JOURNAL OF COMBINATORIAL THEORY, Series A 31, 56–65 (1981)

Two Combinatorial Applications of the Aleksandrov–Fenchel Inequalities*

RICHARD P. STANLEY

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

Communicated by the Managing Editors

Received August 1, 1980

The Aleksandrov–Fenchel inequalities from the theory of mixed volumes are used to prove that certain sequences of combinatorial interest are log concave (and therefore unimodal).

1. MIXED VOLUMES

We wish to show how the Aleksandrov–Fenchel inequalities from the theory of mixed volumes can be used to prove that certain sequences of combinatorial interest are log concave (and therefore unimodal). In particular, we prove the following two results (all terminology will be defined later):

(a) Let M be a unimodular (= regular) matroid of rank n on a finite set S, and let $T \subseteq S$. Let f_i be the number of bases B of M satisfying $|B \cap T| = i$, and set $g_i = f_i / \binom{n}{i}$. Then the sequence $g_0, g_1, ..., g_n$ is log concave.

(b) Let P be a finite poset (= partially ordered set) with n elements, and let $x \in P$. Let N_i be the number of order-preserving bijections $\sigma: P \to \{1, 2, ..., n\}$ satisfying $\sigma(x) = i$. Then the sequence $N_1, N_2, ..., N_n$ is log concave. This confirms a conjecture of Chung *et al.* [5], which is a strengthening of an unpublished conjecture of R. Rivest that $N_1, ..., N_n$ is unimodal.

We first review the salient facts from the theory of mixed volumes. Let $K_1, ..., K_s$ be convex bodies (= non-empty compact convex sets) in $\mathbb{R}^n$. If $\lambda_1, ..., \lambda_s \geqslant 0$ then define the convex body

$$K = \{\lambda_1 v_1 + \cdots + \lambda_s v_s : v_i \in K_i\}.$$

* Partially supported by NSF Grant MCS 78-02743 and by Bell Telephone Laboratories, Murray Hill, N. J.

0097-3165/81/040056-10$02.00/0

The n-dimensional measure or volume of K is denoted by $V(K)$. As a function of $\lambda_1,..., \lambda_s$, the volume of K is a homogeneous polynomial of degree n,

$$V(K) = \sum_{i_1=1}^{s} \sum_{i_2=1}^{s} \cdots \sum_{i_n=1}^{s} V_{i_1 \cdots i_n} \lambda_{i_1} \cdots \lambda_{i_n},$$

where the coefficients $V_{i_1 \cdots i_n}$ are uniquely determined by requiring that they are symmetric in their subscripts. Then $V_{i_1 \cdots i_n}$ depends only on the bodies $K_{i_1},..., K_{i_n}$ and not on the remaining bodies K_j, so we may write $V(K_{i_1},..., K_{i_n})$ for $V_{i_1 \cdots i_n}$ and call it the *mixed volume* of $K_{i_1},..., K_{i_n}$. Equivalently, we have

$$V(K) = \sum_{a_1 + \cdots + a_s = n} \frac{n!}{a_1! \cdots a_s!} V(\underbrace{K_1,..., K_1}_{a_1},..., \underbrace{K_s,..., K_s}_{a_s}) \lambda_1^{a_1} \cdots \lambda_s^{a_s}. \quad (1)$$

We also mention that $V(K_{i_1},..., K_{i_n}) \geqslant 0$. A good survey of these and other facts about mixed volumes appears in [4, Chap. II] or [8, Chap. 5]. A more comprehensive reference is [3].

The basic result we need about mixed volume was proved independently by Fenchel [9, 10] and Aleksandrov [1]. See also [4(7.7, 13, 15, 18, 19] for various ramifications and extensions. Given $0 \leqslant k \leqslant m \leqslant n$ and convex bodies $C_1,..., C_{n-m}, K, L \subset \mathbb{R}^n$, define $\mathbf{C} = (C_1,..., C_{n-m})$ and

$$V_k(\mathbf{C}, K, L) = V(C_1,..., C_{n-m}, \underbrace{K,..., K}_{m-k}, \underbrace{L,..., L}_{k}).$$

A sequence $a_0, a_1,..., a_m$ of non-negative real numbers is said to be *log concave* if $a_i^2 \geqslant a_{i-1} a_{i+1}$ for $1 \leqslant i \leqslant m-1$. In particular, a log concave sequence is *unimodal*, i.e., for some j we have $a_0 \leqslant a_1 \leqslant \cdots \leqslant a_j$ and $a_j \geqslant a_{j+1} \geqslant \cdots \geqslant a_m$.

