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Plane Partitions: Past, Present, and Future[a]

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It is not our purpose here to give a comprehensive survey of the theory of plane partitions. Such a survey up to 1971 was given in [21], while summaries of more recent work appear in [22] and [23]. We will instead sketch some basic definitions, results, and conjectures that should provide an entry into more detailed work for readers interested in pursuing the subject further.

A *plane partition* is an array $\pi = (\pi_{ij})_{i,j\geq 1}$ of nonnegative integers π_{ij} that is weakly decreasing in rows and columns, and for which $|\pi| := \sum \pi_{ij} < \infty$. When writing a plane partition the zero entries are usually suppressed. Thus, for instance,

$$\begin{matrix} 5 & 5 & 4 & 3 & 3 & 2 & 2 & 1 \\ 5 & 4 & 4 & 3 & 1 & 1 & & \\ 3 & 3 & 2 & 1 & 1 & & & \end{matrix}$$

is a plane partition π with 3 rows, 8 columns, largest part 5, and with $|\pi| = 53$. The study of plane partitions began with MacMahon (who collected his results in this area in [12, secs. IX and X]). MacMahon's principal result is equivalent to the remarkable formula

$$\sum_{\pi} q^{|\pi|} = \prod_{i=1}^{r} \prod_{j=1}^{s} \prod_{k=1}^{t} \frac{1 - q^{i+j+k-1}}{1 - q^{i+j+k-2}}, \tag{1}$$

summed over all plane partitions π with $\leq r$ rows, $\leq s$ columns, and largest part $\leq t$. In particular, letting $r, s, t \to \infty$ yields the beautiful result

$$\sum q^{|\pi|} = \prod_{k\geq 1} (1 - q^k)^{-k},$$

summed over all plane partitions.

MacMahon's proof involved intricate combinatorial arguments together with manipulations of determinants. At present there are three basic techniques (all closely interrelated) for dealing with plane partitions: (a) the Robinson–Schensted correspondence and its variants, (b) the theory of symmetric functions and related results from representation theory, and (c) manipulations of determinants. In particular, (a) can be used to give an elegant combinatorial proof of (1) in the limiting case $t \to \infty$, while either (b) or (c) can be used to prove (1) in its full generality. For the

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basic connections between plane partitions and (a)–(c), see [21]. For a comprehensive treatment of the theory of symmetric functions, with many applications (mostly implicit) to plane partitions, see Macdonald [11]. The connection between plane partitions and determinants is given an elegant development by Gessel and Viennot [6–8]. Some additional references related to enumerating plane partitions in general include [2–5, 10, 17, 19, 20, 24, 26].

A currently very active topic within the theory of plane partitions is that of plane partitions satisfying certain symmetry conditions. This work goes back to a conjecture of MacMahon [12, sec. 520] on symmetric plane partitions (i.e., $\pi_{ij} = \pi_{ji}$), later proved by Andrews [1] and Gordon [9] using complicated manipulations of determinants, by Macdonald [11, ex. 17, p. 52] using root systems and symmetric functions, and by Proctor [18] using algebraic methods (Seshadri's standard monomial theory for minuscule representations). This result may be stated as follows:

$$\sum_{\pi} q^{|\pi|} = \prod_{1 \le i \le j \le r} \prod_{k=1}^{t} \frac{1 - q^{(2-\delta_{ij})(i+j+k-1)}}{1 - q^{(2-\delta_{ij})(i+j+k-2)}}, \tag{2}$$

where π ranges over all symmetric plane partitions with $\le r$ rows and largest part $\le t$, and where δ_{ij} denotes the Kronecker delta.

