 1^{[3]} & 2^{[2]} & & & 1^{[2]} & 2^{[2]} & & \\ 1 & 4 & & & 1^{[3]} & 2 & & \\ 5 & & & & 3 & & & \end{array} \right), \\ & \left( \begin{array}{cccccc} 1^{[1]} & 2^{[1]} & 2 & 3 & 1^{[1]} & 2^{[1]} & 1 & 4 \\ 1^{[2]} & 2^{[2]} & & & 1^{[2]} & 2^{[2]} & & \\ 1^{[3]} & 4 & & & 1^{[3]} & 2 & & \\ 1 & & & & 3 & & & \\ 5 & & & & 5 & & & \end{array} \right) = (P', Q'). \end{aligned}$$

Comparing these pairs from  $(P'_s, Q'_s)$  on with those computed in (2.2) illustrates the proof of Lemma 3.1.

We now must show that  $[\cdot]$  behaves well with respect to inversion of biwords and interchange of tableaux.

LEMMA 3.2.  $[\pi, T, U]^{-1} = [\pi^{-1}, U, T]$ .

*Proof.* We must show that  $\binom{t}{s} \in [\pi, T, U]$  if and only if we have  $\binom{t}{s} \in [\pi^{-1}, U, T]$ . Actually it suffices to prove only the forward implication, since both sets of pairs have the same cardinality. In what follows (i), (ii), and (iii) refer to those parts of the definition of the bracket of a triple.

Consider first the case where  $s = j^{[b]}$  so that  $t$  is the element  $b$ th from the bottom in  $[T]_\alpha$  by (i). If  $t$  is in  $T$  itself then  $t$  is a member of the second tableau of the triple  $(\pi, U, T)$  and, by (iii),  $\binom{t}{j^{[b]}}$  appears in its bracket. If  $t \notin T$  then  $t$  is in cell  $(i, j) \in \mu$  and we must have  $t = j^{[c]}$  for some  $c$ . Now because  $i = \alpha'_j - b + 1 = \mu'_j - (\alpha'_j - c)$ , we have

$$b + c = 2\alpha'_j - \mu'_j + 1. \tag{3.3}$$

Also  $(i, j) \notin \mu$  iff  $\alpha'_j - \mu'_j + 1 \leq b \leq \alpha'_j$  which is equivalent, by (3.3), to  $\alpha'_j - \mu'_j + 1 \leq c \leq \alpha'_j$ . Since both (3.3) and the accompanying boundary conditions are symmetric in  $b$  and  $c$ ,  $\binom{j^{[c]}}{j^{[b]}}$  will be in  $[\pi^{-1}, U, T]$ .

Now suppose  $s$  is a positive integer. If  $t$  is also then by the definition of inverse  $\binom{t}{s} \in \pi$  iff  $\binom{t}{s} \in \pi^{-1}$  and this is also true of the bracket by (ii). If  $t = j^{[b]}$  then  $\binom{j^{[b]}}{s}$  is in  $[\pi^{-1}, U, T]$  by an argument similar to that of the first case above with the roles of  $s$  and  $t$  reversed. ■

Combining the two lemmas we have the main result of this section.

THEOREM 3.3. *If  $(\pi, T, U) \leftrightarrow (P, Q)$  by skew Robinson–Schensted then  $(\pi^{-1}, U, T) \leftrightarrow (Q, P)$ .*

*Proof.* This follows immediately from Lemmas 3.1 and 3.2 and the fact that the theorem holds for the ordinary Robinson–Schensted map. ■

Now suppose that  $\pi$  is an involution, i.e.,  $\pi = \pi^{-1}$  as biwords. By the preceding theorem we have a sequence of bijections

$$(\pi, T) \leftrightarrow (\pi, T, T) \leftrightarrow (P, P) \leftrightarrow P,$$

where the middle map is skew Robinson–Schensted. We have proved:

COROLLARY 3.4. *If  $\pi$  is an involution then the above composition of maps gives a bijection*

$$(\pi, T) \leftrightarrow P$$

between  $\pi \in PS_n$  with  $T \in PT(\alpha/\mu)$  such that  $\tilde{\pi} \cup T = \{1, 2, \dots, n\}$  on the one hand and  $P \in ST(\lambda/\alpha)$  such that  $\lambda/\alpha \vdash n$  on the other.

**COROLLARY 3.5.** *Let  $n$  be a fixed positive integer and  $\alpha$  be a fixed partition. Then*

$$\sum_{\lambda/\alpha \vdash n} f_{\lambda/\alpha} = \sum_{k=0}^n \binom{n}{k} \text{Inv}(k) \sum_{\alpha/\mu \vdash n-k} f_{\alpha/\mu}.$$

A theorem of Schützenberger [Scü2] states that if  $\pi$  is an involution and  $\pi \leftrightarrow P$  by normal Robinson–Schensted then the number of fixed points of  $\pi$  equals the number of columns of  $P$  of odd length. This fact generalizes to the skew case, but first we will introduce some notation.

If  $\pi$  is a biword let  $\text{fix}(\pi)$  be the number of fixed points of  $\pi$ , i.e., the number of pairs  $\binom{i}{i}$  in  $\pi$ . If  $\lambda$  is a partition then we will denote the number of odd columns of  $\lambda$  by  $\text{odd}(\lambda)$ . If we have  $\text{odd}(\lambda) = 0$  then we say  $\lambda'$  is even, meaning all of the parts of  $\lambda'$  are even.

**COROLLARY 3.6.** *In the bijection of Corollary 3.4 we always have*

$$\text{fix}(\pi) + \text{odd}(\mu) = \text{odd}(\lambda).$$

*In particular, we still have a bijection when we restrict to the case where  $\pi$  is a fixed point free involution and  $\mu', \lambda'$  are both even.*

*Proof.* Let  $[\pi] = [\pi, T, T]$ , then  $\text{fix}([\pi]) = \text{fix}(\pi) + \text{odd}(\mu)$ , where the first summand takes care of all the pairs  $\binom{i}{i}$  in  $[\pi]$  and the second counts fixed points of the form  $\binom{j^{[k]}}{j^{[k]}}$ . Since the shape of the image of  $[\pi]$  under the original Robinson–Schensted map is the same as the outer shape of  $P$ , the theorem follows from Schützenberger’s result. ■

The equations arising from Corollary 3.6 are best described in terms of generating functions. Let  $\alpha$  be fixed and define

$$F^\alpha(z, t) = \sum_n \left( \sum_{\alpha/\lambda \vdash n} f_{\alpha/\lambda} z^{\text{odd}(\lambda)} \right) t^n/n!$$

and

$$F_\alpha(z, t) = \sum_n \left( \sum_{\lambda/\alpha \vdash n} f_{\lambda/\alpha} z^{\text{odd}(\lambda)} \right) t^n/n!.$$

Also count involutions using

$$\text{Inv}(z, t) = \sum_n \left( \sum_\pi z^{\text{fix}(\pi)} \right) t^n/n!,$$

where the inner sum is over all involutions  $\pi \in S_n$ . Thus Corollary 3.5 merely states that  $F_\alpha(1, t) = \text{Inv}(1, t) F^z(1, t)$  while Corollary 3.6 translates as:

**COROLLARY 3.7.** *If  $\alpha$  is a fixed partition then with the notation above  $F_\alpha(z, t) = \text{Inv}(z, t) F^z(z, t)$ . In particular, if  $n$  is a fixed integer,*

$$\sum_{\substack{\lambda/\alpha \vdash n \\ \lambda \text{ even}}} f_{\lambda/\alpha} = \sum_{k=0}^n \binom{n}{2k} (2k-1)!! \sum_{\substack{\alpha/\mu \vdash n-2k \\ \mu' \text{ even}}} f_{\alpha/\mu},$$

where  $(2k-1)!! = 1 \cdot 3 \cdot 5 \cdots (2k-1)$ .

#### 4. ITERATED SKEW MAPS

Some of the formulae in this and subsequent sections were first derived by one of us [Sta2] to count chains in Young’s lattice (the lattice of partitions ordered by containment). The techniques used there were algebraic and one of the principal motivations of the present work was to find bijective proofs of these results. In all future identities, sums (resp., products) will always be over the non-negative (resp., positive) integers unless otherwise indicated.