1.1. THEOREM. (The Aleksandrov–Fenchel inequalities.) *The sequence $V_0(\mathbf{C}, K, L)$, $V_1(\mathbf{C}, K, L)$,..., $V_m(\mathbf{C}, K, L)$ is log concave.* ∎

2. AN APPLICATION TO MATROID THEORY

Let M be a matroid on a (finite) set $S = \{x_1,..., x_l\}$. By definition [6, 20], M is a pair $(S, \mathcal{I})$, where $\mathcal{I}$ is a non-empty collection of subsets of S satisfying (a) if $X \in \mathcal{I}$ and $Y \subseteq X$, then $Y \in \mathcal{I}$, and (b) if $X, Y \in \mathcal{I}$ with $|X| = |Y| + 1$ then there exists $x \in X - Y$ such that $Y \cup x \in \mathcal{I}$. Elements of $\mathcal{I}$ are called *independent sets*, and maximal independent sets are called *bases*. All bases have the same cardinality n, called the *rank* of M.

The matroid M is called *unimodular* (or *regular*) if there exists a mapping $\phi: S \to \mathbb{R}^n$ such that (a) a subset T of S is independent if and only if the $|T|$ column vectors $\phi(x)$, $x \in T$, are linear independent, and (b) the $n \times l$ matrix $[\phi(x_1) \cdots \phi(x_l)]$ is *totally unimodular*, i.e., every minor has determinant 0 or ± 1. The mapping ϕ is called a *unimodular coordinatization* of M. The most familiar class of unimodular matroids are the *graphic matroids*. Here S is the set of edges of a graph G, and a subset T of S is independent if it contains no cycle. Thus if G is connected, then a basis for the corresponding graphic matroid is just a spanning tree. For further properties of unimodular matroids and for any undefined matroid theory terminology used below, see [6] or [10].

If M is a matroid of rank n on S, then let $T_1,\dots, T_n$ be any n subsets (not necessarily distinct) of S. Define $B(T_1,\dots, T_n)$ to be the number of sequences $(y_1,\dots, y_n) \in S^n$ such that (a) $y_i \in T_i$ for $1 \leqslant i \leqslant n$, and (b) $\{y_1,\dots, y_n\}$ is a basis of M. Let $0 \leqslant k \leqslant m \leqslant n$ and $T_1,\dots, T_{n-m}$, Q, $R \subseteq S$. Set $\mathbf{T} = (T_1,\dots, T_{n-m})$ and define

$$B_k(\mathbf{T}, Q, R) = B(T_1,\dots, T_{n-m}, \underbrace{Q,\dots, Q}_{m-k}, \underbrace{R,\dots, R}_{k}).$$

We now come to the main result of this section.

2.1. THEOREM. *Let M be a unimodular matroid rank n on the set S. Fix $\mathbf{T}$, Q, R as above. Then the sequence $B_0(\mathbf{T}, Q, R)$, $B_1(\mathbf{T}, Q, R),\dots,$ $B_m(\mathbf{T}, Q, R)$ is log concave.*

Theorem 2.1 will be proved by finding convex polytopes $K_1,\dots, K_n$ for which $n!\,V(K_1,\dots, K_n) = B(T_1,\dots, T_n)$, so that Theorem 1.1 applies. Let $v_1,\dots, v_l$ be any (column) vectors (not necessarily distinct) in $\mathbb{R}^n$, and define a convex polytope

$$Z(v_1,\dots, v_l) = \{\alpha_1 v_1 + \cdots + \alpha_l v_l : 0 \leqslant \alpha_i \leqslant 1\}.$$

Thus $Z(v_1,\dots, v_l)$ is a vector sum of line segments and hence by definition a *zonotope*. The following result is attributed by Shephard [16, p. 321] to McMullen.

