Modular Elements of Geometric Lattices

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1. Modular Elements

Let L be a finite geometric lattice of rank n with rank function r. (For definitions, see e.g., [3, Chapter 2], [4], or [1, Chapter 4].) An element $x \in L$ is called a *modular element* if it forms a modular pair with every $y \in L$, i.e., if $a \leqslant y$ then $a \vee (x \wedge y) = (a \vee x) \wedge y$. Recall that in an upper semimodular lattice (and thus in a geometric lattice) the relation of being a modular pair is symmetric; in fact (x, y) is a modular pair if and only if $r(x)+r(y)=r(x \vee y)+r(x \wedge y)$ [1, p. 83]. Every point (atom) of a geometric lattice is a modular element. If every element of L is modular, then L is a modular lattice. The main object of this paper is to show that a modular element of L induces a factorization of the characteristic polynomial of L. This is done in Section 2. First we discuss some other aspects of modular elements.

The following theorem provides a characterization of modular elements.

THEOREM 1. *An element $x \in L$ is modular if and only if no two complements of x are comparable.*

Proof. If x is modular and x' is a complement of x, then $r(x')=n-r(x)$. Hence all the complements of x have the same rank and are incomparable.

Conversely, assume x is not modular. Then there are elements $y<z$ such that $x \wedge y = x \wedge z$ and $x \vee y = x \vee z$. Let $p_1, p_2, \ldots, p_n$ be a basis for L such that $p_1, p_2, \ldots, p_s$ is a basis for $x \wedge y$; $p_1, p_2, \ldots, p_t$ is a basis for y; $p_1, p_2, \ldots, p_u$ is a basis for z; and $p_1, p_2, \ldots, p_v$ is a basis for $x \vee y$. Thus $0 \leqslant s<t<u<v \leqslant n$, $r(x \wedge y)=s$, $r(y)=t$, $r(z)=u$, $r(x \vee y)=v$. Let

$$x' = p_{s+1} \vee p_{s+2} \vee \cdots \vee p_t \vee p_{v+1} \vee p_{v+2} \vee \cdots \vee p_n$$
$$x'' = x' \vee p_{t+1} \vee p_{t+2} \vee \cdots \vee p_u .$$

It is easily seen that x' and x'' are both complements of x with $x'<x''$. □

In particular, an element x with a unique complement x' is modular. For a stronger result, recall that an element x is said to be in the *center* of an ordered set P with 0 and 1 if $P=[0, x] \times [0, x']$ for some x' [1, p. 67]. For an element x to be in the center of a geometric lattice L, each of the following conditions is necessary and sufficient:

(1) x is *distributive* (because x is complemented, cf. [1, p. 69]), i.e., for any y, $z \in L$, the sublattice generated by x, y, z is distributive.

(2) x has a *unique complement*. This result appears to be new; Curtis Greene has

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in fact proved the more general result (unpublished) that the intersection (meet) of all the complements of any element of a geometric lattice L is in the center of L.

(3) x is a *separator* of L [3, Chapter 12], i.e., for any point p and any copoint q not containing p, either $p \leqslant x$ or $x \leqslant q$.

(4) x is a *standard element* of L [1, p. 69] (this follows from (2) since a complemented standard element has a unique complement). The concept of standard elements is due to G. Grätzer.

2. The Characteristic Polynomial

The characteristic polynomial $p_L(\lambda)$ [4] of a geometric lattice L is defined by

$$p_L(\lambda) = \sum_{y \in L} \mu(0, y)\, \lambda^{n-r(y)},$$

where μ denotes the Möbius function of L (see [4]).

This polynomial was first considered by G. D. Birkhoff, while its connection with Möbius functions was noted by Garrett Birkhoff. If $x \in L$, the characteristic polynomial of the segment $[0, x]$ is denoted $p_x(\lambda)$.

The main result of this paper is the following factorization theorem:

THEOREM 2. *If x is a modular element of a finite geometric lattice L of rank n, then*

$$p_L(\lambda) = p_x(\lambda) \left[\sum_{b:\, x \wedge b = 0} \mu(0, b)\, \lambda^{n-r(x)-r(b)} \right]$$

The expression in brackets may be thought of as the characteristic polynomial of the order ideal $C(x) = \{b \mid x \wedge b = 0\}$. $C(x)$ will have a 1 if and only if x has a unique complement x' in L. In this case $C(x) = [0, x']$ and $L = [0, x] \times [0, x']$. Thus when x has a unique complement Theorem 2 is trivial, since $p_{L_1 \times L_2} = p_{L_1} p_{L_2}$.

To prove Theorem 2, we first prove two lemmas. The first lemma is a special case of some results of Schwan on modular pairs (see [1, Section IV.2]), but for the sake of completeness we include a proof. It is to be assumed throughout that x is a modular element of the finite geometric lattice L, and that L has rank n.

LEMMA 1. *For any $a \in L$, the map*

$$\sigma_a : [a, a \vee x] \to [a \wedge x, x]$$

defined by $\sigma_a(y) = x \wedge y$ is an isomorphism with inverse $\tau_a(y) = a \vee y$.

Proof. Clearly σ_a and τ_a are order-preserving. By modularity of x, it is immediate that if $y \in [a \wedge x, x]$, then $\sigma_a \tau_a(y) = x \wedge (a \vee y) = (x \wedge a) \vee y = y$. Also if $y \in [a, a \vee x]$, then $\tau_a \sigma_a(y) = a \vee (x \wedge y) = (a \vee x) \wedge y = y$, and the proof follows. $\square$

LEMMA 2. For any $y \in L$, $x \wedge y$ is a modular element of $[0, y]$.