One of the formulae which Stanley had demonstrated was

$$\sum_{k, n} \left( \sum_{\substack{\lambda/\mu \vdash n \\ \mu \vdash k}} f_{\lambda/\mu} \right) q^k t^n / n! = e^{\frac{t}{1-q} + \frac{t^2}{2(1-q^2)}} \prod_i (1-q^i)^{-1}, \tag{4.1}$$

which is reminiscent of Corollary 3.5; i.e., it was hoped that a Robinson–Schensted algorithm could be found that when applied to involutions would yield (4.1). This led Sagan to wonder what the corresponding identity would be for the same algorithm applied to an arbitrary permutation and, based on numerical evidence, he conjectured that

$$\sum_{k, n} \left( \sum_{\substack{\lambda/\mu \vdash n \\ \mu \vdash k}} f_{\lambda/\mu}^2 \right) q^k t^n / n! = \frac{1}{1-t/(1-q)} \prod_i (1-q^i)^{-1}. \tag{4.2}$$

To prove (4.2) combinatorially we must see what the coefficient of  $t^n/n!$  counts on each side. In both cases the objects enumerated will be weighted by a power of  $q$ . On the left we clearly have pairs  $(P, Q)$  with  $P, Q \in \text{SYT}(\lambda/\mu)$  with the weight of a pair being  $\text{wt}(P, Q) = q^{|\mu|}$ .

The right side counts pairs  $(\pi, \nu)$  consisting of a permutation and a partition both given weights as follows. From the product we see that  $\text{wt}(\nu) = q^{|\nu|}$ . The fraction in  $t$  permits each pair  $(j)$  in  $\pi$  to have weight  $q^k$  for any non-negative integer  $k$ , giving  $\pi$  itself total weight  $\text{wt}(\pi) = \prod \text{wt}(j)$ , where the product is over all pairs in  $\pi$ . To denote the fact that  $\text{wt}(j) = q^k$  we will write the pair as  $(j^{(k)})$ , where the symbol  $^{(k)}$  is omitted if  $k = 0$ . So, for example, if

$$\pi = \begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 5^{(4)} & 2^{(2)} & 4 & 1^{(7)} & 3^{(2)} \end{matrix}$$

then  $\text{wt}(\pi) = q^4 q^2 q^0 q^7 q^2 = q^{15}$ . Finally  $\text{wt}(\pi, \nu) = \text{wt}(\pi) \text{wt}(\nu)$  so that with  $\pi$  as above and  $\nu = (4, 3, 1) \vdash 8$  we have  $\text{wt}(\pi, \nu) = q^{23}$ .

With these preliminaries we can provide a bijection to prove (4.1) and (4.2) by iterating the skew Robinson–Schensted map of Section 2.

Suppose  $(\pi, \nu)$  is given. Let  $m$  be the maximum weight of a pair  $(j) \in \pi$ . We will construct a sequence of tableaux pairs  $(P_k, Q_k)$ ,  $0 \leq k \leq m + 1$ , and let  $(P, Q) = (P_{m+1}, Q_{m+1})$ .

Initially  $(P_0, Q_0) = (\emptyset_\nu, \emptyset_\nu)$  both tableaux being empty Ferrers diagrams for  $\nu$ . To construct the next pair after  $(P_k, Q_k)$ , let  $\pi_k$  be the partial permutation consisting of all the pairs of weight  $m - k$  in  $\pi$ . Now apply skew Robinson–Schensted to the triple  $(\pi_k, P_k, Q_k)$  to obtain  $(P_{k+1}, Q_{k+1})$ . Note that the elements of  $\pi$  get inserted in order of decreasing weight.

To see that the map is well defined, note first of all that since  $\pi \in S_n$  the final tableaux are certainly standard with  $n$  entries. What about the weights? We start with a pair of tableaux of weight  $\text{wt}(\nu)$ . When an element of weight  $k$  is inserted externally from  $\pi$  it is at step  $m - k$ . Thus this element will be internally inserted in the remaining  $k$  steps and so add exactly  $k$  empty cells to  $\nu$  while forming  $\mu$ . Hence at the end of the procedure we have  $\text{wt}(P, Q) = \text{wt}(\mu) = \text{wt}(\pi) \text{wt}(\nu)$ .

The construction of the inverse correspondence is straightforward and is left to the reader. We have proved:

**THEOREM 4.1.** *The map above is a weight preserving bijection*

$$(\pi, \nu) \leftrightarrow (P, Q)$$

between pairs  $\pi \in S_n$  with  $\nu$  a partition and pairs  $P, Q \in \text{ST}(\lambda/\mu)$ , where  $\lambda/\mu \vdash n$ .

By way of illustration, let  $\pi = \begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 3^{(3)} & 5 & 4^{(1)} & 1^{(1)} & 2 \end{matrix}$  and also  $\nu = \begin{matrix} \square & \square \\ \square & \square \end{matrix}$ ; then the steps of the above algorithm are as follows:

$$\begin{aligned}
 & \left( \begin{array}{cccccc} 1 & \square & \square & \square & \square \\ 3' & \square & \square & \square & \square \end{array} \right) \rightarrow \left( \begin{array}{cccccc} \square & \square & 3 & \square & \square & 1 \\ \square & \square & \square & \square & \square & \square \end{array} \right) \\
 & \left( \begin{array}{cccccc} \emptyset & \square & \square & 3 & \square & \square & 1 \\ \square & \square & \square & \square & \square & \square & \square \end{array} \right) \rightarrow \left( \begin{array}{cccccc} \square & \square & \square & \square & \square & \square & \square \\ \square & \square & 3' & \square & \square & \square & 1 \end{array} \right) \\
 & \left( \begin{array}{cccccc} 3 & 4 & \square & \square & \square & \square & \square & \square \\ 4 & 1' & \square & \square & 3' & \square & \square & 1 \end{array} \right) \rightarrow \left( \begin{array}{cccccc} \square & \square & \square & 1 & \square & \square & \square & 3 \\ \square & \square & \square & 4, & \square & \square & \square & 4 \\ 3 & & & & 1 & & & \end{array} \right) \\
 & \left( \begin{array}{cccccc} 2 & 5 & \square & \square & \square & 1 & \square & \square & \square & 3 \\ 5 & 2' & \square & \square & \square & 4, & \square & \square & \square & 4 \\ & & 3 & & & & 1 & & & \end{array} \right) \rightarrow \left( \begin{array}{cccccc} \square & \square & \square & \square & 2 & \square & \square & \square & \square & 2 \\ \square & \square & \square & \square & 5 & \square & \square & \square & \square & 5 \\ \square & 1 & & & & \square & 3 & & & \\ 3 & 4 & & & & & 1 & 4 & & \end{array} \right) \\
 & = (P, Q).
 \end{aligned}$$

As advertised, this bijection gives us a combinatorial proof of Eq. (4.2).

**COROLLARY 4.2.**

$$\sum_{k, n} \left( \sum_{\substack{\lambda/\mu \vdash n \\ \mu \vdash k}} f_{\lambda/\mu}^2 \right) q^k t^n / n! = \frac{1}{1 - t/(1 - q)} \prod_i (1 - q^i)^{-1}.$$

It is an easy matter to see how to extend our results on inverses of permutations to this setting. If  $\pi$  is weighted then  $\pi^{-1}$  is the weighted permutation defined by  $(j^{(k)}) \in \pi$  if and only if  $(i^{(k)}) \in \pi^{-1}$ . Of course,  $\pi$  is an involution whenever  $\pi = \pi^{-1}$ . The following five corollaries now follow immediately from the last five results of Section 3 and the fact that our new algorithm is merely an iteration of the old one.

**COROLLARY 4.3.** *If  $(\pi, \nu) \leftrightarrow (P, Q)$  by iterated skew Robinson–Schensted then  $(\pi^{-1}, \nu) \leftrightarrow (Q, P)$ .*

**COROLLARY 4.4.** *If  $\pi$  is a weighted involution, then the mapping of Theorem 4.1 restricts to a weight preserving bijection*

$$(\pi, \nu) \leftrightarrow P$$

*between  $\pi \in S_n$  with  $\nu$  a partition and  $P \in \text{ST}(\lambda/\mu)$  with  $\lambda/\mu \vdash n$ .*

**COROLLARY 4.5.**

$$\sum_{k, n} \left( \sum_{\substack{\lambda/\mu \vdash n \\ \mu \vdash k}} f_{\lambda/\mu} \right) q^k t^n / n! = \exp \left( \frac{t}{1 - q} + \frac{t^2}{2(1 - q^2)} \right) \prod_i (1 - q^i)^{-1}.$$

COROLLARY 4.6. *In the bijection of Corollary 4.4 we always have*

$$\text{fix}(\pi) + 2 \text{ odd}(v) = \text{odd}(\mu) + \text{odd}(\lambda).$$

*In particular, we still have a bijection when we restrict to the case where  $\pi$  is a weighted fixed point free involution and  $v'$ ,  $\mu'$ , and  $\lambda'$  are all even.*

COROLLARY 4.7.

$$\begin{aligned} & \sum_{k, n} \left( \sum_{\substack{\lambda/\mu \vdash n \\ \mu \vdash k}} f_{\lambda/\mu} z^{\text{odd}(\mu) + \text{odd}(\lambda)} \right) q^k t^n / n! \\ &= \exp \left( \frac{tz}{1-q} + \frac{t^2}{2(1-q^2)} \right) \prod_i [(1 - z^2 q^{2i-1})(1 - q^{2i})]^{-1}. \end{aligned}$$

*In particular, if  $n$  is fixed,*

$$\sum_{\substack{\lambda/\mu \vdash 2n \\ \lambda', \mu' \text{ even}}} f_{\lambda/\mu} q^{|\mu|} = \frac{(2n-1)!!}{(1-q^2)^n} \prod_i (1 - q^{2i})^{-1}.$$

### 5. ANOTHER GENERALIZATION

In the skew Robinson–Schensted bijection of Section 2 it is not necessary for both tableaux to have exactly the same shape. Note how the roles of the partitions  $\alpha$  and  $\beta$  change in passing from  $T, U$  to  $P, Q$  in the following theorem.