2.2. THEOREM. *The volume of the zonotope $Z = Z(v_1,\dots, v_l)$ is given by*

$$V(Z) = \sum_{1 \leqslant i_1 < \cdots < i_n \leqslant l} |\det[v_{i_1},\dots, v_{i_n}]|. \quad \blacksquare$$

2.3. COROLLARY. *Let $\phi: S \to \mathbb{R}^n$ be a unimodular coordinatization of the unimodular matroid M of rank n on the set S. If $T = \{y_1,\dots, y_t\} \subseteq S$,* *then set* $Z_T = Z(\phi(y_1),..., \phi(y_t))$. *Then for any subsets* $T_1,..., T_n$ *of* S *we have*

$$n!V(Z_{T_1},...,Z_{T_n}) = B(T_1,..., T_n).$$

Proof. Let $\lambda_1,..., \lambda_n \geqslant 0$. Then

$$\lambda_1 Z_{T_1} + \cdots + \lambda_n Z_{T_n} = Z(\lambda_1 \phi(x_{11}),..., \lambda_1 \phi(x_{1t_1}), \lambda_2 \phi(x_{21}),..., \lambda_2 \phi(x_{2t_2}),..., \lambda_n \phi(x_{n1}),..., \lambda_n \phi(x_{nt_n})),$$

where $T_i = \{x_{i1}, x_{i2},..., x_{it_i}\}$. Hence by Theorem 2.2, and the unimodularity of M,

$$V(\lambda_1 Z_{T_1} + \cdots + \lambda_n Z_{T_n}) = \sum_{a_1 + \cdots + a_n = n} C(a_1,..., a_n) \lambda_1^{a_1} \cdots \lambda_n^{a_n}, \tag{2}$$

where $C(a_1,..., a_n)$ is the number of ways of choosing subsets $Q_i \subseteq T_i$ such that $|Q_i| = a_i$ and $Q_1 \cup \cdots \cup Q_n$ is a basis of M. Thus $B(T_1,..., T_n) = C(1,..., 1)$. Comparing (1) and (2), we see $C(1,..., 1) = n!V(Z_{T_1},..., Z_{T_n})$, completing the proof. ∎

Proof of Theorem 2.1. According to Corollary 2.3, $n!V_k(\mathbf{Z}, Z_Q, Z_R) = B_k(\mathbf{T}, Q, R)$, where $\mathbf{Z} = (Z_{T_1},..., Z_{T_{n-m}})$. The proof follows from Theorem 1.1 (the factor $n!$ being irrelevant). ∎

A special case of Theorem 2.1 deserves a separate statement.

2.4. COROLLARY. *Let* M *be a unimodular matroid of rank* n *on the set* S, *and let* $T_1,..., T_r$, Q, R *be pairwise disjoint subsets of* S *whose union is* S. *Fix non-negative integers* $a_1,..., a_r$ *such that* $m = n - a_1 - \cdots - a_r \geqslant 0$, and for $0 \leqslant k \leqslant m$ define f_k to be the number of bases B of M such that $|B \cap T_i| = a_i$ for $1 \leqslant i \leqslant r$, and $|B \cap R| = k$ (so $|B \cap Q| = m - k$). Set $g_k = f_k / \binom{m}{k}$. *Then the sequence* $g_0, g_1,..., g_m$ (*and hence a fortiori* $f_0,..., f_m$) *is log concave.*

Proof. Set

$$\mathbf{T} = (\underbrace{T_1,..., T_1}_{a_1}, \underbrace{T_2,..., T_2}_{a_2},..., \underbrace{T_r,..., T_r}_{a_r}).$$

Clearly $B_k(\mathbf{T}, Q, R) = a_1! \cdots a_r!\, k!(m-k)! f_k$, so $g_k = B_k(\mathbf{T}, Q, R)/a_1! \cdots a_r!\, m!$. Since a constant non-negative multiple of a log concave sequence is log concave, the proof follows from Theorem 2.1. ∎

Further inequalities involving the numbers $B(T_1,..., T_n)$ (for unimodular matroids) can be obtained by using known mixed volume inequalities other than Theorem 1.1, such as those in [4, 13, 15]. We will not state these inequalities here.

It is natural to ask when equality can hold in Theorem 2.1, i.e., when $B_k(\mathbf{T}, Q, R)^2 = B_{k-1}(\mathbf{T}, Q, R)\, B_{k+1}(\mathbf{T}, Q, R)$. A partial answer to this question is given by a result of Minkowski–Süss–Bonnesen (e.g., [3, p.91; 4, p. 48], but stated carelessly in these two references in the degenerate case (ii) below), which may be stated as follows:

2.5. THEOREM. *Let K and L be convex bodies in* $\mathbb{R}^n$ *(the case* $n = m$ *of Theorem* 1.1). *The following two conditions are equivalent*:

(a) $V_1(K, L)^n = V_0(K, L)^{n-1} V_n(K, L)$.

