UNIMODAL SEQUENCES ARISING FROM LIE ALGEBRAS

Richard P. Stanley

We wish to discuss a connection between combinatorial theory and the representation theory of complex semisimple Lie algebras, known in principle since 1950 but not previously explicitly formulated. Define a sequence $a_0, a_1, \ldots, a_d$ of real numbers to be *symmetric* if $a_i = a_{d-i}$ for $0 \le i \le d$, and *unimodal* if for some i we have $a_0 \le a_1 \le \cdots \le a_i \ge a_{i+1} \ge \cdots \ge a_d$. Of course if the sequence $a_0, a_1, \ldots, a_d$ is both symmetric and unimodal, then

$$a_0 \le a_1 \le \cdots \le a_e \ge a_{e+1} \ge \cdots \ge a_d, \text{ if } d = 2e$$

$$a_0 \le a_1 \le \cdots \le a_e = a_{e+1} \ge \cdots \ge a_d, \text{ if } d = 2e + 1$$

A polynomial $a_0 + a_1q + \dots + a_dq^d$ of degree d is said to be *symmetric* (resp., *unimodal*) if the corresponding sequence $a_0,a_1,\dots,a_d$ is symmetric (resp., unimodal). Perhaps the best-known unimodal polynomial is $(1+q)^d$. In this case, unimodality is easy to prove because we have a simple explicit formula for the coefficients. In general, it may be quite difficult to show that a sequence is unimodal (e.g., [19]), and there are an abundance of sequences conjectured to be unimodal but for which no idea of a proof is known.

Here we will describe a large class of symmetric unimodal polynomials arising from Lie algebras. We will describe these polynomials in a way which does not require knowledge of Lie algebras. The proof of unimodality will be omitted, since it requires a knowledge of the representation theory of Lie algebras such as may be found in [11]. The proof may be attributed essentially to Dynkin [7]. He showed that certain polynomials arising from Lie algebras are unimodal. These polynomials had previously been known (e.g., by H. Weyl) to have a simple expression as a product. Hughes [10] was the first to realize the relevance of Dynkin's result to combinatorics. Hughes was unaware of the product formula and worked out some special cases on his own. The only contribution of the present paper is to describe these results in a form which will make them immediately accessible to combinatorialists, and to mention some open problems suggested by the results.

Recall that a *root system* (called by some writers a "reduced root system") is a finite set R of vectors in a real vector space V satisfying certain axioms. What will be of interest to us here is not the actual axioms, but rather the classification of root systems which follows from the axioms. For further information on root systems, see for example [4], [11].

A root system R is said to be *irreducible* if it cannot be written as a non-trivial disjoint union $R_1 \cup R_2$ of two root systems R_1 and R_2 such that every vector in R_1 is orthogonal to every vector in R_2. Every root system is a unique disjoint union of irreducible root systems. To classify all root systems it suffices to classify the irreducible ones. There are four infinite families of irreducible root systems, denoted A_n, B_n, C_n, D_n, and exactly five other irreducible root systems, denoted E_6, E_7, E_8, F_4, G_2. The subscript refers to the dimension of the space spanned by the root system R and is called the *rank* of R. For small values of n some overlap occurs among the root systems A_n, B_n, C_n, D_n. To insure all root systems are distinct, one may take A_n for $n \geq 1$, B_n for $n \geq 2$, C_n for $n \geq 3$, and D_n for $n \geq 4$. For our purposes it is best to describe a root system R as follows. If R has rank n, then there exists a certain subset $\{\alpha_1,\ldots,\alpha_n\}$ of R (not unique) called a *base*. Once a base has been chosen, R decomposes into two subsets R_+ and R_-, whose elements are called the *positive roots* and *negative roots*, respectively. Every vector $\beta \in R_+$ can be written uniquely in the form

$$\beta = \sum_{i=1}^{n} a_i \alpha_i, \tag{1}$$

where a_i is a *non-negative* integer. A vector β lies in R_+ if and only if $-\beta \in R_-$. For future use we introduce the notation

$$x^\beta = x_1^{a_1} \ldots x_n^{a_n},$$

where $x_1,\ldots,x_n$ are independent indeterminates and β is given by (1).

We may now describe a root system R by listing the vectors $\beta \in R_+$ in the form (1). We will give this description for the root systems A_n, B_n, C_n, D_n, G_2, referring the reader to the tables in [4] for the remaining four irreducible root systems. For convenience we denote the vector $\alpha_i + \alpha_{i+1} + \ldots + \alpha_j$ by $[i,j]$. In particular, $[i,i-1] = 0$ and $[i,i] = \alpha_i$.