**THEOREM 5.1.** *Let  $n$  and  $m$  be fixed integers and let  $\alpha$  and  $\beta$  be fixed partitions (it is not necessary to have  $|\alpha| = |\beta|$ ). Then the map*

$$(\pi, T, U) \leftrightarrow (P, Q)$$

*defined below is a bijection between partial permutations  $\pi$  with  $T \in \text{PT}(\alpha/\mu)$ ,  $U \in \text{PT}(\beta/\mu)$ , such that  $\tilde{\pi} \cup T = \{1, 2, \dots, n\}$ ,  $\tilde{\pi} \cup U = \{1, 2, \dots, m\}$ , on the one hand, and  $P \in \text{ST}(\lambda/\beta)$ ,  $Q \in \text{ST}(\lambda/\alpha)$ , such that  $\lambda/\beta \vdash n$ ,  $\lambda/\alpha \vdash m$ , on the other.*

*Proof.* We will construct pairs  $(P_k, Q_k)$  for  $k = 0, 1, \dots, m$  starting with  $(P_0, Q_0) = (T, \emptyset_\alpha)$ . As before, at the  $k$ th step we see whether  $k \in \tilde{\pi}$  or  $k \in U$ . If  $k \in \tilde{\pi}$  or  $k = u_{ij}$ , where  $(i, j)$  is in  $P_{k-1}$ , then we proceed as in the original skew algorithm.

The only other possibility is that there is no element of  $P_{k-1}$  in cell  $(i, j)$ . In this case  $P_k$  is just  $P_{k-1}$  with  $(i, j)$  adjoined but empty, while  $Q_k$  has the

element  $k$  placed in the corresponding square. It is easy to verify by induction that the results will always be skew tableaux.

By the time we reach  $(P, Q) = (P_m, Q_m)$ , the elements of  $U$  will have forced us to empty every square of  $P$  inside  $\beta$  (either by internal insertion or by addition of empty squares). Thus  $P$  will be of shape  $\lambda/\beta$  for some  $\lambda$ . Also the  $Q$ 's are formed by starting with the blank shape  $\alpha$  and placing entries so that the "outer" shapes of  $P_k$  and  $Q_k$  always agree. It follows that  $Q \in \text{ST}(\lambda/\alpha)$  as desired.

As usual, the construction of the inverse map will be left as an exercise. ■

Consider  $n = 4, m = 3, \mu = (2, 1), \alpha = (5, 1), \beta = (3, 1)$ , and  $(\pi, T, U) = (2, \begin{smallmatrix} \square & \square & 1 & 3 & 4 \\ \square & \square & 1 & 3 \end{smallmatrix}, \begin{smallmatrix} \square & \square & 1 & 3 \\ \square & \square & 1 \end{smallmatrix})$ . Our construction yields

$$\begin{array}{cccccccc}
 \square & \square & 1 & 3 & 4 & \square & \square & 1 & 3 & 4 & \square & \square & 1 & 2 & 4 & \square & \square & \square & 2 & 4 \\
 P_k: & \square & & & & , & \square & \square & & & , & \square & \square & 3 & & , & \square & \square & 1 & & = P \\
 & & & & & & & & & & & & & & & & & & & & & 3 \\
 \\
 \square & \square \\
 Q_k: & \square & & & & , & \square & 1 & & & , & \square & 1 & 2 & & , & \square & 1 & 2 & & = Q. \\
 & & & & & & & & & & & & & & & & & & & & & 3
 \end{array}$$

The corresponding identity is

**COROLLARY 5.2.** *Let  $n, m$  be fixed integers and  $\alpha, \beta$  fixed partitions. Then*

$$\sum_{\substack{\lambda/\beta \vdash n \\ \lambda/\alpha \vdash m}} f_{\lambda/\beta} f_{\lambda/\alpha} = \sum_{k \geq 0} \binom{n}{k} \binom{m}{k} k! \sum_{\substack{\alpha/\mu \vdash n-k \\ \beta/\mu \vdash m-k}} f_{\alpha/\mu} f_{\beta/\mu}.$$

It should also be noted that Theorem 3.3 continues to hold in this setting, although "restriction to the diagonal" yields nothing new since  $T = U$  forces  $\alpha = \beta$ .

As an application of the ideas of this section we note that Corollary 5.2 can be used to obtain a special case of a result of Stanley's [Sta2, Theorem 3.7] which interpolates between Eqs. (2.1) and (3.1).

**COROLLARY 5.3.** *Let  $n$  and  $m$  be fixed integers. Then*

$$\sum_{\substack{\lambda \vdash n \\ \alpha \vdash n-m}} f_{\lambda} f_{\lambda/\alpha} = \binom{n}{m} m! \text{Inv}(n-m).$$

*Proof.* In Corollary 5.2 let  $\beta = \emptyset$  and sum over all  $\alpha$ . This clearly gives

the left-hand side of 5.3. On the right the only surviving term is when  $\mu = \emptyset$ . Hence  $k = m$  and the right side reduces to

$$\binom{n}{m} m! \sum_{\alpha \vdash n-m} f_\alpha = \binom{n}{m} m! \text{Inv}(n-m)$$

by Eq. (3.1). ■

The extremal cases  $m = n$  and  $m = 0$  of Corollary 5.3 correspond to Eqs. (2.1) and (3.1), respectively.

Zelevinsky (private communication) noticed the following character-theoretic interpretation of Corollary 5.3. Let  $\chi^\alpha$  be the  $\alpha$ th irreducible character of the corresponding symmetric group and consider  $\chi = \sum_{\alpha \vdash n-m} \chi^\alpha$ . Thus the induced character  $\chi \uparrow S_n$  has dimension equal to the right-hand side of the corollary. Decomposition into irreducibles yields  $\chi \uparrow S_n = \sum_{\lambda \vdash n} m_\lambda \chi^\lambda$ , where

$$m_\lambda = \langle \chi \uparrow S_n, \chi^\lambda \rangle = \langle \chi, \chi^\lambda \downarrow S_{n-m} \rangle = \sum_{\alpha \vdash n-m} f_{\lambda/\alpha}$$

by Frobenius reciprocity and the branching rule for the symmetric group. The equality of Corollary 5.3 follows.

### 6. THE SKEW KNUTH CORRESPONDENCE

Just as Knuth [Knu] was able to extend Robinson–Schensted to tableaux with repetitions, we have an analog of Theorem 2.1 for matrix words and generalized tableaux. Since the objects we will be dealing with are now multisets (sets with repetition)  $A, B$ , etc., we will define  $A \cup B$  to be the multiset where the number of repetitions of an element  $k$  is the sum of the number of copies of  $k$  in  $A$  and  $B$ .

**THEOREM 6.1.** *Let  $\alpha$  be a fixed partition. Then the map*

$$(\pi, T, U) \leftrightarrow (P, Q)$$

*defined below is a bijection between matrix words  $\pi$  with  $T, U \in \text{GT}(\alpha/\mu)$ , on the one hand, and  $P, Q \in \text{GT}(\lambda/\alpha)$ , on the other, such that  $\tilde{\pi} \cup T = P$  and  $\hat{\pi} \cup U = Q$ .*

*Proof.* Let the largest element of  $\hat{\pi} \cup U$  be  $m$ . The pairs  $(P_k, Q_k)$ ,  $k = 0, 1, \dots, m$ , are constructed by starting as usual with  $(T, \emptyset_\alpha)$ . To obtain  $P_k$ , internally insert all the elements of  $P_{k-1}$  corresponding to  $k$ 's in  $Q_{k-1}$  followed by all the elements of  $\tilde{\pi}$  paired with  $k$ 's in  $\hat{\pi}$ . In both cases the

insertions proceed from left to right. Placing  $k$ 's in the appropriate squares of  $Q_{k-1}$  results in  $Q_k$ .