(b) *Either of the following two conditions hold*:

(i) *K and L are homothetic (i.e.,* $K = v + \alpha L$ *for some* $v \in \mathbb{R}^n$ *and* $\alpha > 0$) *and do not lie in parallel hyperplanes. This is equivalent to* $0 \neq V_0(K, L) = V_1(K, L) = V_2(K, L) = \cdots = V_n(K, L)$.

(ii) $V_1(K, L) = 0$ *(so also* $V_0(K, L) = 0$ *or* $V_n(K, L) = 0$). *This is equivalent to the fact that one of the following three conditions hold*: (α) *K and L lie in parallel hyperplanes, or equivalently* $0 = V_0(K, L) = V_1(K, L) = \cdots = V_n(K, L)$, *or* ($\beta$) $\dim K \leqslant n - 2$ (so $V_0(K, L) = 0$), *or* (γ) *L is a point (so* $V_n(K, L) = 0$).

To apply this result to Theorem 2.1, we need the following two lemmas.

2.6. LEMMA. *Let K and L be convex bodies in* $\mathbb{R}^n$, *and let l be the line segment from the origin to a point v. If* $l + K = l + L$ *then* $K = L$.

Proof. Let $x \in \mathbb{R}^n$. We claim $x \in K$ if and only if $x \in l + K$ and $x + v \in l + K$, from which the proof will follow. Clearly, if $x \in K$ then $x \in l + K$ and $x + v \in l + K$. Hence assume $x \in l + K$ and $x + v \in l + K$. Since $x \in l + K$, we have $x = y + sv$ for some $y \in K$ and $0 \leqslant s \leqslant 1$. Thus $x - sv \in K$. Since $x + v \in l + K$, we have $x + v = z + tv$ for some $z \in K$ and $0 \leqslant t \leqslant 1$. Hence $x + (1 - t)v \in K$. But x is on the line segment joining $x - sv$ and $x + (1 - t)v$, so $x \in K$ since K is convex. ∎

2.7. LEMMA. *Let* $v_1, ..., v_r$, $w_1, ..., w_s$ *be vectors in* $\mathbb{R}^n$ *with non-negative coordinates. Let* $v'_1, ..., v'_t$ *be the vectors obtained from* $v_1, ..., v_r$ *by discarding any* $v_i = 0$ *and by adding together all remaining* v_j*'s which are scalar multiples of each other. Similarly define* $w'_1, ..., w'_u$. *The zonotopes* $Z(v_1, ..., v_r)$ *and* $Z(w_1, ..., w_s)$ *are homothetic if and only if* $t = u$ *and after suitable indexing* $v'_i = \gamma w'_i$ *for all* $1 \leqslant i \leqslant t$ *and some fixed* $\gamma > 0$.

Proof. Let $Z_1 = Z(v_1, ..., v_r)$ and $Z_2 = Z(w_1, ..., w_s)$. The "if" part is clear so assume Z_1 and Z_2 are homothetic. Since $Z(\alpha u_1, \beta u_1, u_2, ..., u_q) = Z((\alpha + \beta) u_1, u_2, ..., u_q)$ if $\alpha\beta \geqslant 0$, it suffices to assume that $\{v_1, ..., v_r\} = \{v'_1, ..., v'_t\}$ and $\{w_1, ..., w_s\} = \{w'_1, ..., w'_u\}$. Since the coordinates of each v_i and

w_j are non-negative, the origin is a vertex of Z_1 and Z_2, and Z_1 and Z_2 are homothetic if and only if $Z_1 = \gamma Z_2$ for some $\gamma > 0$. We may then assume (by multiplying Z_2 by a suitable scalar) that $Z_1 = Z_2$.

Let v be a vertex of Z_1 which is connected to the origin by an edge of Z_1. Then $v = v_i$ for some i since no v_i is a scalar multiple of another. Since $Z_1 = Z_2$ we also have $v = w_j$ for some j. Let l be the line segment joining the origin to v. Then $Z_1 = l + Z(v_1,..., v_{i-1}, v_{i+1},..., v_r) = l + Z(w_1,..., w_{j-1}, w_{j+1},..., w_s)$. The proof follows from Lemma 2.6 and induction on r. ∎

We may now directly apply Theorem 2.5 and Lemma 2.7 to the situation of Corollary 2.3 in the case $(T_1,..., T_n) = (Q,..., Q, R,..., R)$.