A_n: $[i,j]$, $\quad 1 \le i \le j \le n$

B_n: $[i,j]$, $\quad 1 \le i \le j \le n$
$[i,j-1] + 2[j,n]$, $\quad 1 \le i < j \le n$

C_n: $[i,j]$, $\quad 1 \le i \le j \le n$
$[i,j-1] + 2[j,n-1] + \alpha_n$, $\quad 1 \le i \le j \le n-1$

D_n: $[i,j]$, $1 \le i \le j \le n$, omitting $(i,j) = (n-1,n)$
$[i,n-2] + \alpha_n$, $\quad 1 \le i \le n-2$
$[i,j-1] + 2[j,n-2] + [n-1,n]$, $\quad 1 \le i < j \le n-2$.

G_2: $[i,j]$, $\quad 1 \le i \le j \le 2$
$2\alpha_1 + \alpha_2$, $3\alpha_1 + \alpha_2$, $3\alpha_1 + 2\alpha_2$.

Given a root system R of rank n, define a polynomial

$$P(R;x_1,\ldots,x_n) = \prod_{\beta\in R_+} (1-x^\beta).$$

For instance,

$$P(A_3;x,y,z) = (1-x)(1-y)(1-z)(1-xy)(1-yz)(1-xyz)$$

$$P(B_3;x,y,z) = (1-x)(1-y)(1-z)(1-xy)(1-yz)(1-xyz) \cdot (1-xyz^2)(1-xy^2z^2)(1-yz^2)$$

$$P(C_3;x,y,z) = (1-x)(1-y)(1-z)(1-xy)(1-yz)(1-xyz) \cdot (1-xy^2z)(1-x^2y^2z)(1-y^2z)$$

$$P(D_3;x,y,z) = (1-x)(1-y)(1-z)(1-xy)(1-xyz)(1-xz)$$

$$P(G_2;x,y) = (1-x)(1-y)(1-xy)(1-x^2y)(1-x^3y)(1-x^3y^2)$$

Note that $P(D_3;x,y,z) = P(A_3;y,x,z)$. This is because A_3 and D_3 are isomorphic, as alluded to above.

We now come to the theorem of Dynkin.

THEOREM 1 *Let* R *be a root system of rank* n, *and let* $m_1,\dots,m_n$ *be any positive integers. Define*

$$Q(R;m_1,m_2,\dots,m_n) = \frac{P(R;q^{m_1},q^{m_2},\dots,q^{m_n})}{P(R;q,q,\dots,q)} .$$

Then $Q(R;m_1,m_2,\dots,m_n)$ *is a symmetric unimodal polynomial in the variable* q *with nonnegative integer coefficients.*

EXAMPLE 1

$$\begin{aligned} Q(A_3;2,2,1) &= \frac{(1-q)(1-q^2)^2(1-q^3)(1-q^4)(1-q^5)}{(1-q)^3(1-q^2)^2(1-q^3)} , \\ &= 1 + 2q + 3q^2 + 4q^3 + 4q^4 + 3q^5 + 2q^6 + q^7. \end{aligned}$$

REMARK Let $[i] = 1 - q^i$ and $[i]! = [i][i-1]\dots[1]$. Then

$$P(R;q,q,\dots,q) = [e_1]!\dots[e_n]!,$$

where $e_1,\dots,e_n$ are certain integers known as the *exponents* of R.

REMARK If R is reducible, say $R = R_1 \cup R_2$, then

$$P(R;x_1,\dots,x_n) = P(R_1;x_1,\dots,x_k)P(R_2;x_{k+1},\dots,x_n).$$

Since the product of two symmetric unimodal polynomials is easily seen to be symmetric and unimodal (e.g., [2]), we could have confined our attention to irreducible root systems.

EXAMPLE 2 There is a combinatorial description of the polynomial (2) when $R = A_n$. Given $m_1,\ldots,m_n$, let λ be the partition with $m_i - 1$ parts equal to i, and let $\lambda' = (\lambda_1',\lambda_2',\ldots,\lambda_n')$ be the conjugate partition to λ. Then

$$Q(A_n;m_1,\ldots,m_n) = \frac{\Pi(1-q^{c_i+n+1})}{\Pi(1-q^{h_i})}, \tag{3}$$

where the c_i's are the contents and the h_i's are the hook lengths of λ', as defined in [16, Def. 15.1]. It follows from [16, Thm. 15.3] that the coefficient of q^j in (3) is equal to the number of column-strict plane partitions of $j + \Sigma i\lambda_i'$ of shape λ' and largest part at most $n + 1$. In the special case $m_1 = k+1$, $m_2 = m_3 = \ldots = m_n = 1$, (3) reduces to the Gaussian coefficient