Note that when inserting an element in a row it still displaces the first entry *strictly* greater than itself and that each insertion during step  $k$  begins in a row at least as high as the one just previous (if any). Thus in constructing  $P_k$  each insertion comes to rest to the right of the preceding one so that  $Q_k$  remains a generalized tableaux. This fact also makes it easy to define the inverse, since we know we must begin deletion with the square of the rightmost  $k$  in  $Q_k$  and proceed to the left. ■

EXAMPLE. Let

$$(\pi, T, U) = \left( \begin{array}{cccccc} 1 & 1 & 3 & \square & 2 & \square & 1 \\ 1 & 5 & 4 & 1 & 3 & 1 & 2 \end{array} \right).$$

Then the output tableaux are

$$(P, Q) = \left( \begin{array}{cccc|cc} \square & \square & 1 & 4 & \square & \square & 1 & 1 \\ \square & \square & 5 & & \square & \square & 3 & \\ 1 & 2 & & & 1 & 1 & & \\ 3 & & & & 2 & & & \end{array} \right).$$

Schur functions count generalized tableaux, so we immediately have a generalization of the famous Cauchy identity [Mac, p. 33]:

COROLLARY 6.2. *Fix a partition  $\alpha$ . Then*

$$\sum_{\lambda} s_{\lambda/\alpha}(\mathbf{x}) s_{\lambda/\alpha}(\mathbf{y}) = \sum_{\mu} s_{\alpha/\mu}(\mathbf{x}) s_{\alpha/\mu}(\mathbf{y}) \prod_{i,j} (1 - x_i y_j)^{-1}.$$

We will now demonstrate that all of the results of Sections 3.4, and 5 can be generalized to tableaux and biwords with repeated elements.

It is easy to show by standard techniques that the results of Section 3 all have analogs in this setting. In particular, Theorem 3.3 holds for the skew Knuth map, where  $\pi^{-1}$  is the biword of the transpose of the corresponding matrix. Thus:

COROLLARY 6.3. *If  $\pi$  is a biword corresponding to a symmetric matrix then the mapping of Theorem 6.1 restricts to a bijection*

$$(\pi, T) \leftrightarrow P,$$

where  $T \in \text{GT}(\alpha/\mu)$  and  $P \in \text{GT}(\lambda/\alpha)$  with  $\alpha$  fixed and  $\tilde{\pi} \circ T = P$ .

The resultant identity is

**COROLLARY 6.4.** *Let  $\alpha$  be a fixed partition. Then*

$$\sum_{\lambda} s_{\lambda/\alpha}(\mathbf{x}) = \sum_{\mu} s_{\alpha/\mu}(\mathbf{x}) \prod_{i < j} (1 - x_i x_j)^{-1} \prod_i (1 - x_i)^{-1}.$$

If  $M$  is the matrix corresponding to  $\pi$  in Corollary 6.3 then the trace of  $M$  counts fixed points of  $\pi$  and we always have

$$\text{tr}(M) + \text{odd}(\mu) = \text{odd}(\lambda)$$

(cf. Corollary 3.6). It follows that

**COROLLARY 6.5.**

$$\sum_{\lambda} s_{\lambda/\alpha}(\mathbf{x}) z^{\text{odd}(\lambda)} = \sum_{\mu} s_{\alpha/\mu}(\mathbf{x}) z^{\text{odd}(\mu)} \prod_{i < j} (1 - x_i x_j)^{-1} \prod_i (1 - z x_i)^{-1}.$$

*In particular,*

$$\sum_{\lambda' \text{ even}} s_{\lambda'/\alpha}(\mathbf{x}) = \sum_{\mu' \text{ even}} s_{\alpha/\mu'}(\mathbf{x}) \prod_{i < j} (1 - x_i x_j)^{-1}.$$

We can also iterate the skew Knuth algorithm to obtain results about weighted matrices and generalized tableaux. The weights of partitions and skew tableaux are the same as in the case with distinct entries.

Now consider all copies of a pair  $\binom{i}{j}$  in a matrix word  $\pi$ . Clearly the order in which these copies are weighted is immaterial since only the weights themselves determine when otherwise identical elements are to be inserted. So we make the convention that among all pairs for fixed  $i, j$  the weights will be listed in weakly decreasing order. This corresponds to a matrix  $M$  where each entry  $m_{ij}$  has been weighted by a partition  $\lambda$  with at most  $m_{ij}$  parts, denoted  $M = (m_{ij}^{\lambda})$ . For example, the matrix word

$$\pi = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 \\ 1^{(2)} & 1^{(2)} & 1^{(1)} & 2 & 2 & 2^{(3)} & 2^{(3)} & 2 & 2 \end{pmatrix}$$

corresponds to the matrix

$$M = \begin{pmatrix} 3^{(2, 2, 1)} & 2 \\ 0 & 4^{(3, 3)} \end{pmatrix}.$$

The next theorem is proved in exactly the same manner as Theorem 4.1.

**THEOREM 6.6.** *The iteration of the mapping of Theorem 6.1 produces a weight-preserving bijection*

$$(\pi, v) \leftrightarrow (P, Q)$$

*between weighted matrix words  $\pi$  with a partition  $v$ , on the one hand, and pairs  $P, Q \in \text{GT}(\lambda/\mu)$ , on the other, where  $\check{\pi} = P$  and  $\hat{\pi} = Q$  as multisets.*

**COROLLARY 6.7.**

$$\sum_k \left( \sum_{\substack{\lambda \\ \mu \vdash k}} s_{\lambda/\mu}(\mathbf{x}) s_{\lambda/\mu}(\mathbf{y}) \right) q^k = \prod_n \frac{1}{(1 - q^n) \prod_{i,j} (1 - x_i y_j q^{n-1})}$$

*Proof.* Clearly the left-hand side of this equation counts the pairs  $(P, Q)$  of weight  $k$  in Theorem 6.6. The number of ways to weight a matrix entry  $m_{ij}$  with a partition of at most  $m_{ij}$  parts is

$$\sum_m (x_i y_j)^m / (1 - q)(1 - q^2) \cdots (1 - q^m) = \prod_n (1 - x_i y_j q^{n-1})^{-1}$$

by a well-known identity of Euler [And, p. 19, Corollary 2.2]. ■

Note that Corollary 6.7 is the special case of Theorem 5.1 of [Sta1] obtained by taking  $v = (q, 0, 0, \dots)$  and replacing  $x_i$  by  $x_i/q$ . We can also rederive Corollary 4.2 from 6.7 by taking the coefficient of  $x_1 x_2 \cdots x_n y_1 y_2 \cdots y_n$  on both sides.

In fact, 6.7 itself is a simple consequence of Corollary 6.2, derived as follows. Let  $L(\mathbf{x}, \mathbf{y}, q)$  and  $R(\mathbf{x}, \mathbf{y}, q)$  denote the left- and right-hand sides of 6.7, respectively. Clearly  $R(\mathbf{x}, \mathbf{y}, q)$  is defined by the recurrence

$$R(q^{-1}\mathbf{x}, \mathbf{y}, q) = R(\mathbf{x}, \mathbf{y}, q) \prod_{i,j} (1 - x_i y_j)^{-1},$$

together with the initial condition,

$$R(\mathbf{0}, \mathbf{y}, q) = \prod_i (1 - q^i)^{-1}.$$

One can now use 6.2 to show that  $L(\mathbf{x}, \mathbf{y}, q)$  satisfies the same recurrence and boundary conditions, thus proving that  $L(\mathbf{x}, \mathbf{y}, q)$  and  $R(\mathbf{x}, \mathbf{y}, q)$  are equal as desired.

Transposing weighted matrices presents no surprises. If  $\pi$  is the word of the weighted matrix  $M = (m_{ij}^\lambda)$  then  $\pi^{-1}$  is the word of  $M^t = (m_{ji}^\lambda)$  and we obtain

**THEOREM 6.8.** *If  $(\pi, \nu) \leftrightarrow (P, Q)$  is the iterated skew Knuth map then  $(\pi^{-1}, \nu) \leftrightarrow (Q, P)$ . Furthermore, if the matrix  $M$  of  $\pi$  is symmetric, i.e.,  $m_{ij}^\lambda = m_{ji}^\lambda$ , then restriction of the above correspondence gives a weight-preserving bijection*

$$(\pi, \nu) \leftrightarrow P$$

with  $\tilde{\pi} = P$  as multisets.

**COROLLARY 6.9.**

$$\sum_k \sum_{\substack{\lambda \\ \mu \vdash k}} s_{\lambda/\mu}(\mathbf{x}) q^k = \prod_n \frac{1}{(1 - q^n) \prod_{i < j} (1 - x_i x_j q^{n-1}) \prod_i (1 - x_i q^{n-1})}$$

Keeping track of fixed points and odd column lengths gives:

**COROLLARY 6.10.**

$$\sum_k \sum_{\substack{\lambda \\ \mu \vdash k}} s_{\lambda/\mu}(\mathbf{x}) z^{\text{odd}(\lambda) + \text{odd}(\mu)} q^k = \prod_n \frac{1}{(1 - z^2 q^{2n-1})(1 - q^{2n}) \prod_{i < j} (1 - x_i x_j q^{n-1}) \prod_i (1 - x_i z q^{n-1})}$$

In particular,

$$\sum_{\lambda', \mu' \text{ even}} s_{\lambda'/\mu'}(\mathbf{x}) q^{|\mu'|} = \prod_n \frac{1}{(1 - q^{2n}) \prod_{i < j} (1 - x_i x_j q^{n-1})}$$

Corollaries 4.5 and 4.7 can be obtained from 6.9 and 6.10, respectively, by equating coefficients of  $x_1 x_2 \cdots x_n y_1 y_2 \cdots y_n$ . In their turn, 6.9 and 6.10 can be derived from 6.4 and 6.5 by setting up appropriate recurrences (cf. the comments after Corollary 6.7).