2.8. THEOREM. *Let M be unimodular matroid of rank n on the finite set S, and let $R \subseteq S$. Without loss of generality assume that M has no loops. Let f_i be the number of bases B of M satisfying $|B \cap R| = i$, and set $g_i = f_i/\binom{n}{i}$. (Thus by Corollary 2.4 we have $g_i^2 \geqslant g_{i-1}g_{i+1}$ so in particular $g_1^n \geqslant g_0^{n-1}g_{n-1}$). The following two conditions are equivalent*:

(a) $g_1^n = g_0^{n-1}g_n$.

(b) *One of the following two conditions hold*:

(i) $f_1 = 0$ (*so either* $f_0 = 0$ *or* $f_n = 0$).

(ii) *For some integer $k > 1$, the closure $\bar{x}$ of every point x of S has ka_x elements for some positive integer a_x. Moreover, for some j satisfying $0 < j < k$ and for all $x \in S$, $|R \cap \bar{x}| = ja_x$.*

Proof. Let $Q = S - R$. By Corollary 2.3 we have $g_i = B_i(Q, R) = n!V_i(Z_Q, Z_R)$. It now follows from Theorem 2.5 that $g_1^n = g_0^{n-1}g_n$ if and only if either Z_Q and Z_R are homothetic (note that Z_Q and Z_R cannot lie in parallel hyperplanes since $Z_Q + Z_R = Z_S$), or else $f_1 = 0$. If $\phi: S \to \mathbb{R}^n$ is a unimodular coordinatization of M, then $y \in \bar{x}$ if and only if $\phi(x) = \phi(y)$. Moreover, x is a loop if and only if $\phi(x) = (0, 0,..., 0)$. Hence, by Lemma 2.7, Z_Q and Z_R are homothetic if and only if for some $\beta > 0$ and for every $x \in S$, we have $|\bar{x} \cap Q| = \beta |\bar{x} \cap R|$. From this the proof is immediate. ∎

If I_i denotes the number of i-element independent sets of the finite matroid M of rank n, then Mason [12; 14, p. 491; 20, p. 298] has conjectured that the sequence $I_0, I_1,..., I_n$ is log concave. For some recent progress on this conjecture, see [7]. We remark that this conjecture would follow if Theorem 2.1 were valid for all finite matroids. More precisely, let us say that M has *Property P* if the conclusion of Corollary 2.4 holds in the case $i = 0$. In other words, for any fixed $R \subseteq S$ define (as in Theorem 2.8) f_i, $0 \leqslant i \leqslant n$, to be the number of bases B of M satisfying $|B \cap R| = i$, and set $g_i = f_i/\binom{n}{i}$. Then M has Property P if the sequence $g_0, g_1,..., g_n$ is log concave for all choices $R \subseteq S$. Thus unimodular matroids have Property P by Corollary 2.4.

2.9. THEOREM. *Let M be a matroid of rank n on the set S, with I_i i-element independent sets. Let F_n be a free matroid of rank n and define N to be the rank n truncation of the direct sum $M + F_n$. If N has Property* P, *then* $I_0, I_1, \ldots, I_n$ *is log concave.*

Proof. A basis B for N is obtained by taking the union of an independent set of M, say with i elements, with any $n - i$ points of F_n. Hence the number f_i of bases B of N satisfying $|B \cap S| = i$ is $I_i\binom{n}{i}$. The proof follows from the definition of Property P. ∎

In conclusion we mention a strengthening of a special case of Corollary 2.4. Let G be a graph, and let H be the graph obtained from G by adjoining a new vertex x and connecting it to each vertex of G. Let R be the set of edges of H incident to x. The number f_k of spanning trees of H which intersect R in k elements is just the number of rooted forests of G (i.e., spanning forests in which every component is a rooted tree) with $p - k$ edges, where p is the number of vertices of G. It follows from [2, Theorem 7.5] that the polynomial $f_p x^p + f_{p-1} x^{p-1} + \cdots + f_0$ is the characteristic polynomial of a symmetric matrix and hence has real roots. This is stronger than the statement that the numbers $g_i = f_i/\binom{n}{i}$ are log concave, where n is the rank of the cycle matroid of G. We have been unable to decide whether the numbers $f_0, f_1, \ldots, f_m$ of Corollary 2.4 have in general the property that the polynomial $f_m x^m + f_{m-1} x^{m-1} + \cdots + f_0$ has real roots.