$$\begin{bmatrix} k+n \\ n \end{bmatrix} = \frac{[k+n]!}{[n]![k]!}$$

Hence we recover the known result that the Gaussian coefficients are unimodal polynomials. This was first mentioned explicitly by Elliott [8,§129] as a consequence of the Cayley-Sylvester theorem of invariant theory, stated by Cayley [5,p. 265] in 1856 and proved by Sylvester [18] in 1878. A modern treatment of the Cayley-Sylvester theorem appears in [15, Thm. 3.3.4] and uses the same basic ideas necessary to prove Theorem 1. For further aspects of the Cayley-Sylvester theorem, see [12]. The unimodality of the Gaussian coefficients was one of the special cases of Theorem 1 worked out by Hughes [10]. No purely combinatorial proof of this unimodality result is known, though several other noncombinatorial proofs have been given besides those mentioned above. For instance, there is a proof using Hodge theory applied to the Grassmann manifold, and a proof of Almkvist-Fossum using invariant theory in characteristic p [1, p. 111.5, Rmk. 1.8]. All these proofs are closely related to one another and are based on the same underlying principle, viz., the representation theory of the Lie algebra $sl(2,\mathbb{C})$ or the Lie group $SL(2,\mathbb{C})$. Recently R. Proctor has analyzed the above proofs and from this has produced a proof (to be published) of the unimodality of the Gaussian coefficients using only elementary linear algebra.

EXAMPLE 3 A simple computation shows

$$Q(C_n;1,1,\ldots,1,2) = (1+q)(1+q^2)\ldots(1+q^n).$$

This is another case worked out by Hughes [10]. More generally, the coefficient of q^j in $Q(C_n;1,1,\ldots,1,m)$ is the number of column-strict plane partitions of j with less than m columns and largest part at most n. This follows from the Bender-Knuth conjecture, proved by Andrews in [3]. Alternatively, I. G. Macdonald (unpublished) has given a Lie-theoretic proof of this interpretation of the coefficients of $Q(C_n;1,1,\ldots,1,m)$, thereby yielding another proof of the Bender-Knuth conjecture.

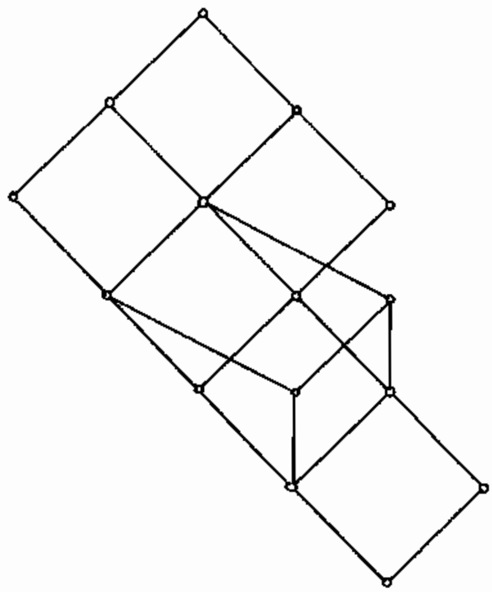
PROBLEM 1 Let S be a finite set of non-zero vectors in $\mathbb{R}^n$ with non-negative integer coordinates. Form the polynomial $F(S;x_1,\ldots,x_n) = \prod_{\beta\in S} (1-x^\beta)$. For what sets S is the rational functions $F(S;q^{m_1},\ldots,q^{m_n})/F(S;q,\ldots,q)$ a polynomial in q for all positive integers $m_1,\ldots,m_n$? This is similar to a problem considered by MacMahon [14, Section VIII, Ch. 5].