Finally, we can use two different partitions  $\alpha$  and  $\beta$ .

**THEOREM 6.11.** *Let  $\alpha$  and  $\beta$  be fixed partitions. Then the skew Knuth correspondence can be extended to a map*

$$(\pi, T, U) \leftrightarrow (P, Q),$$

which is a bijection between matrix words  $\pi$  such that  $T \in \text{GT}(\alpha/\mu)$ ,  $U \in \text{GT}(\beta/\mu)$ , on the one hand, and  $P \in \text{GT}(\lambda/\beta)$ ,  $Q \in \text{GT}(\lambda/\alpha)$ , on the other, with  $\tilde{\pi} \cup T = P$  and  $\tilde{\pi} \cup U = Q$ .

COROLLARY 6.12. *Fix partitions  $\alpha$  and  $\beta$ . Then*

$$\sum_{\lambda} s_{\lambda/\beta}(\mathbf{x}) s_{\lambda/\alpha}(\mathbf{y}) = \sum_{\mu} s_{\alpha/\mu}(\mathbf{x}) s_{\beta/\mu}(\mathbf{y}) \prod_{i,j} (1 - x_i y_j)^{-1}.$$

As usual, Corollary 5.2 can be reproved by taking square-free terms in 6.12. It should also be noted that Corollaries 6.2, 6.4, and 6.12 as well as Corollaries 7.2 and 7.6 of the next section were proved independently by Towber, Lascoux, Macdonald, and Zelevinsky using symmetric function techniques. This approach will appear in the second edition of [Mac].

## 7. THE SKEW DUAL

Knuth [Knu] also discovered a dual to his generalization of Robinson–Schensted which mapped 0–1 matrices ( $m_{ij}=0$  or 1 for all  $i, j$ ) to pairs of tableaux of conjugate shape. One can do the same thing in the skew case.

THEOREM 7.1. *Let  $\alpha$  be a fixed partition. Then the map*

$$(\pi, T, U) \leftrightarrow (P, Q)$$

*defined below is a bijection between 0–1 matrix words  $\pi$  such that  $T \in \text{GT}(\alpha/\mu)$ ,  $U \in \text{GT}(\alpha'/\mu')$ , on the one hand, and  $P \in \text{GT}(\lambda/\alpha)$ ,  $Q \in \text{GT}(\lambda'/\alpha')$ , on the other, with  $\tilde{\pi} \circ T = P$  and  $\tilde{\pi} \circ U = Q$ .*

*Proof.* We start with  $(P_0, Q_0) = (T^t, \emptyset_{\alpha}, )$ , where  $T^t$  is the transpose of the tableau  $T$  which now has strictly increasing rows. Since  $P_0$  and  $Q_0$  have the same shape  $\alpha'$ , we can proceed with the insertion process. The one difference is that when an element  $m$  is inserted into a row, it replaces the smallest entry greater than or equal to  $m$ . This preserves row-strictness of  $P_k$ , while the fact that  $\pi$  comes from a 0–1 matrix ensures column-strictness of  $Q_k$ . If  $(P_n, Q_n)$  is the last pair then let  $P = P'_n$  and  $Q = Q_n$ . ■

COROLLARY 7.2. *Let the partition  $\alpha$  be fixed. Then*

$$\sum_{\lambda} s_{\lambda/\alpha}(\mathbf{x}) s_{\lambda'/\alpha'}(\mathbf{y}) = \sum_{\mu} s_{\alpha/\mu}(\mathbf{x}) s_{\alpha'/\mu'}(\mathbf{y}) \prod_{i,j} (1 + x_i y_j).$$

The analogs of the results in Sections 4 and 5 follow the same pattern as before so we will merely state them here.

THEOREM 7.3. *Iteration of the mapping of Theorem 7.1 produces a weight preserving bijection*

$$(\pi, \nu) \leftrightarrow (P, Q)$$

between 0–1 matrix words  $\pi$  with a partition  $\nu$ , on the one hand, and  $P \in \text{GT}(\lambda/\mu)$ ,  $Q \in \text{GT}(\lambda'/\mu')$ , on the other, such that  $\tilde{\pi} = P$  and  $\hat{\pi} = Q$  as multisets.

COROLLARY 7.4.

$$\sum_k \left( \sum_{\substack{\lambda \\ \mu \vdash k}} s_{\lambda/\mu}(x) s_{\lambda'/\mu'}(y) \right) q^k = \prod_n (1 - q^n)^{-1} \prod_{i,j} (1 + x_i y_j q^{n-1}).$$

THEOREM 7.5. Fix partitions  $\alpha$  and  $\beta$ . Then the map of Theorem 7.1 can be extended to a bijection

$$(\pi, T, U) \leftrightarrow (P, Q)$$

between 0–1 matrix words  $\pi$  with  $T \in \text{GT}(\alpha/\mu)$ ,  $U \in \text{GT}(\beta'/\mu')$ , on the one hand, and  $P \in \text{GT}(\lambda/\beta)$ ,  $Q \in \text{GT}(\lambda'/\alpha')$ , on the other, such that we have  $\tilde{\pi} \circ T = P$  and  $\hat{\pi} \circ U = Q$ .

COROLLARY 7.6. Let  $\alpha$  and  $\beta$  be fixed partitions. Then

$$\sum_{\lambda} s_{\lambda/\beta}(\mathbf{x}) s_{\lambda'/\alpha'}(\mathbf{y}) = \sum_{\mu} s_{\alpha/\mu}(\mathbf{x}) s_{\beta'/\mu'}(\mathbf{y}) \prod_{i,j} (1 + x_i y_j).$$

### 8. SKEW SHIFTED TABLEAUX

There is another family of arrays, the shifted tableaux, that exhibit many of the nice properties of their “left-justified” cousins. It is therefore not surprising that skew shifted tableaux are relatively well behaved. Below we will outline the necessary definitions and notation for this case. Consult [Sag or Wor] for a more detailed description.

A partition  $\lambda = (\lambda_1, \dots, \lambda_k)$  is *strict* if  $\lambda_1 > \lambda_2 > \dots > \lambda_k$ . Given a strict partition the corresponding *shifted* Ferrers diagram  $\lambda^*$  has row  $i$  shifted  $i - 1$  spaces to the right. For example, if we have  $\lambda = (5, 4, 2)$  then

$$\lambda^* = \begin{array}{cccccc} \square & \square & \square & \square & \square & \\ & \square & \square & \square & \square & \\ & & \square & \square & & \\ & & & \square & \square & \end{array}$$

The cells  $(i, i) \in \lambda^*$  are called the *diagonal*; all other cells are *off-diagonal*. The number of diagonal cells is called the *length* of  $\lambda$  and denoted  $l(\lambda)$ . In the example above  $l(\lambda) = 3$  corresponding to the diagonal squares  $(1, 1)$ ,  $(2, 2)$ , and  $(3, 3)$ .

The definitions of skew shifted shape, as well as partial and standard skew shifted tableau, are the obvious analogs of those in Section 1. To illustrate, if  $\lambda$  is as above and  $\mu = (3, 2)$  then one standard tableau of shape  $\lambda^*/\mu^*$  is

$$P^* = \begin{array}{ccccc} \square & \square & \square & 1 & 3 \\ & & \square & \square & 2 & 6. \\ & & & 4 & 5 \end{array}$$

As might be expected, the sets of partial and standard shifted tableaux of shape  $\lambda^*/\mu^*$  will be denoted  $PT(\lambda^*/\mu^*)$  and  $ST(\lambda^*/\mu^*)$ , respectively. In addition, it will be necessary to be able to distinguish certain elements of a shifted tableau  $P^*$ . In previous papers [Sag, Wor] this was done by circling these elements. Since this is not the most convenient convention typographically, we will underline numbers which are distinguished, e.g.,  $\underline{k}$ . If we do not wish to commit ourselves as to whether an integer is distinguished or not, we will write  $\tilde{k}$ . Of particular interest will be the tableaux, where only off-diagonal entries can be underlined. Let  $PT_0(\lambda^*/\mu^*)$  and  $ST_0(\lambda^*/\mu^*)$  be the sets of such tableaux of the partial and standard varieties, respectively; e.g., a shifted array  $P^* \in ST_0(\lambda^*/\mu^*)$  might look like

$$P^* = \begin{array}{ccccc} \square & \square & \square & 1 & \underline{3} \\ & & \square & \square & \underline{2} & 6. \\ & & & 4 & 5 \end{array}$$

Also let  $g_{\lambda/\mu} = |ST(\lambda^*/\mu^*)|$  so that  $|ST_0(\lambda^*/\mu^*)| = 2^{n-l(\lambda-\mu)} g_{\lambda/\mu}$ , where  $\lambda/\mu \vdash n$  and  $l(\lambda/\mu) = l(\lambda) - l(\mu)$ .