Proof. Let $a \in [0, y]$ and let $b \leqslant a$. We need to show $b \vee ((x \wedge y) \wedge a) = (b \vee (x \wedge y)) \wedge a$. Using the modularity of x, we have

$$\begin{aligned}(b \vee (x \wedge y)) \wedge a &= ((b \vee x) \wedge y) \wedge a = (b \vee x) \wedge a = b \vee (x \wedge a)\\ &= b \vee (x \wedge (y \wedge a)) = b \vee ((x \wedge y) \wedge a). \quad \square\end{aligned}$$

Proof of Theorem 2. By Crapo's Complementation Theorem [2], if $a \in [0, y]$ then

$$\mu(0, y) = \sum_{a', a''} \mu(0, a') \zeta(a', a'') \mu(a'', y),$$

where a' and a'' are complements of a in $[0, y]$, and ζ is the zeta function of $[0, y]$. Choosing $a = x \wedge y$, then by Lemma 2 all the complements of a have the same rank and hence are incomparable. Thus

$$\mu(0, y) = \sum \mu(0, b) \mu(b, y), \tag{1}$$

where the sum is over all complements b of $x \wedge y$ in $[0, y]$, i.e., over all $b \in L$ satisfying $0 \leqslant b \leqslant y$, $b \wedge (x \wedge y) = 0$, $b \vee (x \wedge y) = y$. Now $b \wedge (x \wedge y) = b \wedge x$, and by the modularity of x, $b \vee (x \wedge y) = (b \vee x) \wedge y$. Thus the sum (1) is over all $b \in L$ satisfying $b \wedge x = 0$ and $y \in [b, b \vee x]$. Hence

$$\begin{aligned}p_L(\lambda) &= \sum_{y \in L} \mu(0, y) \lambda^{n-r(y)}\\ &= \sum_{y \in L} \sum_{\substack{b \wedge x = 0\\ y \in [b, b \vee x]}} \mu(0, b) \mu(b, y) \lambda^{n-r(y)}\\ &= \sum_{b \wedge x = 0} \sum_{y \in [b, b \vee x]} \mu(0, b) \mu(b, y) \lambda^{n-r(y)}\end{aligned}$$

Now by Lemma 1, as y ranges over $[b, b \vee x]$, $z = x \wedge y$ ranges over the isomorphic interval $[b \wedge x, x] = [0, x]$, and $\mu(b, y) = \mu(0, z)$. Moreover $r(y) = r(b) + r(z)$. Therefore

$$\begin{aligned}P_L(\lambda) &= \sum_{b \wedge x = 0} \sum_{z \in [0, x]} \mu(0, a) \mu(0, z) \lambda^{n-r(b)-r(z)}\\ &= \left[\sum_{z \in [0, x]} \mu(0, z) \lambda^{r(x)-r(z)}\right] \left[\sum_{b \wedge x = 0} \mu(0, b) \lambda^{n-r(b)-r(x)}\right]\\ &= p_x(\lambda) \left[\sum_{b \wedge x = 0} \mu(0, b) \lambda^{n-r(b)-r(x)}\right]. \quad \square\end{aligned}$$

3. Examples

As a special case of Theorem 2, suppose x is modular copoint, and that exactly α points (atoms) a of L do not lie below x. Then these points a, together with 0, are the only elements b of L satisfying $b \wedge x = 0$. Moreover, $\mu(0, a) = -1$, $\mu(0, 0) = 1$, so

$$p_L(\lambda) = p_x(\lambda) \left[\sum_a (-1) + \lambda \right] = p_x(\lambda)(\lambda - \alpha).$$

Thus if L contains a maximal chain $0 = x_0 < x_1 < \cdots < x_n = 1$ such that each x_{i-1} is modular in $[0, x_i]$ and such that exactly α_i atoms of $[0, x_i]$ do not lie below x_{i-1}, then $p_L(\lambda) = (\lambda - \alpha_1)(\lambda - \alpha_2)\ldots(\lambda - \alpha_n)$. It is easy to show that the condition that each x_{i-1} is modular in $[0, x_i]$ is equivalent to the condition that each x_i is modular in L. A class of geometric lattices with such a 'modular maximal chain' is the following: Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be any collection of positive integers with $\alpha_n = 1$. Let $p_1, \ldots, p_n$ be n independent points, and on the line $p_i p_n$ $(1 \leqslant i \leqslant n-1)$ insert an additional $\alpha_i - 1$ points. The geometric lattice L of flats of this geometry contains a modular maximal chain, and $p_L(\lambda) = (\lambda - \alpha_1)(\lambda - \alpha_2)\ldots(\lambda - \alpha_n)$.

At this point it is natural to ask for a characterization of the modular elements of various geometric lattices. We state such a characterization when L is the lattice of contractions of a finite graph. Recall that a contraction of a graph G may be regarded as a partition π of the vertices of G, such that the subgraph H induced by each block B of π is connected [3, Chapter 6].

THEOREM 3. *Let L be the lattice of contractions of a doubly connected finite graph G. Then $\pi \in L$ is a modular element of L if and only if the following conditions hold:*

(i) *At most one block B of π contains more than one vertex of G.*

(ii) *Let H be the subgraph induced by the block B of* (i). *Let K be any connected component of the subgraph induced by $G - B$, and let H_1 be the graph induced by the set of vertices in H which are connected to some vertex in K. Then H_1 is a clique (complete subgraph) of G.* □

The proof is of a routine nature and will be omitted.

If G is not doubly connected, then the lattice of contractions of G is a direct product of the lattices of contractions of the maximal doubly connected subgraphs of G, so Theorem 3 easily extends to arbitrary finite graphs G.

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