PROBLEM 2 Let $R = A_n$. Given $m_1,\ldots,m_n$, let λ' be defined as in Example 2. Let $L(\lambda')$ be the poset (partially ordered set) whose elements are column-strict plane partitions $\pi = (\pi_{ij})$ of shape λ' and largest part at most $n + 1$, ordered by setting $\pi \leq \pi'$ if $\pi_{ij} \leq \pi'_{ij}$ for all (i,j). It is easily seen that L

is a distributive lattice. If L has a_i elements of rank i, then it follows from Example 2 that

$$\Sigma a_i q^i = Q(A_n;m_1,\ldots,m_n)$$

In other words, (3) is the "rank generating function" of $L(\lambda')$. Whenever we have a poset L whose rank generating function is symmetric and unimodal, we can ask whether L has a *symmetric chain decomposition*, i.e., a partition into saturated chains $y_1 < y_2 < \ldots < y_k$, where if ρ is the rank function of L, then $\rho(y_1) + \rho(y_k) = \rho(L)$. Such symmetric chain decompositions have been found for a variety of posets (a survey of this subject appears in [9]), but it remains open for the lattices $L(\lambda')$. A somewhat weaker result in the special case that λ' has one part is proved in [17]. One unusual feature of this problem is that although the rank generating function of $L(\lambda')$ is symmetric and unimodal, it need not be true that $L(\lambda')$ is self-dual. For instance, if $n = 2$, $(m_1,m_2) = (2,3)$, $\lambda' = (3,2)$, then $L(\lambda')$ is given by



In a similar manner one can interpret $Q(C_n;1,1,\ldots,1,m)$ as the rank generating function of a certain finite distributive lattice.

PROBLEM 3 Which other of the polynomials (2) are the rank generating functions for distributive lattices (or perhaps just posets) "naturally associated" with the root system R? The results of [13] are closely related to this question.

REFERENCES

1. G. Almkvist and R. Fossum, Decompositions of exterior and symmetric powers of indecomposable $\mathbb{Z}/p\mathbb{Z}$-modules in characteristic p and relations to invariants, Københavns Universitet, Matematisk Institut, Preprint Series no. 19, August, 1977.

2. G. E. Andrews, A theorem on reciprocal polynomials with applications to permutations and compositions, *Amer. Math. Monthly* 82 (1975), 830-833.

3. G. E. Andrews, Plane partitions (II): The equivalence of the Bender-Knuth and MacMahon conjectures, *Pacific J. Math.* 72 (1977), 283-291.

4. N. Bourbaki, Groupes et algèbres de Lie, Ch. 4,5, et 6, Hermann, Paris, 1968.

5. A. Cayley, A second memoir upon quantics, *Coll. Math. Papers,* II, 250-275, Cambridge University Press, 1889.

6. L. Comtet, Advanced Combinatorics, D. Reidel Pub. Co., Dordrecht, 1974.

7. E. B. Dynkin, Some systems of weights of linear representations of semisimple Lie groups (Russian), *Dokl. Akad. Nauk SSSR (N.S.)* 71 (1950), 221-224.

8. E. B. Elliott, An Introduction to the Algebra of Quantics, Oxford University Press, 1895.

9. J. R. Griggs, Sufficient conditions for a symmetric chain order, *SIAM J. Appl. Math.* 32 (1977), 807-809.

10. J. W. B. Hughes, Lie algebraic proofs of some theorems on partitions, 135-155, in *Number Theory and Algebra* (H. Zassenhaus, ed.), Academic Press, New York, 1977.

11. J. E. Humphreys, Introduction to Lie Algebras and Representation Theory, *Graduate Texts in Math.* 9, Springer, New York/Heidelberg/Berlin, 1972.

12. A. Iarrobino, Notes in invariant theory I: Cayley-Sylvester theorem, preprint.

13. V. Lakshmibai, C. Musili and C. S. Seshadri, Geometry of G/P, *Bull. Amer. Math. Soc.,* (new series) 1 (1979), 432-435.

14. P. A. MacMahon, Combinatory Analysis, Vols. 1-2, Cambridge University Press, 1915, 1916, reprinted by Chelsea, New York, 1960.

15. T. A. Springer, Invariant Theory, Lecture Notes in Math., No. 585, Springer, Berlin/Heidelberg/New York, 1977.

16. R. Stanley, Theory and application of plane partitions, Parts 1 and 2, *Studies in Applied Math.* 50 (1971), 167-188, 259-279.

17. R. Stanley, Weyl groups, the hard Lefschetz theorem, and the Sperner property, *SIAM J. Algebraic and Discrete Methods,* to appear.

18. J. J. Sylvester, Proof of the hitherto undemonstrated fundamental theorem of invariants, *Coll. Math. Papers,* III, 117-126, Chelsea, New York, 1973.

19. G. Szekeres, An asymptotic formula in the theory of partitions, II, *Quart. J. Math. Oxford* (2) 4 (1953), 96-111.

Added in proof. The unpublished proof of Macdonald mentioned in Example 3 now appears in I. G. Macdonald, *Symmetric Functions and Hall Polynomials,* Oxford University Press, 1979, 50-52.