We must now modify the definitions of the insertion operators  $R_m$  and  $R_{i,j}$  from Section 2 to apply to a shifted tableau  $P^*$ . If a diagonal element is never displaced during insertion then the  $R$ 's behave as before and the insertion is called a *Schensted insertion*. If, at some point,  $p_{ii}^*$  is displaced then it is inserted into the  $i + 1$ th column (replacing the smallest element larger than itself if one exists). The process continues column by column until some entry comes to rest at the lower end of a column. Since entries are moving to the right and up during column insertion, the diagonal will never be intersected again. An insertion of this type is called *non-Schensted*. For example, a non-Schensted internal insertion might look like

$$R_{1,4} \left( \begin{array}{ccccc} \square & \square & \square & 1 & 3 \\ & \square & \square & 2 & 6 \\ & & & 4 & 5 \end{array} \right) = \begin{array}{ccccc} \square & \square & \square & \square & 3 & 6 \\ & \square & \square & 1 & 5 & . \\ & & & 2 & 4 \end{array}$$

Finally, we will wish to consider internal insertions that only displace entries by columns in  $P^*$ . Let  $(i, j)$  be an inner corner of the shape  $\lambda^*/\mu^*$  of  $P^*$ .  $C_{i,j}(P^*)$  will be the tableau obtained by removing  $p_{ij}^*$  from column  $j$  and inserting it in column  $j + 1$ , etc., until an element comes to rest at the end of some column. Such an operation will always be considered non-Schensted. By way of illustration,

$$C_{1,4} \left( \begin{array}{ccccc} \square & \square & \square & 1 & 3 \\ & \square & \square & 2 & 6 \\ & & 4 & 5 & \end{array} \right) = \begin{array}{ccccc} \square & \square & \square & \square & 1 & 3 \\ & \square & \square & 2 & 6 & \\ & & 4 & 5 & \end{array} .$$

One last bit of terminology before stating the main result of this section. If  $A, B$  are sets of integers, some of which have been underlined, we write  $A \equiv B$  to mean that  $A$  and  $B$  are the same up to underlining, e.g.,  $\{1, \underline{3}, 4, 5, 7\} \equiv \{1, 3, \underline{4}, 5, \underline{7}\}$ . We do not permit a set  $A$  to have both  $k \in A$  and  $\underline{k} \in A$ . Thus we say  $A$  and  $B$  are disjoint if  $A \equiv C$  and  $B \equiv D$ , where  $C$  and  $D$  have no underlined elements and  $C \cap D = \emptyset$ . If  $A$  and  $B$  are disjoint then the set  $A \cup B$  is defined as having every element which is in either  $A$  or  $B$  with the underlining preserved (the fact that the sets are disjoint makes the underlining of the union well-defined).

**THEOREM 8.1.** *Let  $n$  be a fixed positive integer and  $\alpha$  a fixed strict partition. Then the map*

$$(\pi, T^*, U^*) \leftrightarrow (P^*, Q^*)$$

*defined below is a bijection between  $\pi \in \text{PS}_n$ ,  $T^* \in \text{PT}(\alpha^*/\mu^*)$ ,  $U^* \in \text{PT}_0(\alpha^*/\mu^*)$  such that  $\tilde{\pi} \cup T^* = \{1, 2, \dots, n\} \equiv \tilde{\pi} \cup U^*$ , on the one hand, and  $P^* \in \text{ST}(\lambda^*/\alpha^*)$ ,  $Q^* \in \text{ST}_0(\lambda^*/\alpha^*)$ , on the other.*

*Proof.* We will construct  $(P_0^*, Q_0^*) = (T^*, \phi_\alpha^*)$ ,  $(P_1^*, Q_1^*)$ , ...,  $(P_n^*, Q_n^*) = (P^*, Q^*)$  as follows. At the  $k$ th step we determine whether  $k \in \tilde{\pi}$  or  $\underline{k} \in U^*$ .

In the former case we have  $\binom{k}{m}$  in  $\pi$  and set  $P_k^* = R_m(P_{k-1}^*)$ . If the insertion is Schensted then we place  $k$  in the corresponding position of  $Q_{k-1}^*$  to form  $Q_k^*$ . However, if it was non-Schensted then  $\underline{k}$  is put in that position.

Now suppose  $\underline{k} \in U^*$  appearing, say, in cell  $(i, j)$ . If  $u_{ij}^* = k$  then  $P_k^* = R_{i,j}(P_{k-1}^*)$  with  $Q_k^*$  formed as before. If instead  $u_{ij}^* = \underline{k}$  then  $P_k^* = C_{i,j}(P_{k-1}^*)$  with  $\underline{k}$  being added to form  $Q_k^*$ .

Note that we will have  $\underline{k} \in Q^*$  if and only if the  $k$ th insertion was non-Schensted, i.e., step  $k$  ended in column insertion. This guarantees that  $Q^*$  has no underlined elements on the diagonal. It also makes construction of the inverse map possible by flagging which deletions are to be started by columns rather than by rows. ■

The following example will illustrate all five possible insertions (Schensted and non-Schensted  $R_m$  and  $R_{i,j}$  as well as  $C_{i,j}$ ). Let

$$(\pi, T^*, U^*) = \left( \begin{array}{cccccccc} 2 & 3 & \square & \square & \square & 4 & \square & \square & \square & 1 \\ 3 & 2 & & \square & 1 & , & & \square & 4 & \\ & & & & 5 & & & & 5 & \end{array} \right).$$

Then

$$\begin{aligned} (P_k^*, Q_k^*) : & \left( \begin{array}{cccccccc} \square & \square & \square & 4 & \square & \square & \square & \square \\ & \square & 1 & , & \square & \square & & \\ & & 5 & & \square & & & \end{array} \right), \left( \begin{array}{cccccccc} \square & \square \\ & \square & 1 & 4, & \square & \square & 1 & \\ & & 5 & & \square & & & \end{array} \right), \\ & \left( \begin{array}{cccccccc} \square & \square & \square & \square & 3 & \square & \square & \square & \square & 2 \\ & \square & 1 & 4 & , & \square & \square & 1 & & \\ & & 5 & & \square & & & & & \end{array} \right), \left( \begin{array}{cccccccc} \square & \square & \square & \square & 2 & \square & \square & \square & \square & 2 \\ & \square & 1 & 3 & , & \square & \square & 1 & & \\ & & 4 & 5 & & \square & 3 & & & \end{array} \right), \\ & \left( \begin{array}{cccccccc} \square & \square & \square & \square & 2 & \square & \square & \square & \square & 2 \\ & \square & \square & 1 & 3, & \square & \square & 1 & 4 & \\ & & 4 & 5 & & \square & 3 & & & \end{array} \right), \left( \begin{array}{cccccccc} \square & \square & \square & \square & 2 & \square & \square & \square & \square & 2 \\ & \square & \square & 1 & 3, & \square & \square & 1 & 4 & \\ & & \square & 4 & 5 & & \square & 3 & 5 & \end{array} \right) = (P^*, Q^*). \end{aligned}$$

Translating the bijection into an equation yields:

**COROLLARY 8.2.** *Let the positive integer  $n$  and strict partition  $\alpha$  be fixed. Then*

$$\sum_{\lambda/\alpha \vdash n} 2^{n-l(\lambda/\alpha)} g_{\lambda/\alpha}^2 = \sum_k \binom{n}{k}^2 k! \sum_{\alpha/\mu \vdash n-k} 2^{n-k-l(\alpha/\mu)} g_{\alpha/\mu}^2.$$

Composing this map with itself or using two fixed strict partitions produces the expected results which we record in the next four corollaries.