3. An Application to Posets

Let P be a finite poset (= partially ordered set) with n elements, and let x be a fixed element of P. Let N_i be the number of order-preserving bijections $\sigma: P \to \{1, 2, \ldots, n\}$ satisfying $\sigma(x) = i$. R. Rivest conjectured (unpublished) that the sequence $N_1, N_2, \ldots, N_n$ is unimodal. Chung *et al.* [5] conjectured the stronger result that $N_1, N_2, \ldots, N_n$ is log concave and proved this in the case that P is a union of two chains. For some related results, see [11, 17]. Here we will prove the conjecture of Chung, *et al.* In fact, we have the following more general result.

3.1. THEOREM. *Let $x_1 < \cdots < x_k$ be a fixed chain in the n-element poset P. If $1 \leqslant i_1 < \cdots < i_k \leqslant n$, then define $N(i_1, \ldots, i_k)$ to be the number of order-preserving bijections $\sigma: P \to \{1, 2, \ldots, n\}$ such that $\sigma(x_j) = i_j$ for $1 \leqslant j \leqslant k$. Suppose $1 \leqslant j \leqslant k$ and $i_{j-1} + 1 < i_j < i_{j+1} - 1$, where we set $i_0 = 0$ and $i_{k+1} = n + 1$. Then*

$$N(i_1, \ldots, i_k)^2 \geqslant N(i_1, \ldots, i_{j-1}, i_j - 1, i_{j+1}, \ldots, i_k)\, N(i_1, \ldots, i_{j-1}, i_j + 1, i_{j+1}, \ldots, i_k).$$

In particular, the case $k=1$ *yields* $N_i^2 \geqslant N_{i-1}N_{i+1}$, *which is the conjecture of Chung et al.*

The proof of Theorem 3.1 is an immediate consequence of Theorem 1.1 and the next result.

3.2. THEOREM. *Preserve the notation of Theorem* 3.1. *Suppose* $P=\{x_1,\ldots,x_k, y_1,\ldots,y_{n-k}\}$. *If* $0\leqslant i\leqslant k$, *let* K_i *be the convex polytope of all points* $(t_1,\ldots,t_{n-k})\in\mathbb{R}^{n-k}$ *such that* (a) $0\leqslant t_j\leqslant 1$, (b) $t_j\leqslant t_l$ *if* $y_j\leqslant y_l$ in P, (c) $t_j=0$ *if* $y_j<x_i$ in P (*this condition being vacuous when* $i=0$), *and* (d) $t_j=1$ *if* $y_j>x_{i+1}$ *in* P (*this condition being vacuous when* $i=k$). *Then*

$$(n-k)!\,V(\underbrace{K_0,\ldots,K_0}_{i_i-1},\underbrace{K_1,\ldots,K_1}_{i_2-i_1-1},\underbrace{K_2,\ldots,K_2}_{i_3-i_2-1},\ldots,\underbrace{K_k,\ldots,K_k}_{n-i_k})=N(i_1,i_2,\ldots,i_k).$$

Proof. Let $\lambda_0,\ldots,\lambda_k\geqslant 0$ and set $K=\lambda_0K_0+\cdots+\lambda_kK_k$. For each order-preserving bijection $\sigma: P\to\{1,2,\ldots,n\}$, define Δ_σ to be the set of all $(t_1,\ldots,t_{n-k})\in K$ such that (a) $t_i\leqslant t_j$ if $\sigma(y_i)\leqslant\sigma(y_j)$ and (b) $\lambda_0+\lambda_1+\cdots+\lambda_{j-1}\leqslant t_i\leqslant\lambda_0+\lambda_1+\cdots+\lambda_j$ if $\sigma(x_j)<\sigma(y_i)<\sigma(x_{j+1})$, where $0\leqslant j\leqslant k$ and where we set $\sigma(x_0)=0$, $\sigma(x_{k+1})=n+1$. Suppose $\sigma(x_j)=i_j$ and let π be the permutation of $\{1,2,\ldots,n-k\}$ defined by $\sigma(y_{\pi(1)})<\sigma(y_{\pi(2)})<\cdots<\sigma(y_{\pi(n-k)})$. Then Δ_σ consists of all points $(t_1,\ldots,t_{n-k})\in\mathbb{R}^{n-k}$ such that

