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Quotients of Peck Posets

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Abstract. An elementary, self-contained proof of a result of Pouzet and Rosenberg and of Harper is given. This result states that the quotient of certain posets (called unitary Peck) by a finite group of automorphisms retains some nice properties, including the Sperner property. Examples of unitary Peck posets are given, and the techniques developed here are used to prove a result of Lovász on the edge-reconstruction conjecture.

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Key words. Poset, Peck poset, Sperner property, automorphism group, quotient poset, edge-reconstruction.

Let P be a finite graded poset of rank n, i.e., P is a disjoint union of subsets P_0, $P_1, \ldots, P_n$, called *ranks*, such that if $x \in P_i$ and y covers x, then $y \in P_{i+1}$. Let $p_i = |P_i|$, where P is said to be *rank-symmetric* if $p_i = p_{n-i}$ for all i, and *rank-unimodal* if $p_0 \leqslant p_1 \leqslant \cdots \leqslant p_j \geqslant p_{j+1} \geqslant p_{j+2} \geqslant \cdots \geqslant p_n$ for some j. P satisfies the *Sperner property* if no antichain (= set of pairwise incomparable elements) of P is bigger than the largest rank. More generally, P is *k-Sperner* if no union of k antichains is larger than the union of the k largest ranks, and is *strongly Sperner* if it is k-Sperner for $1 \leqslant k \leqslant n+1$. P is a *Peck poset* if it is rank-symmetric, rank-unimodal, and strongly Sperner.

Let V_i be the complex vector space with basis P_i. It is known [13], Lemma 1.1, that a finite graded poset P of rank n is Peck if and only if there exist linear transformations $\phi_i\colon V_i \to V_{i+1}$, $0 \leqslant i < n$, satisfying:

(A) If $x \in P_i$ then

$$\phi_i(x) = \sum_{\substack{y \in P_{i+1} \\ y > x}} c_y \cdot y$$

for some $c_y \in \mathbb{C}$.

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(B) For all $0 \leqslant i < \frac{1}{2}n$, the linear transformation

$$\phi_{n-i+1} \cdots \phi_{i+1}\phi_i : V_i \to V_{n-i}$$

is invertible.

Let us call a Peck poset P *unitary* if the above linear transformations ϕ_i can be taken to be

$$\phi_i(x) = \sum_{\substack{y \in P_{i+1} \\ y > x}} y, \quad x \in P_i. \tag{1}$$

Let G be a group of automorphisms of a poset P, and let P/G be the quotient poset. The elements of P/G are the orbits of G, and $\mathcal{O} \leqslant \mathcal{O}'$ in P/G if there exist $x \in \mathcal{O}$, $x' \in \mathcal{O}'$ such that $x \leqslant x'$ in P.

The purpose of this paper is to give a simple, straightforward proof that if P is unitary Peck then P/G is Peck. A somewhat weaker result was first obtained by M. Pouzet [9], last sentence on p. 118, at least when P is a Boolean algebra. Namely, if V_i/G denotes the vector space with basis P_i/G (the orbits of G on P_i), then for $0 \leqslant i \leqslant j \leqslant n$ there are linear transformations ψ_{ij}: $V_i/G \to V_j/G$ such that: (a) if $\mathcal{O} \in P_i/G$ then

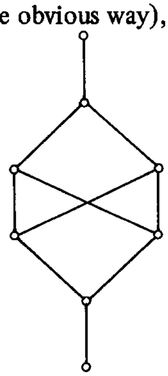
$$\psi_i(\mathcal{O}) = \sum_{\substack{\mathcal{O} \leqslant \mathcal{O}' \\ \mathcal{O}' \in P_j/G}} c_{\mathcal{O}'} \circ \mathcal{O}' \quad \text{for some } c_{\mathcal{O}'} \in \mathbf{C}, \text{ and (b)}$$

$$\text{rank } \psi_{ij} = \min\{|P_i/G|, |P_j/G|\}.$$

This result implies that P/G has the Sperner property, but it is not strong enough to imply the strong Sperner property. Pouzet and Rosenberg [10] have gone on to generalize the argument of [9] and to obtain our main result (Theorem 1) as a special case. Independently, Harper [3] has generalized Theorem 1 using category theory. Both these proofs involve considerable background not really necessary if only Theorem 1 is desired. Thus, the proof given here, while basically the same as those in [3] and [10], should be more accessible.