**COROLLARY 8.3.** *Iteration of the bijection of Theorem 8.1 produces a weight-preserving bijection*

$$(\pi, \nu) \leftrightarrow (P^*, Q^*)$$

*between weighted  $\pi \in S_n$  with  $\nu$  a strict partition and pairs  $P^* \in \text{ST}(\lambda^*/\mu^*)$ ,  $Q^* \in \text{ST}_0(\lambda^*/\mu^*)$ , where  $\lambda/\mu \vdash n$ .*

**COROLLARY 8.4.**

$$\sum_{k,n} \left( \sum_{\substack{\lambda/\mu \vdash n \\ \mu \vdash k}} 2^{n-l(\lambda/\mu)} g_{\lambda/\mu}^2 \right) q^k t^n / n! = \frac{1}{1-t/(1-q)} \prod_n (1+q^n).$$

COROLLARY 8.5. *Let positive integers  $n, m$  and strict partitions  $\alpha, \beta$  be fixed. Then the mapping of Theorem 8.1 can be extended to a bijection*

$$(\pi, T^*, U^*) \leftrightarrow (P^*, Q^*)$$

*between partial permutations  $\pi, T^* \in \text{PT}(\alpha^*/\mu^*), U^* \in \text{PT}_0(\beta^*/\mu^*)$ , such that  $\hat{\pi} \cup T^* = \{1, 2, \dots, n\}$ ,  $\hat{\pi} \cup U^* \equiv \{1, 2, \dots, m\}$ , on the one hand, and  $P^* \in \text{ST}(\lambda^*/\beta^*), Q^* \in \text{ST}_0(\lambda^*/\alpha^*)$ , such that  $\lambda/\beta \vdash n, \lambda/\alpha \vdash m$ , on the other.*

COROLLARY 8.6.

$$\sum_{\substack{\lambda/\beta \vdash n \\ \lambda/\alpha \vdash m}} 2^{m-l(\lambda/\alpha)} g_{\lambda/\beta} g_{\lambda/\alpha} = \sum_{k \geq 0} \binom{n}{k} \binom{m}{k} k! \sum_{\substack{\alpha/\mu \vdash n-k \\ \beta/\mu \vdash m-k}} 2^{m-k-l(\beta/\mu)} g_{\alpha/\mu} g_{\beta/\mu}.$$

Shifted tableaux with repeated entries are defined somewhat differently from the column-strict arrays of Section 6. A *generalized shifted tableau* of shape  $\lambda^*/\mu^*$  is an array,  $P^*$ , with entries from the alphabet  $\{\underline{1} < \underline{1} < \underline{2} < \underline{2} < \underline{3} < \underline{3} < \dots\}$  such that

- (i) the rows and columns of  $P^*$  are weakly increasing (as with any Young tableau),
- (ii) for any  $\tilde{k}$ , there is at most one  $\underline{k}$  in any row and at most one  $k$  in any column. For example,

$$P^* = \begin{array}{ccccccccc} \square & \square & \square & \square & \underline{1} & \underline{1} & \underline{5} & 5 \\ & \square & \underline{1} & \underline{1} & \underline{3} & \underline{5} & 5 & \underline{6} \\ & & \underline{2} & \underline{3} & 3 & \underline{6} & 6 & \underline{6} \\ & & & & 4 & 4 & \underline{6} & \end{array}$$

Note that conditions (i) and (ii) imply that the  $\tilde{k}$ 's in  $P^*$  will form a union of skew hooks (a collection of cells containing no diagram of the form  $\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}$ ). Furthermore, the underlining of all  $\tilde{k}$ 's in a given hook is completely determined by (ii), except for the lower leftmost one which can be either underlined or not. Since the elements on the diagonal are all lower leftmost in their respective rim hooks, they may be underlined or not. Let  $\text{GT}(\lambda^*/\mu^*)$  and  $\text{GT}_0(\lambda^*/\mu^*)$  denote the sets of all generalized tableaux and generalized tableaux without diagonal underlines, respectively. The corresponding generating functions are the *Schur Q- and P-functions* (which are special cases of the Hall–Littlewood symmetric functions [Mac, p. 104 and ff]) defined by

$$Q_{\lambda/\mu}(\mathbf{x}) = \sum_{P^* \in \text{GT}(\lambda^*/\mu^*)} (x_1^{n_1(P^*)} x_2^{n_2(P^*)} \dots),$$

$$P_{\lambda/\mu}(\mathbf{x}) = \sum_{P^* \in \text{GT}_0(\lambda^*/\mu^*)} (x_1^{n_1(P^*)} x_2^{n_2(P^*)} \dots),$$

where  $n_k(P^*)$  is the number of elements equal to  $\tilde{k}$  in  $P^*$ . In the example above  $x_1^{n_1(P^*)}x_2^{n_2(P^*)}\dots = x_1^4x_2x_3^3x_4^2x_5^4x_6^5$ . Because of the relationship between the underlining permitted in  $GT(\lambda^*/\mu^*)$  and  $GT_0(\lambda^*/\mu^*)$  discussed above, we see that these two functions are multiples of each other:  $Q_{\lambda/\mu}(\mathbf{x}) = 2^{l(\lambda/\mu)}P_{\lambda/\mu}(\mathbf{x})$ .

We will also permit underlining in our biwords. An *underlined matrix*  $M$  is a non-negative integral matrix with some of its positive entries underlined. If  $m_{ij} = k$  in  $M$  then in the corresponding *underlined matrix word*  $\pi$ , the first occurrence of  $j$  in the  $k$  pairs  $(i_j)$  is underlined while the remaining  $k - 1$  are ordinary integers. To illustrate, let

$$M = \begin{pmatrix} 0 & \underline{3} & 0 \\ 2 & \underline{1} & 0 \\ 1 & 0 & \underline{2} \end{pmatrix}, \quad \text{then } \pi = \begin{matrix} 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 \\ \underline{2} & 2 & 2 & 1 & 1 & \underline{2} & 1 & \underline{3} & 3 \end{matrix}$$

We can now describe the insertion process. Although the rules may seem strange at first, they are forced on us in order to preserve conditions (i) and (ii) in the definition of a generalized shifted tableaux. The options are as follows:

- (a) if  $k$  is being inserted in a row (resp., column) it displaces the smallest element greater than (resp., greater than or equal to) itself.
- (b) if  $\underline{k}$  is being inserted in a row (resp., column) it displaces the smallest element greater than or equal to (resp., greater than) itself.

It is convenient to record these possibilities in a table:

	Row	Column
$k$	$>$	$\geq$
$\underline{k}$	$\geq$	$>$

The operators  $R_m$ ,  $R_{i,j}$ , and  $C_{i,j}$  are defined by iterating (a) and (b) in the obvious way, with two exceptions. Before considering the exceptional cases we will compute the effect of applying  $R_1$  to the example tableau  $P^*$  above. The resulting tableau is

□	□	□	□	<u>1</u>	1	<u>5</u>	5	5
	□	<u>1</u>	1	1	4	<u>5</u>	<u>6</u>	
		<u>2</u>	<u>3</u>	3	<u>6</u>	6	6	
			<u>3</u>	4	<u>6</u>			

The only modifications of the above insertion rules occur when a diagonal element is displaced:

(c) if a  $\underline{k}$  displaces  $p_{ii}^* = k$  then the incoming  $\underline{k}$  loses its underline after being inserted on the diagonal in row  $i$ ,

(d) if a  $\underline{k}$  displaces  $p_{ii}^* = k$  then the outgoing  $\underline{k}$  loses its underline before being inserted (off the diagonal) in column  $i + 1$ .

These exceptions are illustrated by

$$R_1 \left( \begin{array}{ccc} \square & \square & \underline{2} \\ & & 2 \end{array} \right) = \begin{array}{ccc} \square & \square & 1 \\ & & 2 \end{array}, \quad R_{1,3} \left( \begin{array}{ccc} \square & \square & \underline{2} \\ & & \underline{2} \end{array} \right) = \begin{array}{ccc} \square & \square & \square \\ & & \underline{2} \end{array}.$$

Note that (c) and (d) do not apply to internal insertions of the form  $R_{i,i}$ , since there is no incoming  $\underline{k}$  in this case.

Finally everything is in place to prove:

**THEOREM 8.7.** *Let  $\alpha$  be a fixed strict partition. Then the map*

$$(\pi, T^*, U^*) \leftrightarrow (P^*, Q^*)$$

*defined below is a bijection between underlined matrix words  $\pi, T^* \in \text{GT}(\alpha^*/\mu^*), U^* \in \text{GT}_0(\alpha^*/\mu^*)$ , on the one hand, and  $P^* \in \text{GT}(\lambda^*/\alpha^*), Q^* \in \text{GT}_0(\lambda^*/\alpha^*)$ , on the other, such that  $\hat{\pi} \circ T^* \equiv P^*$  and  $\hat{\pi} \circ U^* \equiv Q^*$  as multisets.*

*Note.* If  $A$  and  $B$  are multisets of underlined integers then their disjoint union is the multiset where the number of repetitions of each  $k$  (resp.  $\underline{k}$ ) is the sum of the number of copies of  $k$  (resp.,  $\underline{k}$ ) in  $A$  and  $B$ .

*Proof.* The only thing that has not been specified by the above discussion is the order of the insertions at the  $k$ th step, i.e., those corresponding to  $\underline{k}$ 's in  $\hat{\pi} \circ U^*$ . First, we perform the internal insertions that involve a  $\underline{k} \in U^*$  (all  $C_{i,j}$ 's) working from top to bottom in  $U^*$ . Next the  $R_{i,j}$ 's corresponding to  $k$ 's in  $U^*$  are performed working from left to right. Last, we externally insert the elements below a  $k$  in  $\pi$  (remember: only  $k$ 's appear in  $\hat{\pi}$ ). ■

It is an unfortunate historical accident of notation that the generating function for the  $P$ -tableaux in the above bijection is the Schur  $Q$ -function and vice versa. However, this should not prevent us from writing the correct identity.

