AN APPLICATION OF ALGEBRAIC GEOMETRY TO EXTREMAL SET THEORY

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§1. Introduction In 1928, E. Sperner showed that if $\mathcal{S}$ is a collection of subsets of an n-element set such that no two distinct elements S, T of $\mathcal{S}$ satisfy $S \subset T$, then $\operatorname{card} \mathcal{S} \leq \binom{n}{[n/2]}$, and that this inequality is best possible. Sperner's result gave rise to the subject of "extremal set theory." We will indicate how the hard Lefschetz theorem implies an analogue of Sperner's theorem whenever we have a nonsingular irreducible complex projective variety X with a cellular decomposition into affine spaces. The most interesting special case arises when $X = G/P$, where G is a complex semisimple algebraic group and P a parabolic subgroup. In particular, choosing P to be a certain maximal parabolic subgroup of $G = SO(2n + 1)$, we can deduce the following conjecture of Erdös and Moser (1963): Let S be a set of $2\ell + 1$ distinct real numbers, and let $T_1, \ldots, T_k$ be subsets of S whose element sums are all equal. Then k does not exceed the

Supported in part by NSF Grant MCS 7701947.

middle coefficient of the polynomial $2(1+q)^2(1+q^2)^2\ldots(1+q^{\ell})^2$, and this inequality is best possible.

Let P be a finite partially ordered set (or poset, for short), and assume that every maximal chain of P has length n. We say that P is graded of rank n. Thus, P has a unique rank function $\rho : P \to \{0,1,\ldots,n\}$ satisfying $\rho(x) = 0$ if x is a minimal element of P and $\rho(y) = \rho(x) + 1$ if y covers x in P (i.e., if $y > x$ and no $z \in P$ satisfies $y > z > x$). If $\rho(x) = i$ then we say that x has rank i. Define $P_i = \{x \in P : \rho(x) = i\}$ and set $p_i = p_i(P) = \text{card } P_i$. The polynomial $F(P,q) = p_0 + p_1 q + \cdots + p_n q^n$ is called the rank-generating function of P. We say that P is rank-symmetric if $p_i = p_{n-i}$ for all i, and that P is rank-unimodal if $p_0 \le p_1 \le \cdots \le p_i \ge p_{i+1} \ge \cdots \ge p_n$ for some i.

An antichain (also called a Sperner family or clutter) is a subset A of P such that no two distinct elements of A are comparable. The poset P is said to have the Sperner property (or Property S_1) if the largest size of an antichain is equal to $\max\{p_i : 0 \le i \le n\}$. More generally, if k is a positive integer then P is said to have the k-Sperner property (or Property S_k) if the largest subset of P containing no $(k+1)$-element chain has cardinality $\max\{p_{i_1} + \cdots + p_{i_k} : 0 \le i_1 < \cdots < i_k \le n\}$. If P has Property S_k for all $k \le n+1$ then we say that P has Property S. See, for instance, [7], [8] for additional information. Using some results from algebraic geometry, we will give several new classes of graded posets which have Property S. These posets will all be rank-symmetric and rank-unimodal.

We will simply state here our main results without proofs. Complete details can be found in the paper "Weyl groups, the hard Lefschetz theorem, and the Sperner property."

§2. Some Algebraic Geometry Let X be a complex projective variety of complex dimension n . Suppose that there are finitely many pairwise-disjoint subsets C_i of X , each isomorphic as an algebraic variety to complex affine space of some dimension n_i , such that (a) the union of the C_i's is X , and (b) $\bar{C}_i - C_i$ is a union of some of the C_j's . (Here, $\bar{C}_i$ denotes the closure of C_i either in the Hausdorff or Zariski topology - under the present circumstances the two closures coincide.) We then say that the C_i's form a cellular decomposition of X [1, p.500]. The simplest and most familiar example is complex projective space $\mathbb{P}^n$ itself. This has the cellular decomposition $\mathbb{C}^n \cup \mathbb{C}^{n-1} \cup \cdots \cup \mathbb{C}^0$, where $\mathbb{C}^n$ is $\mathbb{P}^n$ minus some hyperplane $\mathbb{P}^{n-1}$, then where $\mathbb{C}^{n-1}$ is $\mathbb{P}^{n-1}$ minus some hyperplane $\mathbb{P}^{n-2}$, etc.

Now given a cellular decomposition $X = \cup C_i$ define a partial ordering $Q^X = Q^X(C_1, C_2, \ldots)$ on the C_i's by setting $C_i \geq C_j$ in Q^X if $C_i \subset \bar{C}_j$. If X is irreducible of dimension n , then Q^X is graded of rank n . If, moreover, X is nonsingular, then Poincaré duality implies that Q^X is rank-symmetric. Our main result about varieties with cellular decompositions is the following.

Theorem Let X be a nonsingular irreducible complex projective variety of complex dimension n with a cellular decomposition $X = \cup C_i$. Then, Q^X is graded of rank n , rank-symmetric, rank-unimodal, and has Property S.

This theorem is proved by first using [1, p.501] to identify Q^X with a certain basis for the cohomology ring $H^*(X,\mathbb{C})$, and then using results about the cohomology of complex projective varieties (in particular, the hard Lefschetz theorem [10]) to construct chains in Q^X . A result of Griggs [8, Thm. 3] then yields Property S.