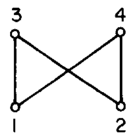
THEOREM 1. *If P is a unitary Peck poset then P/G is Peck.*

NOTE: (a) If P is unitary Peck then P/G need not be unitary Peck. For instance, if P is the Boolean algebra B_5 (which is unitary Peck) and G the cyclic group of order 5 (acting in the obvious way), then P/G is given by

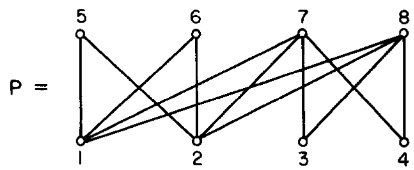


which is not unitary Peck.

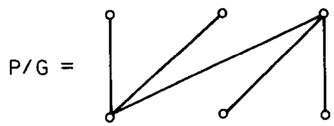
(b) If P is Peck then P/G need not be Peck. For instance, let P be the Peck poset



Let G be generated by the transposition $(1,2)$. Then P/G isn't Peck (or even rank-symmetric), although it does have the strong Sperner property. For an example where P/G does not even have the Sperner property, take P to be the Peck poset



Let G be generated by the permutation $(1,2)(7,8)$. Then



which lacks the Sperner property.

We now turn to the proof of Theorem 1. The action of $w \in G$ on P_i extends to a linear transformation $w\colon V_i \to V_i$. Let

$$V_i^G = \{f \in V_i \mid wf = f \text{ for all } w \in G\},$$

the space of G-invariant elements of V_i. For any subset S of P_i we identify S with the element $\Sigma_{x \in S}\, x$ of V_i. The following lemma is standard, but for the sake of completeness we include a proof.

LEMMA 1. *A basis for V_i^G consists of the orbits $\mathcal{O} \in P_i/G$.*

Proof. Clearly the $\mathcal{O}$'s are linearly independent elements of V_i^G. Moreover, if $f = \Sigma_{x \in P_i} f(x)x \in V_i^G$, then

$$f = \frac{1}{|G|} \sum_{w \in G} wf = \frac{1}{|G|} \sum_x f(x) \sum_w wx = \frac{1}{|G|} \sum_x f(x) \frac{|G|}{|\mathcal{O}_x|} \mathcal{O}_x,$$

where $\mathcal{O}_x$ is the orbit containing x. Hence, P_i/G spans V_i^G. □

LEMMA 2. *Let* ϕ_i *be given by* (1). *If* $f \in V_i$ *and* $w \in G$, *then* $\phi_i(wf) = w\phi_i(f)$.

Proof. By linearity, we may assume $f = x \in P_i$. We have

$$\begin{aligned}\phi_i(wx) &= \sum_{\substack{y > wx \\ y \in P_{i+1}}} y \\ &= w \cdot \sum_{w^{-1}y > x} w^{-1}y \quad \text{(since } w \text{ is an automorphism of } P) \\ &= w\phi_i(x).\end{aligned}$$

COROLLARY. *Suppose P is unitary Peck. If* $f \in V_i^G$, *then* $\phi_i(f) \in V_{i+1}^G$.

Proof. For all $w \in G$, we have $w\phi_i(f) = \phi_i(wf) = \phi_i(f)$, so $\phi_i(f) \in V_{i+1}^G$. □

Proof of Theorem 1. Let ϕ_i be given by (1). Pick $0 \neq f \in V_i^G$, $0 \leqslant i \leqslant n/2$. Set

$$g = \phi_{n-i-1}\phi_{n-i-2} \cdots \phi_i(f).$$

By the above corollary, $g \in V_{n-i}^G$. On the other hand, since P is unitary Peck we have $g \neq 0$. Hence, the map $\phi_{n-i-1} \cdots \phi_i\colon V_i^G \to V_{n-i}^G$ is injective.

There are several ways to see that $\phi_{n-i-1} \cdots \phi_i\colon V_i^G \to V_{n-i}^G$ is surjective. For instance, pick $g \in V_{n-i}^G$. Since P is unitary Peck, some $f \in V_i$ satisfies $g = \phi_{n-i-1} \cdots \phi_i(f)$. Let

$$\tilde{f} = \frac{1}{|G|} \sum_{w \in G} wf.$$

Then $\tilde{f} \in V_i^G$ and $g = \phi_{n-i-1} \cdots \phi_i(\tilde{f})$, as desired.

By Lemma 1, we may identify V_i^G with the vector space V_i/G with basis P_i/G. We have shown that the ϕ_i's, when restricted to V_i^G, satisfy condition (B). Since condition (A) is obvious, the proof is complete. □

THEOREM 2. *The following posets are unitary Peck:*

(a) *a product of unitary Peck posets,*

(b) *the lattice* $L(m, n)$ *of Ferrers diagrams fitting in an* $m \times n$ *rectangle* (*see* [13] *for a more detailed definition*),

(c) *the lattice* $M(n)$ *of order ideals of* $L(2, n)$ (*again see* [13]),

(d) *The lattice* $L_n(q)$ *of subspaces of an n-dimensional vector space over the finite field* GF(q).