A total of ten symmetry classes of plane partitions can be defined, and the known results concerning them are summarized in [22]. In particular, seven of the ten classes have simple formulas that enumerate them, while the remaining three have simple conjectured formulas. For instance, it is conjectured that the number of symmetric plane partitions π with largest part $\le n$, such that every row of π is a self-conjugate partition (as defined, e.g., in [11, p. 2]) is equal to

$$\prod_{1 \le i \le j \le k \le n} \frac{i+j+k-1}{i+j+k-2}. \tag{3}$$

Compare Equation (3) with the case $q = 1$ of (1) and (2), namely,

$$A(r, s, t) = \prod_{i=1}^{r} \prod_{j=1}^{s} \prod_{k=1}^{t} \frac{i+j+k-1}{i+j+k-2}$$

$$B(r, t) = \prod_{1 \le i \le j \le r} \prod_{k=1}^{t} \frac{i+j+k-1}{i+j+k-2},$$

where $A(r, s, t)$ [respectively, $B(r, t)$] is the number of plane partitions (respectively, symmetric plane partitions) with $\le r$ rows, $\le s$ columns, and largest part $\le t$ (respectively, $\le r$ rows and largest part $\le t$). The correct "q-analog" of (3) is not so obvious and is given in [22] or [23]. For other recent work related to symmetries of plane partitions, see [13, 15–17].

There is a close connection between plane partitions and certain triangular arrays known as *Gelfand patterns* (see [23]). A Gelfand pattern is an array

$$\begin{array}{ccccccc} a_{11} & & a_{12} & & \cdots & & a_{1n} \\ & a_{22} & & a_{23} & \cdots & a_{2n} & \\ & & & \vdots & & & \\ & & & a_{nn} & & & \end{array}$$

of nonnegative integers satisfying $a_{ij} \le a_{i,j+1}$ and $a_{i-1,j-1} \le a_{ij} \le a_{i-1,j}$. If we require the stronger condition $a_{ij} < a_{i,j+1}$, then we obtain a *strict Gelfand pattern*. A strict Gelfand pattern with first row 1, 2, ..., n is a *monotone triangle* of length n, introduced by Mills *et al.* [14]. There is a simple bijection [14, pp. 354–355] between monotone triangles of length n and $n \times n$ *alternating sign* matrices, that is, $n \times n$ matrices whose entries are 0, ± 1, whose row and column sums all equal 1, and whose nonzero entries in every row and column alternate in sign. The seven monotone triangles of length 3 are

123	123	123	123	123	123	123
12	12	13	13	13	23	23
1	2	1	2	3	2	3

The seven 3×3 alternating sign matrices consist of the six 3×3 permutation matrices, together with

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

There are now a plethora of closely interrelated conjectures, due mostly to Mills–Robbins–Rumsey, concerning the enumeration of monotone triangles and related objects. These conjectures are summarized in [23]. Let us mention a few of these conjectures.

If T is an alternating sign matrix, then let $s(T)$ denote its number of -1s. Define $A_n(x) = \sum_T x^{s(T)}$, where T ranges over all $n \times n$ alternating sign matrices. For instance, $A_3(x) = 6 + x$.

THEOREM (easy): $A_n(0) = n!$.

CONJECTURE: $A_n(1) = \prod_{k=0}^{n-1} \frac{(3k+1)!}{(n+k)!}$.

THEOREM: $A_n(2) = 2^{\binom{n}{2}}$.

CONJECTURE: $A_n(3) = 3^{t(n)} C_n$, where

$$t(n) = \begin{cases} m(m-1), & n = 2m \\ m^2, & n = 2m+1, \end{cases}$$

and where C_n is determined by the recurrence

$$C_0 = 1, \qquad \frac{C_{2n+1}}{C_{2n}} = \frac{\binom{3n}{n}}{\binom{2n}{n}}, \qquad \frac{C_{2n}}{C_{2n-1}} = \frac{4\binom{3n}{n}}{3\binom{2n}{n}}.$$

CONJECTURE: The number C_n is equal to the number of $n \times n$ alternating sign matrices that are invariant under a 180° rotation.

CONJECTURE: There exist polynomials $B_n(x)$ for which

$$A_n(x) = \begin{cases} B_n(x)B_{n+1}(x), & n \text{ odd} \\ 2B_n(x)B_{n+1}(x), & n \text{ even.} \end{cases}$$

Moreover, $B_{2n+1}(x) = \sum_T x^{\sigma(T)}$, summed over all $(2n + 1) \times (2n + 1)$ alternating sign matrices T that are invariant under a reflection in a vertical axis through the center of T, and where T has $2\sigma(T) + n$ entries equal to -1.

For a recent result related to the enumeration of strict Gelfand patterns, see [25].

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