$$\begin{aligned}&0\leqslant t_{\pi(1)}\leqslant t_{\pi(2)}\leqslant\cdots\leqslant t_{\pi(i_1-1)}\leqslant\lambda_0\leqslant t_{\pi(i_1)}\leqslant\cdots\leqslant t_{\pi(i_2-2)}\\&\leqslant\lambda_0+\lambda_1\leqslant t_{\pi(i_2-1)}\leqslant\cdots\leqslant t_{\pi(i_3-3)}\leqslant\lambda_0+\lambda_1+\lambda_2\leqslant\cdots\\&\leqslant t_{\pi(n-k)}\leqslant\lambda_0+\cdots+\lambda_k.\end{aligned}$$

Thus Δ_σ is a simplex of dimension $n-k$ and volume

$$V(\Delta_\sigma)=\frac{\lambda_0^{i_1-1}}{(i_1-1)!}\frac{\lambda_1^{i_2-i_1-1}}{(i_2-i_1-1)!}\cdots\frac{\lambda_k^{n-i_k}}{(n-i_k)!}$$

Moreover, the simplices Δ_σ, as σ ranges over the set $\mathscr{L}(P)$ of all order-preserving bijections $\sigma: P\to\{1,2,\ldots,n\}$, have pairwise disjoint interiors and have union K. (In fact, they form the maximal faces of a triangulation of K.) Hence

$$\begin{aligned}V(K)&=\sum_{\sigma\in\mathscr{L}(P)}V(\Delta_\sigma)\\&=\sum_{1\leqslant i_1<\cdots<i_k\leqslant n}N(i_1,i_2,\ldots,i_k)\frac{\lambda_0^{i_1-1}\cdots\lambda_k^{n-i_k}}{(i_1-1)!\cdots(n-i_k)!}.\end{aligned}$$

Comparing with (1) proves the theorem, and thus also Theorem 3.1. ∎

One special case of Theorem 3.1 is of independent combinatorial interest. Given $n \geqslant 1$, define the descent set $D(\pi)$ of a permutation $\pi = a_1 a_2 \cdots a_n$ of $\{1, 2,..., n\}$ by $D(\pi) = \{i: a_i > a_{i+1}\}$.

3.3. COROLLARY. *Let S be a subset of $\{1, 2,..., n-1\}$ and let $1 \leqslant j \leqslant n$. Define $\omega_i = \omega_i(S, j)$ to be the number of permutations $\pi = a_1 a_2 \cdots a_n$ of $\{1, 2,..., n\}$ such that $D(\pi) = S$ and $a_j = i$. Then the sequence $\omega_1, \omega_2,..., \omega_n$ is log concave.*

Proof. Suppose the elements of S are $1 \leqslant d_1 < d_2 < \cdots < d_k \leqslant n-1$, and define a poset P with elements $x_1,..., x_n$ by

$$x_1 < \cdots < x_{d_1} > x_{d_1+1} < x_{d_1+2} < \cdots < x_{d_2} > x_{d_2+1} < x_{d_2+2} < \cdots < x_{d_3} > \cdots < x_n.$$

An order-preserving bijection $\sigma: P \to \{1, 2,..., n\}$ such that $\sigma(x_j) = i$ corresponds to a permutation $\sigma(x_1), \sigma(x_2),..., \sigma(x_n)$ enumerated by ω_i. The proof follows from Theorem 3.1. ∎

As in the last section, we can ask about the conditions for equality in Theorem 3.1. Although the result analogous to Theorem 2.8 turns out to be trivial (and requiring no facts about convexity), we state it for the sake of completeness.