Proof. (a) This follows from the argument used to prove [2], Theorem 2, or [12], Theorem 3.2, but applied to unitary Peck posets only.

(b) It follows from [13], Section 4, that $L(m, n)$ is Peck, and from [13], bottom of p. 175, that $L(m, n)$ is, in fact, unitary Peck. An elementary proof appears in [11].

(c) It follows from [13], Section 5, that $M(n)$ is Peck, and the unitary property is a consequence of known results from algebraic geometry analogous to the $L(m, n)$ case. (See, e.g., [4], Corollary 3.2, p. 175.) Again, an elementary proof appears in [11].

(d) Let $P = L_n(q)$. If $0 \leqslant i < \frac{1}{2}n$, then let $\psi_i : V_i \to V_{n-i}$ be defined by

$$\psi_i(x) = \sum_{\substack{y \in P_{n-i} \\ y > x}} y, \quad x \in P_i .$$

Kantor [5] shows that ψ_i is invertible. Now if $\phi_i : V_i \to V_{i+1}$ is given by (1) for $0 \leqslant i < n$, then for $0 \leqslant i < \frac{1}{2}n$ it is easily seen that $\phi_{n-i-1} \cdots \phi_{i+1}\phi_i$ is a nonzero scalar multiple of ψ_i and therefore invertible. □

Note, in particular, that since chains are unitary Peck (as can be seen by inspection or because $L(1, n)$ is a chain of length n), it follows from (a) that the Boolean algebra B_n is unitary Peck. We also remark that $L(m, n)$ is easily seen to be of the form B_{mn}/G (e.g., [14], Section 9), so we obtain what seems to be the simplest proof to date that $L(m, n)$ is Peck. Our methods here do not yield the stronger result that $L(m, n)$ is unitary Peck.

Let us finally remark that the linear algebraic machinery we have set up provides a convenient means to prove a theorem of Lovász [6], [7], Section 15.17a, on the edge-reconstruction conjecture. Let Γ be a graph (with no loops or multiple edges) on the vertex set $\{1, 2, \ldots, n\}$. If Γ has q edges, then let $\tilde{\Gamma}_1, \tilde{\Gamma}_2, \ldots, \tilde{\Gamma}_q$ be the *unlabeled* graphs obtained by deleting a single edge from Γ.

THEOREM 3. *If $q > \frac{1}{2}\binom{n}{2}$, then Γ can be recovered up to isomorphism from $\tilde{\Gamma}_1, \tilde{\Gamma}_2, \ldots, \tilde{\Gamma}_q$.*

Proof. Let V_i be the vector space whose basis consists of the set P_i of all graphs with i edges on the vertex set $\{1, 2, \ldots, n\}$. Let $\psi_i : V_i \to V_{i-1}$ be the linear transformation defined by $\psi_i(\Gamma) = \Gamma_1 + \cdots + \Gamma_i$, where $\Gamma_1, \ldots, \Gamma_i$ are the (labeled) graphs obtained from Γ by deleting a single edge. Since Boolean algebras are unitary Peck, ψ_i is injective for $i > \frac{1}{2}\binom{n}{2}$. (Think of ψ_i as adding edges to the *complement* of Γ.)

The symmetric group S_n acts on P_q by permuting vertices, and hence acts on V_q. A basis for $V_q^{S_n}$ consists of the distinct sums $\tilde{\Gamma} = \Sigma_{w \in S_n} w\Gamma$, where $\Gamma \in P_q$. We may identify $\tilde{\Gamma}$ with the *unlabeled* graph isomorphic to Γ. By the arguments used to prove Theorem 1, when we restrict ψ_q to $V_q^{S_n}$ for $q > \frac{1}{2}\binom{n}{2}$, we obtain an injection $\psi_q : V_q^{S_n} \to V_{q-1}^{S_n}$. In particular, for nonisomorphic unlabeled graphs $\tilde{\Gamma}, \tilde{\Gamma}' \in P_q$ we have $\tilde{\Gamma}_1 + \cdots + \tilde{\Gamma}_q = \psi_q(\tilde{\Gamma}) \neq \psi_q(\tilde{\Gamma}') = \tilde{\Gamma}'_1 + \cdots + \tilde{\Gamma}'_q$. Hence the unlabeled graphs $\tilde{\Gamma}_1, \ldots, \tilde{\Gamma}_q$ determine $\tilde{\Gamma}$ as desired. □

We don't know whether the above argument can be extended in some way for $q \leqslant \frac{1}{2}\binom{n}{2}$. In particular, we are unable to obtain Müller's extension [8], [1], Section 2, of Lovász' result.

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