§3. <u>Applications</u> The best known examples of varieties with cellular decompositions are the "generalized partial flag manifolds" $X = G/P$, where G is a complex semisimple algebraic group and P is a parabolic subgroup. The poset Q^X has an explicit description in terms of the Weyl groups of G and P [2, §3], [5]. More concretely, let W be the Weyl group of G . Then W is generated by a set $S = \{s_1,\ldots,s_m\}$ of reflections such that (W,S) is a Coxeter system (as defined, e.g., in [3]). To every parabolic subgroup P of G containing a given Borel subgroup, there corresponds a subset J of S . The Weyl group of P is then the subgroup W_J of W generated by J . Then Q^X may be regarded as a partial order on the set W^J of cosets of W_J in W . Hence, Q^X has cardinality $|W|/|W_J|$.

The precise definition of Q^X will not be given here, but we will mention two interesting special cases. If $G = SL(n,\mathbb{C})$, then $W = S_n$, the group of permutations of $\{1,2,\ldots,n\}$, and S consists of the "adjacent transpositions" $(12),(23),\ldots,(n-1,n)$. Suppose $J = S - \{(k,k+1)\}$. Then, $W_J \cong S_k \times S_{n-k}$, and W^J has order $n!/k!(n-k)! = \binom{n}{k}$. The poset Q^X turns out to be isomorphic to the set of all integer sequences $0 \le a_1 \le \cdots \le a_k \le n-k$, ordered component-wise (i.e., $(a_1,\ldots,a_k) \le (b_1,\ldots,b_k)$ if $a_i \le b_i$ for all i). It follows that Q^X is a distributive lattice, denoted $L(k,n-k)$, with rank-generating function

$$F(L(k,n-k),q) = \left[{n \atop k}\right]_q := \frac{(1-q^n)(1-q^{n-1})\ldots(1-q^{n-k+1})}{(1-q^k)(1-q^{k-1})\ldots(1-q)} .$$

$L(k,n-k)$ may also be described as the set of all Ferrers diagrams [4, §2.4] which fit in a $k \times (n-k)$ rectangle, ordered by inclusion. Hence, as a special case of our main theorem, we conclude the following: If $\varphi_1,\ldots,\varphi_t$ are Ferrers diagrams which fit in a $k \times (n-k)$ rectangle, with no φ_i contained in a φ_j for $i \ne j$, then t does note exceed the middle coefficient of $\left[{n \atop k}\right]_q$, and this bound can be achieved by choosing $\varphi_1,\ldots,\varphi_t$ to be of size $[\frac{1}{2}k(n-k)]$. It would be interesting to find a combinatorial proof of this fact.

Our second special case consists of choosing $G = SO(2n+1)$. Then W is the group of all $n \times m$ signed permutation matrices (i.e., one nonzero entry in every row or column, equal to ± 1) and has order $2^n n!$. For a particular choice of the parabolic subgroup P , W_J turns out to be the group of permutation matrices, so $|W^J| = 2^n n!/n! = 2^n$. The elements of Q^X may be identified with subsets of $\{1,2,\ldots,n\}$. The partial order on Q^X is given by setting $\{a_1,\ldots,a_j\} \le \{b_1,\ldots,b_k\}$ if $a_1 > a_2 > \cdots > a_j$, $b_1 > b_2 > \cdots > b_k$, $k \ge j$, and $b_i \ge a_i$ for

$1 \leq i \leq j$. This partial order makes Q^X into a distributive lattice, denoted $M(n)$, with rank-generating function

$$F(M(n),q) = (1+q)(1+q^2)\dots(1+q^n) .$$

Now suppose $S = \{\alpha_1,\dots,\alpha_n\}$ is a set of n distinct positive real numbers with $\alpha_1 < \cdots < \alpha_n$. If $\{\alpha_{i_1},\dots,\alpha_{i_r}\}$ and $\{\alpha_{j_1},\dots,\alpha_{j_s}\}$ are unequal subsets of S with the same element sums, then Lindström [9] observed that $\{i_1,\dots,i_r\}$ and $\{j_1,\dots,j_s\}$ are incomparable elements of $M(n)$. From this it is possible to deduce the following result.

<u>Corollary</u> Let S be a set of distinct real numbers. Assume that r elements of S are positive, s are equal to zero (so $s = 0$ or 1), and t are negative. Let $S_1,\dots,S_k$ be subsets of S with the same element sums. Then k does not exceed the middle coefficient of the polynomial

$$F_{rst}(q) = 2^s(1+q)(1+q^2)\dots(1+q^r) \cdot (1+q)(1+q^2)\dots(1+q^t) .$$

Moreover, this bound can be obtained by choosing $S = \{-1,-2,\dots,-r\} \cup \{1,2,\dots,t\} \cup Z$, where $Z = \phi$ or $\{0\}$ depending on whether $s = 0$ or 1 .

More generally, our main theorem implies that if the element sums of $S_1,\dots,S_k$ take on j distinct values, then k does not exceed the sum of the j middle coefficients of $F_{rst}(q)$.

It is easy to show that for fixed $n = r + s + t$, the middle coefficient of $F_{rst}(q)$ is maximized when $s = 1$ and $|r - t| \leq 1$. From this we deduce the following conjecture of Erdös and Moser [6, Eqn. (12)].

<u>Corollary</u> Let S be a set of $n = 2\ell + 1$ distinct real numbers, and let $f(S,m)$ be the number of subsets of S with element sum m . Then

$$f(S,m) \leq f(\{-\ell,-\ell + 1,\ldots,\ell\},0) .$$

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