**COROLLARY 8.8.** *Fix a strict partition  $\alpha$ . Then*

$$\sum_{\lambda} Q_{\lambda/\alpha}(\mathbf{x}) P_{\lambda/\alpha}(\mathbf{y}) = \sum_{\mu} Q_{\alpha/\mu}(\mathbf{x}) P_{\alpha/\mu}(\mathbf{y}) \prod_{i,j} (1 + x_i y_j)/(1 - x_i y_j).$$

The iterated analog of the shifted Knuth map carries over from the unshifted case mutatis mutandis. The only point that might need clarification is the definition of a *weighted underlined matrix word*: for each  $i, j, k$  we permit only the first occurrence of  $(j^{(k)})$  to have an underlined  $j$ , e.g.,

$$\pi = \left( \begin{array}{cccccccc} 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 \\ \underline{1}^{(2)} & \underline{1}^{(2)} & \underline{1}^{(1)} & 2 & 2 & \underline{2}^{(3)} & \underline{2}^{(3)} & \underline{2} & \underline{2} \end{array} \right).$$

This ensures that the subword consisting of all pairs of fixed weight  $k$  is an underlined matrix word in the sense of Theorem 8.7.

**COROLLARY 8.9.** *Iteration of the map of Theorem 8.7 produces a weight-preserving bijection*

$$(\pi, \nu) \leftrightarrow (P^*, Q^*)$$

between weighted underlined matrix words  $\pi$  with  $\nu$  a strict partition, and pairs  $P^* \in \text{GT}(\lambda^*/\mu^*)$ ,  $Q^* \in \text{GT}_0(\lambda^*/\mu^*)$ , where  $\tilde{\pi} \equiv P^*$  and  $\hat{\pi} \equiv Q^*$  as multisets.

**COROLLARY 8.10.**

$$\begin{aligned} \sum_k \left( \sum_{\substack{\lambda \\ \mu \vdash k}} Q_{\lambda/\mu}(\mathbf{x}) P_{\lambda/\mu}(\mathbf{y}) \right) q^k \\ = \prod_n (1 + q^n) \prod_{i,j} (1 + x_i y_j q^{n-1}) / (1 - x_i y_j q^{n-1}). \end{aligned}$$

The two-partition case is disposed of with similar dispatch.

**COROLLARY 8.11.** *Let partitions  $\alpha, \beta$  be fixed. Then the map of Theorem 8.7 can be extended to a bijection*

$$(\pi, T^*, U^*) \leftrightarrow (P^*, Q^*)$$

between underlined matrix words  $\pi$ ,  $T^* \in \text{GT}(\alpha^*/\mu^*)$ ,  $U^* \in \text{GT}_0(\beta^*/\mu^*)$ , on the one hand, and  $P^* \in \text{GT}(\lambda^*/\beta^*)$ ,  $Q^* \in \text{GT}_0(\lambda^*/\alpha^*)$ , on the other, such that  $\tilde{\pi} \cup T^* \equiv P^*$  and  $\hat{\pi} \cup U^* \equiv Q^*$  as multisets.

**COROLLARY 8.12.** *Fix strict partitions  $\alpha$  and  $\beta$ . Then*

$$\sum_{\lambda} Q_{\lambda/\beta}(\mathbf{x}) P_{\lambda/\alpha}(\mathbf{y}) = \sum_{\mu} Q_{\alpha/\mu}(\mathbf{x}) P_{\beta/\mu}(\mathbf{y}) \prod_{i,j} (1 + x_i y_j) / (1 - x_i y_j).$$

## 9. COMMENTS AND OPEN QUESTIONS

The amazingly rich structure of the original Robinson–Schensted algorithm suggests that we have only begun to scratch the surface in the skew case. Below we outline some directions for future research.

(1) One of Schensted’s original results [Sch] was that when  $\pi \leftrightarrow (P, Q)$  (the case  $\alpha = \emptyset$  of Theorem 2.1) then the length of the longest increasing (resp., decreasing) subsequence of  $\tilde{\pi}$  is the length of the first row (resp., column) of  $P$ . It is not at all clear what the analogous result is if  $\alpha/\mu \neq \emptyset$ . In fact, Schensted proved that if one reverses the order of the elements of  $\tilde{\pi}$  then the  $P$ -tableau is transposed. A generalization of this result to the skew case would be most welcome.

(2) Viewing the biword  $\pi$  as an  $n$  by  $n$  matrix we can let the dihedral group of the square act on  $\pi$  and ask what happens to the output tableaux  $P, Q$  (with appropriate changes to the accompanying tableaux  $T, U$  or partition  $\nu$ ). Exchanging  $\pi$  for  $\pi^{-1}$  or reversing  $\tilde{\pi}$  are special cases. In fact, the complete answer is known for the original algorithm for any symmetry of the square [Gan, Scü].

(3) Two other important tools in tableau theory whose skew analogs are missing are Schützenberger’s jeu de taquin [Scu1; Scü2] and the Knuth relations [Knu]. The jeu gives an alternative way to define the  $P$ -tableau and can be used to demonstrate various symmetric function identities [Wor]. The Knuth relations characterize all biwords  $\pi$  which have the same  $P$ -tableau and have been used, among other things, to generalize Schensted’s theorem about increasing and decreasing subsequences [Gre].

(4) Note that we did not derive any results about inverting biwords in the shifted case. This is because even when  $\alpha = \emptyset$  and  $\pi \leftrightarrow (P^*, Q^*)$  it is not true that inverting  $\pi$  will “interchange”  $P^*$  and  $Q^*$  in any reasonable sense of the term. However, Haiman [Hai] has discovered another process, dubbed mixed insertion and denoted  $m$ , such that  $\pi^{-1} \xrightarrow{m} (Q^*, P^*)$ . It is an easy matter to find a skew analog of Haiman’s algorithm, but it does not result in any new identities, so has been omitted.

(5) As mentioned in the previous section, the Hall–Littlewood symmetric functions  $P_{\lambda/\mu}(\mathbf{x}; t)$  specialize to the ordinary Schur functions or Schur  $P$ -functions when  $t$  is 0 or  $-1$ , respectively. In fact, one can algebraically derive a general formula for arbitrary  $t$  that has Corollaries 6.2 and 8.8 as special cases. It would be interesting to obtain a Robinson–Schensted type proof in this general setting using the combinatorial definition of  $P_{\lambda/\mu}(\mathbf{x}; t)$  given in [Mac, pp. 119–121].

(6) Stanley [Sta2] was led to identities like (4.1) by counting chains

in Young's lattice  $Y$  (the lattice of partitions ordered by inclusion). To see the connection, merely note that  $f_{\lambda/\mu}$  is just the number of maximal chains from  $\mu$  to  $\lambda$  in  $Y$ . What is more, Stanley has shown that there is a general class of partially ordered sets, called differential posets, where similar equations hold. However, the lattice of shifted partitions is not differential (which is not surprising, since we have no analog of (4.1) in the shifted case). Nonetheless, we can still derive identities like Corollary 8.4 for counting pairs of maximal chains. It seems probable that the techniques of Stanley, which involve solving certain partial differential equations, can also be applied to the shifted case. Is there another class of posets where analogous formulae can be proved?

(7) Butler [But] has used a different approach to obtain generating functions for chains in  $Y$  that go through specified ranks. Specifically she  $q$ -counts linear extensions of certain partially ordered sets to obtain her results. Can the same thing be done in the shifted lattice or in an arbitrary differential poset? This enumeration problem is made harder because, as Butler has noted, the generating functions involved need not be rational as they are for  $Y$ .

(8) Oscillating tableaux are another family of arrays that are important in combinatorics and representation theory. In particular, there are a number of Robinson–Schensted algorithms for such tableaux [Ber; McL; Pro; Sun] and formulae for certain oscillating chains in differential posets [Sta2]. This suggests that a skew oscillating Robinson–Schensted map might be found.

(9) Rim-hook tableaux appear in the determination of the ordinary and  $p$ -modular characters of the symmetric group. White [Whi] and Stanton and White [SW] have obtained a Robinson–Schensted correspondence in this case. Furthermore, the poset of all partitions with given  $p$ -core ordered by rim-hook removal is differential. These two facts point to the existence of a rim-hook version of our algorithm. This matter is currently under investigation by Sagan.

*Note added in proof.* For further information on part (6) of Section 9, see R. Stanley, Variations on differential posets, in "Invariant Theory and Tableaux" (D. Stanton, Ed.), The IMA Volumes in Mathematics and Its Applications, Vol. 19, pp. 145–165, Springer-Verlag, New York, 1990, pp. 145–165.

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