3.4. THEOREM. *Let P be a finite n-element poset ($n \geqslant 3$), and let $x \in P$. Let N_i be the number of order-preserving bijections $\sigma: P \to \{1, 2,..., n\}$ satisfying $\sigma(x) = i$. The following four conditions are equivalent*:

(i) $N_1 \neq 0$ *and* $N_n \neq 0$,

(ii) $N_1 = N_2 = \cdots = N_n$,

(iii) $N_2^{n-1} = N_1^{n-2} N_n$ *and* $N_2 \neq 0$,

(iv) *x is comparable to no other elements of P.*

Proof. Clearly $N_1 \neq 0$ if and only if x is a minimal element of P, and $N_n \neq 0$ if and only if x is maximal. Hence (i) and (iv) are equivalent. But the implications (iii) ⇒ (i), (iv) ⇒ (ii), (ii) ⇒ (iii) are trivial, and the proof follows. ∎

REFERENCES

1. A. D. ALEKSANDROV, "Zur Theorie der Gemischten Volumina von konvexen Körpern," Russian, German summaries, Parts I, II, III, IV. I. Verallgemeinerung einiger Begriffe der Theorie der Konvexen Körper, *Mat. Sbornik N.S.* 2 (1937), 947–972. II. Neue Ungleichungen zwischen den gemischten Volumina und ihre Anwendungen, *Mat. Sbornik N.S.* 2 (1937), 1205–1238. III. Die Erweiterung zweier Lehrsätze Minkowskis über die Konvexen Polyeder auf beliebige Konvexe Flächen, *Mat. Sbornik N.S.* 3 (1938), 27–48. IV. Die gemischten Diskriminanten und die gemischten Volumina, *Mat. Sbornik N.S.* 3 (1938), 227–251.

2. N BIGGS, "Algebraic Graph Theory," Cambridge Univ. Press, Cambridge, 1974.
3. J. BONNESEN AND W. FENCHEL, "Theorie der konvexen Körper," Springer, Berlin, 1934, or New York, 1948.
4. H. BUSEMANN, "Convex Surfaces," Interscience, New York, 1958.
5. F. R. K. CHUNG, P. C. FISHBURN, AND R. L. GRAHAM, On unimodality for linear extensions of partial orders, *SIAM J. Algebraic and Discrete Methods* 1 (1980), 405–410.
6. H. H. CRAPO AND G.-C. ROTA, "On the Foundations of Combinatorial Theory: Combinatorial Geometries," MIT Press, Cambridge, Mass., 1976.
7. T. A. DOWLING, On the independent set numbers of a finite matroid, preprint.
8. H. G. EGGLESTON, "Convexity," Cambridge Univ. Press, Cambridge, 1958.
9. W. FENCHEL, Inégalités quadratiques entre les volumes mixtes des corps convexes, *C. R. Acad. Sci. Paris* **203** (1936), 647–650.
10. W. FENCHEL, Généralizations du théorème de Brunn et Minkowski concernant les corps convexes, *C. R. Acad. Sci. Paris* **203** (1936), 764–766.
11. R. L. GRAHAM, A. C. YAO, AND F. F. YAO, Some monotonicity properties of partial orders, *SIAM J. Algebraic and Discrete Methods* **1** (1980), 251–258.
12. J. H. MASON, Matroids:Unimodal conjectures and Motzkin's theorem, *in* "Combinatorics" (D. J. A. Welsh and D. R. Woodall, Eds.), pp. 207–221, Oxford Univ. Press, Oxford, 1972.
13. V. P. PHEDOTOV, A new method of proving inequalities between mixed volumes, and a generalization of the Aleksandrov–Fenchel–Shephard inequalities, *Soviet Math. Dokl.* **20** (1979), 268–271.
14. P. D. SEYMOUR AND D. J. A. WELSH, Combinatorial applications of an inequality from statistical mechanics, *Math. Proc. Cambridge Philos. Soc.* **77** (1975), 485–495.
15. G. C. SHEPHARD, Inequalities between mixed volumes of convex sets, *Mathematika* 7 (1960), 125–138.
16. G. C. SHEPHARD, Combinatorial properties of associated zonotopes, *Canad. J. Math.* **26** (1974), 302–321.
17. L. A. SHEPP, The FKG inequality and some monotonicity properties of partial orders, *SIAM J. Algebraic and Discrete Methods* **1** (1980), 295–299.
18. B. TEISSIER, Du théorème de l'index de Hodge aux inégalités isopérimétriques, *C. R. Acad. Sci. Paris* **288** (1979), 287–289.
19. B. TEISSIER, Bonnesen-type inequalities in algebraic geometry, I: Introduction to the problem, preprint.
20. D. J. A. WELSH, "Matroid Theory," Academic Press, London/New York, 1976.